PROBABILITY DISTRIBUTIONS RELATED TO RANDOM MAPPINGS

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1. Introduction and Summary. A Random Mapping Space \((X, \pi, P)\) is a triplet, where \(X\) is a finite set of elements \(x\) of cardinality \(n\), \(\pi\) is a set of transformations \(T\) of \(X\) into \(X\), and \(P\) is a probability measure over \(\pi\).

In this paper, four choices of \(\pi\) are considered

(I) \(\pi\) is the set of all transformations of \(X\) into \(X\).
(II) \(\pi\) is the set of all transformations of \(X\) into \(X\) such that for each \(x \in X\)

\[ Tx \neq x. \]

(III) \(\pi\) is the set of one-to-one mappings of \(X\) onto \(X\).
(IV) \(\pi\) is the set of one-to-one mappings of \(X\) onto \(X\), such that for each

\[ x \in X, \ Tx \neq x. \]

In each case \(P\) is taken as the uniform probability distribution over \(\pi\).

If \(x \in X\) and \(T \in \pi\), we will define \(T^k\) as the \(k\)th iteration of \(T\) on \(x\), where \(k\) is an integer, i.e. \(T^0x = T(T^{k-1}x)\), and \(T^0x = x\) for all \(x\). The reader should note that, in general, \(T^k, k < 0\), may not exist or may not be uniquely determined.

If for some \(k \geq 0\), \(T^kx = y\), then \(y\) is said to be a \(k\)th image of \(x\) in \(T\). The set of successors of \(x\) in \(T\), \(S_T(x)\) is the set of all images of \(x\) in \(T\), i.e.,

\[ S_T(x) = \{x, Tx, T^2x, \ldots, T^{n-1}x\}, \]

which need not be all distinct elements.

If for some \(k \leq 0\), \(T^kx = y\), \(y\) is said to be a \(k\)th inverse of \(x\) in \(T\). The set of all \(k\)th inverses of \(x\) in \(T\) is \(T^{(k)}(x)\) and

\[ P_T(x) = \bigcup_{k=-n}^{0} T^{(k)}(x) \]

is the set of predecessors of \(x\) in \(T\).

If there exists an \(m > 0\), such that \(T^m x = x\), then \(x\) is a cyclical element of \(T\) and the set of elements \(x, Tx, T^2x, \ldots, T^{m-1}x\) is the cycle containing \(x\), \(C_T(x)\). If \(m\) is the smallest positive integer for which \(T^m x = x\), then \(C_T(x)\) has cardinality \(m\).

We note further an interesting equivalence relation induced by \(T\). If there exists a pair of integers \(k_1, k_2\) such that

\[ T^{k_1}x = T^{k_2}y, \]

then \(x \sim y\) under \(T\).

It is readily seen that this is in fact an equivalence, and hence decomposes \(X\)

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into equivalence classes, which we shall call the components of \( X \) in \( T \); and designate by \( K_T(x) \) the component containing \( x \).

We define \( s_T(x) \) to be the number of elements in \( S_T(x) \), \( p_T(x) \) to be the number of elements in \( P_T(x) \), and \( l_T(x) \) to be the number of elements in the cycle contained in \( K_T(x) \) (i.e. \( l(x) \) = the number of elements in \( C_T(x) \) if \( x \) is cyclical). We designate by \( q_T \) the number of elements of \( X \) cyclical in \( T \), and by \( r_T \) the number of components of \( X \) in \( T \).

Rubin and Sitgreaves [9] in a Stanford Technical Report have obtained the distributions of \( s, p, l, q, \) and have given a generating function for the distribution of \( r \) in case I. Folkert [3], in an unpublished doctoral dissertation has obtained the distribution of \( r \) in cases I and II. The distribution of \( r \) in case III is classical and may be found in Feller [2], Gontcharoff [4], and Riordan [8]. In the present paper, a number of these earlier results are rederived and extended. Specifically, for cases I and II, we compute the probability distributions of \( s, p, l, q, \) and \( r \). In cases III and IV the distributions of \( l \) and \( r \) are given. In addition some asymptotic distributions and low order moments are obtained.

For the convenience of the reader, an index of notations having a fixed meaning is provided in the appendix to the paper.

2. Representation of \( T \) as a directed graph. It will be convenient to represent elements of \( X \) as directed graphs. For example, if \( n = 10, X = \{1, 2, 3, 4, \ldots, 10\} \),

\[
T(1) = 4, \quad T(2) = 5, \quad T(3) = 9, \quad T(4) = 8, \quad T(5) = 5, \\
T(6) = 8, \quad T(7) = 9, \quad T(8) = 1, \quad T(9) = 4, \quad T(10) = 8,
\]

Then \( T \) has the representation below:

![Directed Graph](image)

3. Probability Distribution for Case I. In case I, \( P(T) = 1/n^n \) for all \( T \in \mathcal{F} \). We now turn to the computation of the probability distributions of \( s \) and \( l \), the number of elements in \( S_T(x) \) and the number of elements in the cycle contained in \( K_T(x) \) respectively.

Then, for any choice of \( x \), we have:

\[
P(s = k, l = j) = P[T^r x \neq x, Tx, \ldots, T^{r-1} x (0 < r \leq k - 1); T^k x = T^{k-j} x]
\]
Hence

\[(3.1) \quad P(s = k, l = j) = \frac{(n - 1)!}{(n - k)!n^k}, \quad 1 \leq j \leq k \leq n,\]

and summing over \(j\), we have

\[(3.2) \quad P(s = k) = \frac{(n - 1)!k}{(n - k)!n^k}, \]

\[(3.3) \quad P(l = j) = \sum_{k=j}^{n} \frac{(n - 1)!}{(n - k)!n^k}. \]

From consideration of symmetry, we note trivially that

\[(3.4) \quad E(l) = E((s + 1)/2).\]

