THE TRANSIENT BEHAVIOUR OF A COINCIDENCE VARIATE IN TELEPHONE TRAFFIC

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1. Introduction. We consider the following problem. Calls arrive at a telephone exchange at the instants $t_1, t_2, \cdots, t_n$, where the inter-arrival intervals $(t_n - t_{n-1})$, $n \geqq 1$, $t_0 = 0$, are independently and identically distributed non-negative random variables with common distribution function $A(x)$ and finite expectation $\alpha = \int_0^\infty x\, dA(x)$. Introduce the Laplace-Stieltjes transform $a(s)$ defined by

$$a(s) = \int_0^\infty e^{-sx}\, dA(x). \tag{1}$$

There are m channels available and a connection is realised if the incoming call finds an idle channel. If all the channels are busy, then the incoming call is lost. Denote by β_n the holding time of the call at t_n if that call is not lost. We suppose that the β_n are non-negative independent random variables, independent also of the input process $\{t_n\}$, with common distribution function $B(x)$ given by

$$B(x) = 1 - e^{-\mu x}, \qquad x \geqq 0. \tag{2}$$

Denote by $\eta(t)$ the number of busy channels at time t and put $\eta_n = \eta(t_n - 0)$. We say that the system is in the state E_k, $k = 0, 1, \cdots, m$ if k channels are busy. Write $P_{k,n} = P(\eta_n = k)$, $k = 0, 1, \cdots, m$, $n = 1, 2, \cdots$, and write $P_k = \lim_{n\to\infty} P_{k,n}$. The limiting distribution $\{P_k\}$ has been obtained by a number of authors, J. W. Cohen [1], C. Palm [2], F. Pollaczek [3], and L. Takács [4]. Introduce the generating function $P_k(w)$, $k = 0, 1, \cdots, m$, defined by

$$P_k(w) = \sum_{n=1}^{\infty} P_{k,n} w^{n-1}, \qquad k = 0, 1, \cdots, m, \ |w| < 1. \tag{3}$$

In this paper we obtain the generating function $P_k(w)$. When $m = \infty$ we obtain the probabilities $P_{k,n}$ explicitly. Our method is a slight generalisation of that of Takács [4]. We remark that in [3] Pollaczek obtained the transient solution in the case $P_{0,1} = 1$ as an application of a very general analytic result.

2. The distribution $\{P_{k,n}\}$. We prove the following theorem.

Theorem 1. *Under the assumptions of Section 1 we have*

$$P_k(w) = \sum_{r=k}^{m} (-)^{r-k} \binom{r}{k} B_r(w), \qquad |w| < 1, \tag{4}$$

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where

$$B_r(w) = C_r(w)\left[(1-w)^{-1} + \sum_{j=1}^{m} D_j C_j^{-1}(w)(1-a_j w)^{-1}\right] \cdot \sum_{j=r}^{m}\binom{m}{j} C_j^{-1}(w) \Big/ \sum_{j=0}^{m}\binom{m}{j} C_j^{-1}(w), \tag{5}$$

$$C_r(w) = \prod_{j=1}^{r} a_j w(1-a_j w)^{-1}, \qquad r > 1, C_0(w) \equiv 1, \tag{6}$$

and where D_j *is the* jth *binomial moment of the initial distribution* $\{P_{k,1}\}$, *that is,*

$$D_j = (j!)^{-1}\left[d^j/dz^j \sum_{k=0}^{m} P_{k,1} z^k\right]_{z=1}, \tag{7}$$

and

$$a_r = a(r\mu) = \int_0^\infty e^{-r\mu x}\, dA(x). \tag{8}$$

PROOF. The sequence of random variables η_n, $n = 1, 2, \cdots$, forms a Markov chain with transition probabilities $p_{j,k} = P(\eta_{n+1} = k \mid \eta_n = j)$, where $p_{m,k} = p_{m-1,k}$ and

$$p_{j,k} = \binom{j+1}{k}\int_0^\infty e^{-k\mu x}(1-e^{-\mu x})^{j+1-k}\, dA(x), \quad 0 \leqq j < m, j < k \leqq m, \tag{9}$$