We now obtain the asymptotic probability densities of \(s\) and \(l\). In (3.1) let \(k = \sqrt{nx}, j = \sqrt{ny}\), and replace factorials by Stirling’s approximation. Then we have

\[(3.5) \quad P(s = \sqrt{nx}, l = \sqrt{ny}) \sim \frac{n^{n-\sqrt{nx}-\frac{1}{2}} e^{-\sqrt{nx}}}{(n - \sqrt{nx})^{n-\sqrt{nx}+\frac{1}{2}}} \cdot \frac{n^{n-\sqrt{nx}-\frac{1}{2}} e^{-\sqrt{nx}}}{n^{n-\sqrt{nx}+\frac{1}{2}} \left(1 - \frac{x}{\sqrt{n}}\right)^{n-\sqrt{nx}+\frac{1}{2}}}.\]

Write

\[\left(1 - \frac{x}{\sqrt{n}}\right)^{n-\sqrt{nx}+\frac{1}{2}} = \exp \left[(n - \sqrt{nx} + \frac{1}{2}) \log \left(1 - \frac{x}{\sqrt{n}}\right)\right],\]

and expand \(\log (1 - x/\sqrt{n})\) in a power series, obtaining

\[P(s = \sqrt{nx}, l = \sqrt{ny}) \sim n^{-1} e^{-\frac{1}{2}x^2}.\]

Thus, the asymptotic density of \((s/\sqrt{n}, l/\sqrt{n})\) is

\[(3.6) \quad f(x, y) = e^{-\frac{1}{2}x^2}, \quad 0 < y \leq x < \infty.\]

The marginal distributions \(f_1(x), f_2(y)\) give the asymptotic densities of \(s/\sqrt{n}, l/\sqrt{n}\) respectively and are easily obtained by integration.

\[(3.7) \quad f_1(x) = xe^{-\frac{1}{2}x^2}, \quad 0 < x,\]

\[(3.8) \quad f_2(y) = \sqrt{2\pi} (1 - \Phi(y)), \quad 0 < y,\]

where \(\Phi(y) = \int_{-\infty}^{y} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} \, dx.\)

In numerical computations, the cumulative distribution function \(F_2(y)\) is probably more useful than the density function \(f_2(y)\) and is therefore given below:

\[(3.9) \quad F_2(y) = P(Y \leq y) = 1 - e^{-\frac{1}{2}y^2} + y\sqrt{2\pi}(1 - \Phi(y)).\]
We note further that

\[(3.10) \quad EY' = \frac{1}{r + 1} \quad EX' = \frac{2^{r/2}}{r + 1} \Gamma \left( \frac{r + 2}{2} \right).\]

Hence

\[(3.11) \quad E(l) \sim \frac{1}{4} (2\pi n)^{\frac{3}{2}}, \quad \sigma_l^2 \sim n[(2/3) - (2\pi/16)].\]

Formulas (3.1), (3.2), (3.3), and (3.7) have been obtained by Rubin and Sitgreaves [9].

Rubin and Sitgreaves have also shown that

\[(3.12) \quad P\{q = j\} = \frac{(n - 1)! j}{(n - j)! n^j}, \quad j = 1, 2, \ldots, n.\]

We now prove this using a partition argument due to Katz [7].

Consider the directed graph representation of $T$ and partition $X$ as follows. Let $M_0(T)$ be those elements of $X$ cyclical under $T$. Define $M_1(T)$ to be those elements of $X$ whose images are cyclical under $T$, but are not themselves cyclical. Let $M_j(T)$ be those elements of $X$ whose images under $T$ are in $M_{j-1}(T)$. Continuing in this manner until $X$ is exhausted, the $n - j$ non-cyclical elements of $X$ are partitioned into $m_j(T)$ sets each non-empty for $j \neq n$. Designate the cardinality of $M_j(T)$ by $n_j(T)$, $j = 1, 2, \ldots, m(T)$.

The number of decompositions of $X$ for $n_1, n_2, \ldots, n_m$ fixed is

\[(3.13) \quad \frac{n!}{j! n_1! n_2! \cdots n_m! j^{n_1} n_1^{n_2} \cdots n_m^{n_{m-1}}},\]

where $\sum_{i=1}^m n_i = n - j$. Hence

\[(3.14) \quad P\{q = j\} = n^{-j} \sum \frac{n!}{n_1! n_2! \cdots n_m! j^{n_1} n_1^{n_2} \cdots n_m^{n_{m-1}}}, \quad j \neq n,\]

where the sum is taken over all non-empty $m$-partitions of $n - j$.

Katz [7] has shown that

\[(3.15) \quad \sum \frac{n!}{j! n_1! n_2! \cdots n_m! j^{n_1} n_1^{n_2} \cdots n_m^{n_{m-1}}} = \frac{n! n^{n-j-1}}{(j - 1)! (n - j)!},\]

from which we obtain (3.12) for $j \neq n$. We have $j = n$, if and only if $T$ is one-to-one and onto; hence

\[(3.16) \quad P\{q = n\} = n!/n^n,\]

which coincides with (3.12) for $j = n$.

It is curious to note that this is exactly the same as the distribution of $s$ given in (3.2), and hence has the same asymptotic distribution and asymptotic moments.

The distribution of $p$ has been obtained by Rubin and Sitgreaves [9],

\[(3.17) \quad P\{p = j\} = \frac{(n - 1)! j^{j-2} (n - j)^{n-j}}{(n - j)! (j - 1)! n^{n-1}}.\]
We establish this as follows:

Let \( X_{j-1} \) be \( j - 1 \) specified elements of \( X \) say \( x_1, x_2, \ldots, x_{j-1} \). Let \( x \) be a distinguished element of \( X \) not in \( X_{j-1} \). Then define \( 3_1 \) as those transformations \( T \) in \( 3 \) such that \( T(X - (X_{j-1} \cup x)) = X - (X_{j-1} \cup x) \). Define \( 3_2 \) as those transformations in \( 3 \) such that \( T(x_{j-1}) = x_{j-1} \cup x \), and \( T^k x_i = x \) for some \( k > 0 \) and \( i = 1, 2, \ldots, j - 1 \). We further define \( 3^* = 3 \cap 3_2 \). Then

\[
\binom{n - 1}{j - 1} P(T \in 3^*) = P(p = j),
\]

and

\[
P(T \in 3^*) = P(T \in 3_1)P(T \in 3_2).
\]

We readily see that

\[
P(T \in 3_1) = \left(\frac{n - j}{n}\right)^n.
\]

Hence we have only to compute \( P(T \in 3_2) \). For any \( T \in 3_2 \), we can, by restricting attention to \( X_{j-1} \) define an associated transformation \( T_{j-1}^* \) which has \( T_{j-1}^* x_i = T x_i, i = 1, 2, \ldots, j - 1 \), and \( T_{j-1}^* x_j = x \). Let \( N_{j-1} \) be the member of distinct transformations which can be constructed in this manner from \( T \in 3_2 \). Since, in \( 3_2, x \) has \( n \) equally likely images under \( T \), we have

\[
P(T \in 3_2) = N_{j-1}/n^{j-1}
\]

and \( N_{j-1} \) is readily obtained by Katz’s Lemma and the partition argument used in (3.13). Hence

\[
N_{j-1} = \frac{1}{j} \sum_{n_1, n_2, \ldots, n_m} \frac{j!}{n_1! \cdots n_m!} \sum_{n_{m+1}} n_{m+1}^{n_{m+1}} = j^{j-2},
\]

\[ j > 1,
\]

and, trivially, \( N_0 = 1 \).

In (3.22), the sum is over all non-empty \( m \)-part partitions of \( j - 1 \), and the factor \( (1/j) \) is obtained by distinguishing the element \( x \). Hence

\[
P(p = j) = \binom{n - 1}{j - 1} \frac{j^{j-2}}{n^{j-1}} \left(\frac{n - j}{n}\right)^{n-j}
\]

and (3.17) is established.

We now note an interesting relationship,

\[
E(S) = E(p).\footnote{This was pointed out by D. Blackwell in a private conversation with the author.}
\]

This is established at once by symmetry. For any \( T \in 3 \) such that \( y \) is a successor of \( x \), there is a corresponding \( T \in 3 \) with \( x \) a predecessor of \( y \); the correspondence is accomplished by interchanging \( x \) and \( y \) in the directed graphs.

We may also note an interesting physical property of directed graphs of this type, which holds for every \( T \in 3 \). For any \( T \in 3 \), let \( r_j \) be the number of ele-
ments $x$ for which $T^{-1}x$ has $j$ elements. Then,

$$\sum_{j=0}^{n} j r_j = n. \tag{3.24}$$

Also,

$$\sum_{j=0}^{n} r_j = n. \tag{3.25}$$

Thus,

$$r_0 = \sum_{j=1}^{n-1} j r_{j+1}. \tag{3.26}$$

From this it follows at once that

$$E(p^{(1)}) = 1, \tag{3.27}$$

where $p^{(1)}$ is the number of elements in $T^{-1}(x)$.

The distribution of $p^{(1)}$ is readily seen to be

$$P\{p^{(1)} = j\} = \binom{n}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{n-j}. \tag{3.28}$$

We proceed now to the question of the probability distribution of $r$, the number of components of $X$ in $T$. Folkert [3] has obtained the distribution and has shown

$$P\{r = j\} = \frac{1}{n^\mu} \sum_{\mu} \frac{1}{n!} \sum_{k_1, k_2, \ldots, k_\mu} \frac{n!}{k_1! k_2! \cdots k_\mu!} k_1^{i_1} k_2^{i_2} \cdots k_\mu^{i_\mu}, \tag{3.29}$$

where $S_\mu^i$ are Stirling's Numbers of the First Kind, and the sum over $k_1, k_2, \ldots, k_\mu$ is over all choices of $k_1, k_2, \ldots, k_\mu$ with $k_i > 0$ ($i = 1, 2, \ldots, \mu$).

In this paper we obtain a probability generating function for the number of components, which has a good deal of intrinsic interest because of its relation to Faa de Bruno's formula (Jordan [6]) and the exponential polynomials of Bell [1].

Let $k_i$ denote the number of components with exactly $i$ elements. Then every $T \in \mathfrak{S}$ determines an $n$-tuple $(k_1, k_2, \ldots, k_n)$. Hence, for every specification of $(k_1, k_2, \ldots, k_n)$ we have a set of transformations $3_{k_1, k_2, \ldots, k_n}$ in $\mathfrak{S}$.

Then

$$P(T \in \mathfrak{S}_{k_1, k_2, \ldots, k_n}) = \frac{n! I_1^{k_1} I_2^{k_2} \cdots I_n^{k_n}}{1^{k_1} 2^{k_2} \cdots n^{k_n} k_1! k_2! \cdots k_n! n^n}, \tag{3.30}$$

where $I_j/j^j$ ($j = 1, 2, \ldots, n$) is the probability that a transformation $T_j$ on $j$ elements $X_j$ is indecomposable, i.e. $K_T(x) = X_j$ for all $x \in X_j$, where $0 \leq k_i \leq n$.
and \( \sum_{i=1}^{n} ik_i = n \). We have

\[
I_i = \sum_{i=1}^{j} P(q = i, K_{ij}(x) = X_j) = \sum_{i=1}^{j} \frac{(j - 1)! i(i - 1)!}{(j - i)! j! i!}.
\]

Hence

\[
I_i = \sum_{i=1}^{j} \frac{(j - 1)! j^i}{i!}.
\]

This result has been obtained earlier by both Katz [7] and Rubin and Sitgreaves [9]. Then, the generating function of \( k_1, k_2, \ldots, k_n \) is given by

\[
G(x_1, x_2, \ldots, x_n) = \sum_{k_1, k_2, \ldots, k_n} \frac{n!(I_1 x_1)^{k_1}(I_2 x_2)^{k_2} \cdots (I_n x_n)^{k_n}}{1! 2! \cdots n! k_1! k_2! \cdots k_n! n^n},
\]

since the coefficient of \( x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} = P(T \in 3_{k_1}, k_2, \ldots, k_n) \) for \( \sum_{i=1}^{n} ik_i = n \).