Thus we have

$$P_{k,n+1} = \sum_{j=k-1}^{m} p_{j,k} P_{j,n}, \qquad 0 \leqq k \leqq m, m \geqq 1 \tag{10}$$

where $p_{k,-1} = 0$, and

$$\sum_{k=0}^{m} P_{k,n} = 1, \qquad n \geqq 1. \tag{11}$$

From equations (3) and (10) we obtain

$$P_k(w) - P_{k,1} = w \sum_{j=k-1}^{m} p_{j,k} P_j(w). \tag{12}$$

Write $P(w, z) = \sum_{k=0}^{m} P_k(w) z^k$; then from equation (12) we obtain

$$\begin{aligned} P(w, z) - P(0, z) &= w\int_0^\infty (1-e^{-\mu x}+ze^{-\mu x})P(w, 1-e^{-\mu x}+ze^{-\mu x})\, dA(x) \\ &\quad + w(1-z)P_m(w)\int_0^\infty e^{-\mu x}(1-e^{-\mu x}+ze^{-\mu x})^m\, dA(x). \end{aligned} \tag{13}$$

Introduce the binomial moments $B_r(w)$, D_r defined by

$$B_r(w) = (r!)^{-1}[d^r/dz^r P(w, z)]_{z=1}, \tag{14}$$

and

$$D_r = B_r(0). \tag{15}$$

From (13) we obtain $B_0(w) = \sum_{j=0}^{m} \sum_{n=1}^{\infty} P_{j,n} w^{n-1} = (1 - w)^{-1}$, $|w| < 1$, and

$$B_r(w) - D_r = a_r w \left[B_r(w) + B_{r-1}(w) - \binom{m}{r-1} P_m(w) \right], \quad r = 1, 2, \cdots, m. \tag{16}$$

where a_r is defined by (8).

Note that $P_m(w) = B_m(w)$ and introduce the quantities $C_r(w)$ defined by (6); then from (16) we obtain

$$B_r(w) = C_r(w) \left[\sum_{j=1}^{r} (1 - a_j w)^{-1} D_j C_j^{-1}(w) + (1 - w)^{-1} - B_m(w) \sum_{j=0}^{r-1} \binom{m}{j} C_j^{-1}(w) \right]. \tag{17}$$

Putting $r = m$ in (17) we obtain

$$B_m(w) = \left[\sum_{j=1}^{m} (1 - a_j w)^{-1} D_j C_j^{-1}(w) + (1 - w)^{-1} \right] \Big/ \sum_{j=0}^{m} \binom{m}{j} C_j^{-1}(w),$$

and thus we obtain equation (5). Finally we have

$$B_r(w) = \sum_{j=r}^{m} \binom{j}{r} P_j(w) \tag{18}$$

Multiplying equation (18) by $(-)^{r-k}\binom{r}{k}$ and summing for $r = k, k + 1, \cdots, m$, we obtain (4) and the theorem is proved. We remark that the limiting distribution $\{P_k\}$ follows easily from Theorem 1. Write $C_r = C_r(1)$ and define B_r, $r = 1, 2, \cdots, m$, by the equation

$$B_r = \lim_{r \to 1} (1 - w) B_r(w) = C_r \sum_{j=r}^{m} \binom{m}{r} C_j^{-1} \Big/ \sum_{j=0}^{m} \binom{m}{j} C_j^{-1}, \quad r = 1, 2, \cdots, m.$$

The limiting distribution $P_k = \lim_{n\to\infty} P_{k,n}$ exists since the process $\{\eta_n\}$ is a finite irreducible aperiodic Markov chain. It follows from Abel's theorem on power series that $\lim_{w\to1} (1 - w)P_k(w) = P_k$. Thus from (4)

$$P_k = \sum_{r=k}^{m} (-)^{r-k} \binom{r}{k} B_r.$$

This is the known solution for the limiting distribution (e.g., Takács [4]).

EXAMPLE. Suppose that $m = 2$ and that $P_{0,1} = 1$ so that $D_r = 0$, $r = 1, 2$.