Since \( r = \sum_{i=1}^{n} k_i \),

\[
G(x_1, x_2, \ldots, x_n) = \sum_{k_1, k_2, \ldots, k_n} \frac{n! r!(I_1 x_1)^{k_1}(I_2 x_2)^{k_2} \cdots (I_n x_n)^{k_n}}{r! 1! 2! \cdots n! k_1! k_2! \cdots k_n! n^n},
\]

and

\[
G(x_1, x_2, \ldots, x_n) = \sum_{r} \frac{n!}{r! n^n} \left( \frac{I_1 x_1}{1!} + \frac{I_2 x_2}{2!} + \cdots + \frac{I_n x_n}{n!} \right)^r.
\]

We can extend the definition to \( G(x_1, x_2, \ldots) \) with no loss of generality, since this will in no way affect the coefficient of \( x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \). Hence

\[
G(x_1, x_2, \ldots) = \frac{n!}{n^n} \exp \sum_{i=1}^{\infty} \frac{I_i x_i}{i!}.
\]

If in (3.34), we replace \( x_i \) by \( x^i \), the coefficient of \( x^n \) in \( G(x, x^2, \cdots) \) is 1 for all \( n \). Thus, we have

\[
\frac{n!}{n^n} \exp \sum_{i=1}^{\infty} \frac{I_i x^i}{i!} = \frac{n!}{n^n} \sum_{i=0}^{\infty} \frac{x^i}{i!}
\]

and

\[
\sum_{i=0}^{\infty} I_i x^i/i! = \log \sum_{i=0}^{\infty} (i^i/i!)x^i.
\]

Replacing \( x_i \) by \( t^i x_i \) in (3.34) we obtain:

\[
G(t x_1, t^2 x_2, \cdots) = \frac{n!}{n^n} \exp \sum_{i=1}^{\infty} \frac{I_i t^i x_i}{i!}.
\]

In (3.37) the coefficient of \( t^n \) gives the probability of any possible decomposition of \( X \) into components with the exponents of \( x_i \) indexing the decomposition.

Finally we observe that, replacing \( x_i \) by \( t x_i \), we get
\[ G(tx, tx^2, \cdots) = \frac{n^l}{n^n} \exp t \sum_{i=1}^{\infty} \frac{I_i x^i}{i!}, \]

or equivalently

\[ G(tx, tx^2, \cdots) = \frac{n^l}{n^n} \left[ \sum_{i=0}^{\infty} \frac{x^i}{i!} \right]^t, \]

and the coefficient of \( t^k x^n \) in \( G(tx, tx^2, \cdots) \) is \( P(r = k) \).

We now employ the generating function given above to obtain Folkert's formula (3.29). From (3.38), we have

\[ \text{coefficient of } t^k = \frac{n^l}{n^n k!} \left( \sum_{i=1}^{\infty} \frac{I_i x^i}{i!} \right)^k, \]

and from (3.36) we have

\[ \text{coefficient of } t^k = \frac{n^l}{n^n k!} \left[ \log \left( 1 + \sum_{i=1}^{\infty} \frac{x^i}{i!} \right)^t \right]^k, \]

Since

\[ \log (1 + u)^k = \sum_{\mu=1}^{\infty} \frac{k!}{\mu!} S_{\mu}^{\mu} u^\mu; \]

see, for example, Jordan [6], p. 146. Employing this in (3.41), we get

\[ \text{coefficient of } t^k = \frac{n^l}{n^n k!} \sum_{\mu=1}^{\infty} \frac{k!}{\mu!} S_{\mu}^{\mu} \left[ \sum_{i=1}^{\infty} \frac{x^i}{i!} \right]^k, \]

and, expansion by the multinomial theorem gives

\[ \text{coefficient of } t^k = \frac{n^l}{n^n k!} \sum_{\mu=1}^{\infty} \frac{k!}{\mu!} S_{\mu}^{\mu} \sum_{k_1, k_2, \ldots, k_n \geq 0, \sum_{\mu=1}^{n} k_\mu = \mu} \frac{1}{k_1! k_2! \cdots k_n!} \left( \frac{1}{1!} \right)^{k_1} \left( \frac{2}{2!} \right)^{k_2} \cdots \left( \frac{n}{n!} \right)^{k_n} \]

To find the coefficient of \( x^n \) in (3.43) it suffices to restrict the second sum to non-negative \( n \)-tuples \((k_1, k_2, \cdots, k_n)\) with \( \sum_{i=1}^{n} k_i = \mu, \sum_{i=1}^{n} ik_i = n; \) hence

\[ P(r = k) = \frac{n^l}{n^n k!} \sum_{\mu=1}^{\infty} \frac{k!}{\mu!} S_{\mu}^{\mu} \sum_{k_1, k_2, \ldots, k_n \geq 0, \sum_{\mu=1}^{n} k_\mu = \mu} \frac{1}{k_1! k_2! \cdots k_n!} \left( \frac{1}{1!} \right)^{k_1} \left( \frac{2}{2!} \right)^{k_2} \cdots \left( \frac{n}{n!} \right)^{k_n}, \]

which coincides with (3.29), except that partitions of \( n \) are enumerated without regard to order in (3.44), and thus we have obtained an alternate form of Folkert's formula. Rubin and Sitgreaves [9] noted that \( n^{-\mu} E(s) = E(s^{-\mu}) \). We remark, further, that it is even more curious that

\[ n^{-\mu} E(s) = P(r = 1) = P(l = 1) = n^{-\mu} E(p) = n^{-\mu} E(q). \]
4. Probability Distribution for Case II. In case II, \( P(T) = (n - 1)^{-n} \) for all \( T \in \mathcal{S} \). As in case I, we first consider the probability distribution of \( s \) and \( l \).

Computing exactly as in Section 3, we obtain

\[
P(s = k, l = j) = \frac{(n - 2)!}{(n - 1)^{k-1}(n - k)!}, \quad 2 \leq j \leq k \leq n,
\]

and

\[
P(s = k) = \frac{(n - 2)! (k - 1)}{(n - 1)^{k-1}(n - k)!}, \quad 2 \leq j \leq k \leq n.
\]

\[
P(l = j) = \sum_{k=j}^{n} \frac{(n - 2)!}{(n - 1)^{k-1}(n - k)!}, \quad 2 \leq j \leq n.
\]

Comparing these results with (3.1), (3.2) and (3.3), we have

\[
P(s = k | I, n) = P(s = k + 1 | II, n + 1)
\]

\[
P(s = k, l = j | I, n) = P(s = k + 1, l = j + 1 | II, n + 1),
\]

and

\[
P(l = j | I, n) = P(l = j + 1 | II, n + 1).
\]

Hence

\[
E(l | II, n + 1) = E(l | I, n) + 1
\]

and

\[
E(s | II, n + 1) = E(s | I, n) + 1.
\]

From (3.4) we have

\[
\frac{1}{2} E(s | II, n) + 1 = E(l | II, n).
\]

Then, by analogy with (3.6), we note that the asymptotic density of

\[
(s/\sqrt{n - 1}, l/\sqrt{n - 1})
\]

is

\[
f(x, y) = e^{-\frac{1}{2}x^2}, \quad 0 \leq y \leq x < \infty,
\]

giving the same marginal density functions as (3.7) and (3.8).

Now consider the probability distribution of the number of elements of \( X \) cyclical under \( T \). We show that

\[
P(q = j) = n^{n-j} D_j \left( \frac{n - 1}{n - 1} \right)^j / (n - 1)^n,
\]

where \( D_j \) is the \( j \)th derangement number, i.e., \( D_j \) is the nearest integer to \( j! / e \), \( j \neq 0 \), and \( D_0 = 1 \).

The proof is identical with the proof of (3.12) except that the \( j! \) in the numerator of (3.13) is replaced by \( D_j \). Hence an application of Katz's lemma
gives

\[ P(q = j) = \frac{1}{(n-1)^n} \sum_{j! n_1! n_2! \cdots n_m!} D_j n^n \sum_{n_{m-1}}^m, \quad j \neq n, \]

the sum being taken over all non-empty \( m \)-part partitions of \( n - j \). Hence

\[ P(q = j) = \frac{1}{(n-1)^n} D_j \frac{n^n n^{n-j-1}}{(j-1)!(n-j)!}, \quad j \neq n, \]

and thus \( P(q = j) \) is given by (4.9). The case \( j = n \), is given trivially by (4.9).

The asymptotic distribution is obtained by replacing \( D_j \) by \( j! e \), and replacing factorials by Stirling’s approximation. Then, letting \( j = \sqrt{n}y \), we get

\[ f(y) = y^{-1/4}, \quad 0 < y < \infty, \]

for the asymptotic density of \( qn^{-1} \). The agreement of (4.12) with (3.7) can hardly be surprising in view of the agreement of (3.2) and (3.12).

We now obtain the distribution of \( p \),

\[ P(p = q) = \frac{(n-1)!(n-j-1)^{n-j-2}}{(j-1)!(n-j)!(n-1)^{n-1}}, \quad j = 1, 2, \ldots, n-2, \]

\[ P(p = n - 1) = 0, \]

\[ P(p = n) = 1 - \sum_{j=1}^{n-2} P(p = j) = \frac{n!}{(n-1)^n} \sum_{j=2}^{n} \frac{n^{n-j-2}}{(n-j)!}. \]

This is established as follows. Define \( X_{j-1}, 3_1, 3_2, \) and \( 3^* \), as in case I. Let \( x \) be a distinguished element of \( X \). Then, as before,

\[ P(p = j) = \binom{n-1}{j-1} P(T \varepsilon 3^*), \]

and

\[ P(T \varepsilon 3^*) = P(T \varepsilon 3_1)P(T \varepsilon 3_2). \]

Then

\[ P(T \varepsilon 3_1) = [(n - j - 1)/(n - 1)]^{n-j}. \]

Exactly as in (3.22), we can employ Katz’s Lemma to obtain

\[ P(T \varepsilon 3_2) = j^{j-2}/(n - 1)^{j-1}. \]

Combining these we have

\[ P(p = j) = \binom{n-1}{j-1} \left( \frac{n-j-1}{n-1} \right)^{n-j} \frac{j^{j-2}}{(n - 1)^{j-1}}. \]
The condition $Tx \neq x$ for all $x \in X$, precludes the possibility of $p = n - 1$. There remains the case $p = n$.

$$P(p = n \mid x) = \sum_{j=2}^{n} P(q = j, K_{\tau}(x) = X, C_{\tau}(x) \neq 0)$$

$$= \sum_{j=2}^{n} \frac{n^{n-j-1}D_j n!}{(n-1)^n (j-1)! (n-j)!} \cdot \frac{(j-1)! j}{D_j} \cdot \frac{n}{n}$$

(4.18)

Inasmuch as (3.23) depends only on invariance under the symmetric group operating on $X$ and (3.26) is a property of the directed graphs in general, both of these apply in $\Pi$.

The distribution of $p^{(1)}$ is obtained trivially,

$$P(p^{(1)} = j) = \left(\frac{n-1}{j}\right) \left(\frac{1}{n-1}\right) \left(\frac{n-2}{n-1}\right)^{n-j}$$

(4.19)

The distribution of $r$, the number of components, has been computed by Folkert [3], and shown to be

$$P(r = k) = \frac{1}{(n-1)^n} \sum_{\mu} \frac{\mu^k}{\mu_1 \mu_2 \cdots \mu_k} \cdot \frac{n!}{k_1! k_2! \cdots k_\mu!} \left(k_1 - 1\right)^{k_1} \left(k_2 - 1\right)^{k_2} \cdots \left(k_\mu - 1\right)^{k_\mu},$$

(4.20)

where the sum over $k_1, k_2, \ldots, k_\mu$ is over all $\mu$-tuples with $k_i > 1$ and $\sum_i k_i = n$.

We will now develop a probability generating function for the number of components, and obtain an alternate derivation of (4.20). The argument parallels the same discussion in Case I and hence will only be sketched briefly.

As in case I,

$$P(T \in 3_{k_1, k_2, \ldots, k_\mu}) = \frac{n! I_{k_1}^{k_1} I_{k_2}^{k_2} \cdots I_{k_\mu}^{k_\mu}}{2^{k_1 k_2} k_2! \cdots n^{k_\mu} k_\mu! k_1! \cdots k_\mu!(n-1)^n},$$

(4.21)

where $I_j/(j-1)^j$ is the probability that a transformation $T_j$ on $j$ elements $X_j$ is indecomposable, i.e., $K_{\tau_j}(x) = X_j$ for all $x \in X_j, 0 \leq k_i \leq n$ and $\sum_i ik_i = n$.

$$\frac{I_j}{(j-1)^j} = \sum_{i=2}^{j} P(q = i, K_{\tau_j}(x) = X_j) = \sum_{i=2}^{j} \frac{j^{j-i-1}}{(j-1)^{(j-1)}}$$

(4.22)

$$= \sum_{i=2}^{j} \frac{(j-1)^{j-1}}{(j-1)^{j-1}}.$$

(4.22) has previously been established by Katz [7] using a somewhat different argument. Then

$$G(x_2, x_3, \ldots, x_n) = \sum_{k_2, k_3, \ldots, k_n} \frac{n! (I_{k_2} x_2)^{k_2} (I_{k_3} x_3)^{k_3} \cdots (I_{k_n} x_n)^{k_n}}{2^{k_2 k_3} k_3! \cdots n^{k_\mu} k_\mu! k_2! \cdots k_\mu!(n-1)^n}$$

is the generating function of $k_2, k_3, \ldots, k_\mu$ in the same manner as (3.32).
Since \( r = \sum_{i=1}^{n} k_i \), we obtain, after extending the definition to \( G(x_2, x_3, \cdots) \),

\[
G(x_2, x_3, \cdots) = \frac{n!}{(n-1)^n} \exp \sum_{i=2}^{\infty} \frac{I_i x_i}{i!}
\]

If, in (4.24), we replace \( x_i \) by \( x^i \), we obtain

\[
\frac{n!}{(n-1)^n} \exp \sum_{i=2}^{\infty} \frac{I_i x^i}{i!} = \frac{n!}{(n-1)^n} \sum_{j=0}^{\infty} \frac{(j-1)^j x^j}{j!}
\]

Thus

\[
\sum_{i=2}^{\infty} \frac{I_i x^i}{i!} = \log \sum_{j=0}^{\infty} \frac{(j-1)^j x^j}{j!}
\]

Replacing \( x_i \) by \( t^i x_i \) in (4.24), we obtain:

\[
G(t^2 x_2, t^3 x_3, \cdots) = \frac{n!}{(n-1)^n} \exp \sum_{i=2}^{\infty} \frac{I_i t^i x_i}{i!}
\]

Then the coefficient of \( t^n \) in (4.27) gives the probability of every possible decomposition of \( X \) into components, in the same manner as (3.37). If we replace \( x_i \) by \( t^i x_i \), we get

\[
G(t^2 x_2, t^3 x_3, \cdots) = \frac{n!}{(n-1)^n} \exp \left[ t \sum_{i=2}^{\infty} \frac{I_i x^i}{i!} \right]
\]

or

\[
G(t^2 x_2, t^3 x_3, \cdots) = \frac{n!}{(n-1)^n} \left[ \sum_{i=2}^{\infty} \frac{(i-1)^i x^i}{i!} \right]^t
\]

giving

\[
\text{coefficient of } t^n x^n \text{ in } G(t^2 x_2, t^3 x_3, \cdots) = P[r = k].
\]

We now employ the generating function to obtain an alternate form of Folkert’s formula (4.20). From (4.28) we have

\[
\text{coefficient of } t^k = \frac{n!}{(n-1)^n k!} \left[ \sum_{i=2}^{\infty} \frac{I_i x^i}{i!} \right]^k
\]

and from (4.26) we have

\[
\text{coefficient of } t^k = \frac{n!}{(n-1)^n k! \sum_{i=1}^{\infty} \frac{(i-1)^i x^i}{i!}}\left[ \sum_{i=1}^{\infty} \frac{(i-1)^i x^i}{i!} \right]^k
\]

Hence

\[
\text{coefficient of } t^k = \frac{n!}{(n-1)^n k!} \sum_{\mu=k}^{\infty} \frac{\mu!}{\mu!} S_\mu \left[ \sum_{i=1}^{\infty} \frac{(i-1)^i x^i}{i!} \right]^k
\]
and, as in (3.43), we get

\[
\text{coefficient of } t^k = \frac{n!}{(n-1)^k k!} \sum_{\mu=0}^{\infty} \frac{k!}{\mu!} s_{\mu}^k \sum_{i_1+i_2+\cdots+i_n = 0} \mu! \left[ \frac{(1 - 1)x^{i_1}}{1!} \right] \left[ \frac{(2 - 1)x^{i_2}}{2!} \right] \cdots \left[ \frac{(n - 1)x^{i_n}}{n!} \right].
\]

(4.32)

To find the coefficient of \( x^n \) in (4.32), it is sufficient to restrict the second sum to non-negative \( n \)-tupules \((k_1, k_2, \cdots, k_r)\) with \( \sum_{i=1}^{n} k_i = \mu \) and \( \sum_{i=1}^{n} ik_i = n \). Thus, we have

\[
P(r = k) = \frac{n!}{(n - 1)^n} \sum_{\mu=0}^{[n/2]} s_{\mu}^k \sum_{k_2, k_3, \cdots, k_r} \frac{1}{k_2! k_3! \cdots k_r!} \left( \frac{1}{2!} \right)^{k_2} \left( \frac{2}{3!} \right)^{k_3} \cdots \left( \frac{(n - 1)}{n!} \right)^{k_r},
\]

with the second sum restricted to \( k_2, k_3, \cdots, k_r \geq 0, \sum_{i=2}^{n} k_i = \mu, \) and \( \sum_{i=2}^{n} ik_i = n \) by the deletion of zero terms, and thus we have an alternate form of Folkert's formula.

5. Probability Distributions for Cases III and IV. In case III, we have \( P(T) = n^{-1} \), and in case IV, we have \( P(T) = D_n^{-1} \). In both cases, every \( x \in X \) is cyclic, since \( S \) is a collection of mappings which are one-to-one and onto in each case. As a consequence

\[
S_T(x) = P_T(x) = C_T(x) = K_T(x).
\]

Therefore, many of the probability distributions considered for cases I and II coincide in cases III and IV. Hence, we consider only the distributions of \( l \) and \( r \). Then, in case III, we have

\[
P(l = j) = \frac{\binom{n}{j} (j - 1)! (n - j)!}{nn!} = 1/n.
\]

Gontcharoff [4] has shown that the probability that the number of components \( r \) of \( T \) is \( k \) is given by

\[
P(r = k) = \text{coefficient of } t^k \text{ in } \frac{t(t + 1) \cdots (t + n - 1)}{n!},
\]

and therefore is given by the well-known result,

\[
P(r = k) = \frac{|S_n^k|}{n!},
\]

which may be found in Riordan [8]. Gontcharoff [4] has also shown that the distribution of \( (r - Er)/\sigma_r \) is asymptotically normally distributed with mean 0 and variance 1. Feller [2] and Greenwood [5] have also computed \( Er \) and \( \sigma_r^2 \). We show an alternative computation using (5.2).
Let $m_j$ be the number of components of $T$ with exactly $j$ elements. Then

\[(5.5) \quad E r = E \sum_{j=1}^{n} m_j = \sum_{j=1}^{n} E m_j.\]

From (5.2) we note that $E m_j = 1/j$, and hence

\[(5.6) \quad E r = \sum_{j=1}^{n} \frac{1}{j} \sim \log n + \gamma,\]

where $\gamma$ is Euler's constant.

The variance has been shown to be

\[(5.7) \quad \sigma_r^2 = \sum_{j=1}^{n} \frac{1}{j} - \left(\sum_{j=1}^{n} \frac{1}{j^2}\right) \sim \log n + \gamma - \frac{\pi^2}{6}.\]

In case IV, we have

\[(5.8) \quad P(l = j) = \left[\binom{n}{j} (j - 1)! D_{n-j}\right] / [n D_n] = \frac{(n - 1)! D_{n-j}}{(n - j)! D_n}.\]

For large $n$; and $j \geq 2$ and sufficiently small compared to $n$,

\[(5.9) \quad P(l = j) \sim 1/n, \quad 2 \leq j.\]

Furthermore

- $P(l = n) \sim e/n$
- $P(l = n - 1) = 0$
- $P(l = n - 2) \sim e/2n$
- $P(l = n - 3) \sim e/3n$.

To get the probability distribution of the number of components, we employ the same type of generating function used earlier. First we note that

\[(5.10) \quad P(K_r(x) = X) = (n - 1)!/D_n\]

since all $(n - 1)!$ $n$-cycles belong to 3. From this, we obtain:

\[(5.11) \quad P(T \in 3_{k_1, k_2, \ldots, k_n}) = \frac{n! 1!^{k_1} 2!^{k_2} \cdots (n - 1)!^{k_n}}{D_n 2^{k_2} 3^{k_3} \cdots n^{k_n}} \frac{k_1! k_2! \cdots k_n!}{k_1! \cdots k_n!}.\]
where $0 \leq k_i \leq n$ and $\sum_{i=2}^{n} i k_i = n$. Then
\begin{equation}
G(x_2, x_3, \cdots, x_n) = \sum_{k_2, k_3, \cdots, k_n} \frac{n!}{D_n} \left( \frac{x_2}{2} \right)^{k_2} \left( \frac{x_3}{3} \right)^{k_3} \cdots \left( \frac{x_n}{n} \right)^{k_n} \frac{1}{k_2! k_3! \cdots k_n!}
\end{equation}
(5.12)
\[= \sum_r \frac{n!}{D_n} \left( \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_n}{n} \right)^r / r!,\]
where $r = \sum_{i=2}^{n} k_i$. Proceeding as before, we have
\begin{equation}
G(x_2, x_3, \cdots) = \frac{n!}{D_n} \exp \sum_{i=2}^{\infty} \frac{x_i}{i}.
\end{equation}
(5.13)
Replacing $x_i$ by $x^i$, we have
\begin{equation}
\frac{n!}{D_n} \exp \sum_{i=2}^{\infty} \frac{x_i}{i} = \frac{n!}{D_n} \sum_{j=0}^{\infty} \frac{D_j}{j!} x^j.
\end{equation}
(5.14)
If we replace $x_i$ by $tx^i$ in (5.13), we obtain
\begin{equation}
G(tx^2, tx^3, \cdots) = \frac{n!}{D_n} \exp t \sum_{i=2}^{\infty} \frac{x^i}{i} = \frac{n!}{D_n} \left[ \sum_{j=0}^{\infty} \frac{D_j}{j!} x^j \right]^t.
\end{equation}
(5.15)
Since
\[\sum_{i=2}^{\infty} \frac{x^i}{i} = - \log (1 - x) - x,
\]
we have
\begin{equation}
G(tx^2, tx^3, \cdots) = \frac{n!}{D_n} \exp -t \log (1 - x) - x
\end{equation}
(5.16)
\[= \frac{n!}{D_n} \frac{e^{-tx}}{(1 - x)^t}.
\]
Then
\begin{equation}
\text{coefficient of } t^k x^n \text{ in } G(tx^2, tx^3, \cdots) = P(r = k).
\end{equation}
(5.17)
From (5.15) we have
\begin{equation}
\text{coefficient of } t^k = \frac{n!}{D_n k!} \left( \sum_{i=2}^{\infty} \frac{x^i}{i} \right)^k,
\end{equation}
(5.18)
and, expanding by the multinomial theorem, we get
\begin{equation}
\text{coefficient of } t^k
\end{equation}
(5.19)
\[= \frac{n!}{D_n k!} \sum_{k_2, k_3, \cdots, k_n \geq 0} \frac{k!}{k_2! k_3! \cdots k_n!} \left( \frac{x^2}{2} \right)^{k_2} \left( \frac{x^3}{3} \right)^{k_3} \cdots \left( \frac{x^n}{n} \right)^{k_n},\]
and thus

\begin{equation}
(5.20) \quad P(r = k) = \frac{n!}{D_n} \sum_{k_2, k_3, \ldots, k_n} [k_2! \cdot k_3! \cdot \ldots \cdot k_n! \cdot 2^{k_2} \cdot 3^{k_3} \cdot \ldots \cdot n^{k_n}]^{-1},
\end{equation}

the sum over all non-negative \( n - 1 \) tuples \( k_2, k_3, \ldots, k_n \) with \( \sum_{i=2}^{n} k_i = k \) and \( \sum_{i=2}^{n} ik_i = n \). Another form of the same result is obtained from (5.16). Here we have

\begin{equation}
(5.21) \quad \text{coefficient of } x^n = n!/D_n \sum_{j=0}^{n} \frac{(-1)^{n-j} \cdot t^{n-j} \cdot (-1)^{k_2} \cdot t(t+1) \cdots (t+j-1)}{(n-j)! \cdot j!}. \quad \text{Hence, we find that the coefficient of } t^k x^n \text{ is}
\end{equation}

\begin{equation}
(5.22) \quad P(r = k) = \frac{n!}{D_n} \sum_{j=0}^{n} \frac{(-1)^{n+j} \cdot S_j^{b-n+j}}{(n-j)! \cdot j!},
\end{equation}

or

\begin{equation}
(5.23) \quad P(r = k) = \frac{n!}{D_n} \sum_{j=0}^{n} \frac{(-1)^{j} \cdot S_{n-j}^{b-j}}{(n-j)! \cdot j!}.
\end{equation}

Since

\begin{equation}
(5.24) \quad Er = \text{coefficient of } x^n \text{ in } (d/dt)G(tx^2, tx^3, \cdots) |_{t=1}
\end{equation}

and

\begin{equation}
(5.25) \quad Er(r - 1) = \text{coefficient of } x^n \text{ in } (d^2/dt^2)G(tx^2, tx^3, \cdots) |_{t=1},
\end{equation}

then

\begin{equation}
(5.26) \quad Er = \text{coefficient of } x^n \text{ in } (n!/D_n)(e^{-x}/(1-x))(-\log(1-x) - x),
\end{equation}

\begin{equation}
(5.27) \quad Er(r - 1) = \text{coefficient of } x^n \text{ in } (n!/D_n)(e^{-x}/(1-x))(\log(1-x) + x)^2.
\end{equation}

Expanding (5.26) and (5.27) in a power series we obtain

\begin{equation}
(5.28) \quad Er = \frac{n!}{D_n} \sum_{s=0}^{n} \frac{D_{n-s}}{s(n-s)!},
\end{equation}

\begin{equation}
(5.29) \quad Er(r - 1) = \frac{n!}{D_n} \sum_{s=0}^{n} \frac{D_{n-s}}{(n-s)!} \sum_{j+k=s+1}^{\frac{s^2}{2}} \frac{1}{jk} = \frac{n!}{D_n} \sum_{s=0}^{n} \frac{D_{n-s}}{(n-s)!} \sum_{j+k=s+1}^{\frac{s^2}{2}} \frac{1}{jk}.
\end{equation}

Since \( D_j = (j!/e) + O(1) \),

\begin{equation}
Er \sim e \sum_{s=0}^{n} \frac{(n-s)!}{s(n-s)!} \frac{1}{e} + O(1) \sim \sum_{s=0}^{n} \frac{1}{s} + O(1).
\end{equation}
Hence

\begin{equation}
E_r \sim \log n + O(1).
\end{equation}

Similarly

\begin{equation}
E_r(r-1) = \frac{n!}{D_n} \sum_{s=4}^{n} \frac{D_{n-s}}{(n-s)!} \sum_{j=s-1}^{s-2} \left[ \frac{1}{j} + \frac{1}{s-j} \right]
\end{equation}

\begin{equation}
\sim e \sum_{s=4}^{n} \left[ \frac{(n-s)!}{e(n-s)!} + O(1) \right] \frac{2}{s} \left[ \log s - 1 - \frac{1}{s-1} - \frac{1}{s} + \frac{1}{s} + \gamma + O\left(\frac{1}{s}\right) \right]
\end{equation}

where \( \gamma \) is Euler's constant. Hence

\begin{equation}
E_r(r-1) \sim \sum_{s=4}^{n} \frac{2}{s} \left[ \log s - 1 - \frac{1}{s-1} - \frac{1}{s} + \gamma + O\left(\frac{1}{s}\right) \right] + O(1),
\end{equation}

and

\begin{equation}
E_r(r-1) \sim \log^2 n + 2(\gamma - 1) \log n + O(1).
\end{equation}

Thus

\begin{equation}
\sigma_r^2 \sim (2\gamma - 1) \log n + O(1).
\end{equation}

6. Miscellaneous Remarks. The problem of random mappings is of interest in various studies of human behaviors. We produce one such example. If we ask each of \( n \) individuals in a group to name his best friend from among the members of the group, the individual asked is the element \( x \), and his choice \( T x \). In this case we have \( T x \neq x \), and the hypothesis of "randomness" leads to case II.

APPENDIX

Index of Notations Having a Fixed Meaning

\( (X, 3, P) \)–random mapping space
\( X \)–a finite set of \( n \) elements
\( 3 \)–a set of transformations \( T \) of \( X \) into \( X \)
\( P \)–a probability measure over \( 3 \)
Case I–is set of all transformations of \( X \) into \( X \), \( P(T) = n^{-n} \) for each \( T \in 3 \)
Case II–is set of all transformations of \( X \) into \( X \) with \( T x \neq x \) for each \( x \in X \), \( P(T) = (n - 1)^{-n} \) for each \( T \in 3 \)
Case III–is set of all one-to-one mappings of \( X \) onto \( X \), \( P(T) = n!^{-1} \) for each \( T \in 3 \)
Case IV–is set of all one-to-one mappings of \( X \) onto \( X \) with \( T x \neq x \) for each \( x \in X \), \( P(T) = D_n^{-1}, D_n \) is the \( n \)th derangement number for each \( T \in 3 \)
\( S_r(x) \)–the set of all images of \( x \) in \( T \).
\( P_r(x) \)–the set of predecessors of \( x \) in \( T \).
\( C_r(x) \)–the cycle containing \( x \).
\( K_r(x) \)–the component containing \( x \).
$s_T(x)$, $s$–the number of elements in $S_T(x)$.
$p_T(x)$, $p$–the number of elements in $P_T(x)$.
$l_T(x)$, $l$–the number of elements in cycle contained in $K_T(x)$.
$q_T(x)$, $q$–the number of cyclical elements of $X$ in $T$.
$r_T$, $r$–the number of components of $T$.
$S_n^k$–Stirling's Numbers of the First Kind.
$S_{k_1,k_2,...,k_n}$ the subset of $S$ with $k_i$ components with exactly $i$ elements,
$i = 1, 2, \cdots, n$.

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