We find that

$$B_1(w) = a_1 w(1 + a_2 w)/(1 - w)\{1 - (a_1 - a_2)w\},$$
$$B_2(w) = a_1 a_2 w^2 / (1 - w)\{1 - (a_1 - a_2)w\},$$

Equating coefficients of powers of w in (4) we obtain

$$P_{0,n} = 1 - a_1\{1 - (a_1 - a_2)^{n-1}\}(1 - a_1 + a_2)^{-1}, \qquad n \geqq 2,$$
$$P_{1,n} = [a_1\{1 - (a_1 - a_2)^{n-1}\} - a_1 a_2\{1 - (a_1 - a_2)^{n-2}\}] \cdot (1 - a_1 + a_2)^{-1}, \qquad n \geqq 2,$$
$$P_{2,n} = a_1 a_2\{1 - (a_1 - a_2)^{n-2}\}(1 - a_1 + a_2)^{-1}, \qquad n \geqq 2.$$

3. The case $m = \infty$. When $m = \infty$ we have the following theorem.

THEOREM 2. *If $m = \infty$ then*

(19) $$P_k(w) = \sum_{r=k}^{\infty} (-)^{r-k} \binom{r}{k} B_r(w), \qquad k \geqq 0, |w| < 1,$$

where $B_0(w) = (1 - w)^{-1}$ and

(20) $$B_r(w) = \left[(1 - w)^{-1} + \sum_{j=1}^{r} (1 - a_j w)^{-1} D_j C_j^{-1}(w)\right] C_r(w), \qquad r \geq 1,$$

where $C_r(w)$, $r \geqq 1$, is given by (6) and D_j, $j \geqq 1$, by (7). If $B_{1,n}$, $B_{2,n}$ are the first and second binomial moments of the distribution $\{P_{k,n}\}$ then

(21) $$B_{1,n} = D_1 a_1^{n-1} + a_1(1 - a_1^{n-1})(1 - a_1)^{-1}, \qquad n \geqq 1,$$

(22) $$B_{2,n} = D_2 a_2^{n-1} + D_1 a_2(a_1 - a_2)^{-1}(a_1^{n-1} - a_2^{n-1}) + a_1 a_2(a_1 - a_2)^{-1} \cdot [a_1(1 - a_1)^{-1}(1 - a_1^{n-2}) - a_2(1 - a_2)^{-1}(1 - a_2^{n-2})], \qquad n \geqq 2.$$

PROOF. The proof is similar to that of Theorem 1. Instead of (16) we have

$$B_r(w) = (1 - a_r w)^{-1}[D_r + a_r w B_{r-1}(w)], \qquad r \geqq 1$$

with $B_0(w) = (1 - w)^{-1}$. Hence we obtain (20). Equation (19) follows from (14) and

$$(\partial^k/\partial z^k)P(w, z) = (k!)^{-1} \sum_{r=k}^{\infty} \binom{r}{k} (z - 1)^{r-k} B_r(w),$$
$$[(\partial^k/\partial z^k)P(w, z)]_{z=0} = (k!)^{-1} P_k(w).$$

Equations (21), (22) follow by equating coefficients of powers of w in the series expansions of $B_1(w)$, $B_2(w)$. The variance V_n of the distribution $\{P_{k,n}\}$ is obtained easily from the equation

$$V_n = 2B_{2,n} + B_{1,n} - (B_1,n)^2.$$

If $P_{0,1} = 1$, that is, if the first call arrives to find all the channels idle we have

$D_j = 0, j \geqq 1$ and equation (20) becomes

$$B_r(w) = (1 - w)^{-1} \prod_{j=1}^{r} a_j w (1 - a_j w)^{-1}, \qquad r \geqq 1. \tag{23}$$

In this case we can obtain the probabilities $\{P_{k,n}\}$ explicitly, namely we have

THEOREM 3. *If* $m = \infty$ *and* $P_{0,1} = 1$ *then*

$$P_{0,n} = 1 + \sum_{m=1}^{n-1} \sum_{r=1}^{m} (-)^r \sum_{j=1}^{r} K_{j,r}\, a_j^m . \qquad n \geqq 2 \tag{24}$$

$$P_{k,n} = \sum_{m=k}^{n-1} \sum_{r=k}^{m} \binom{r}{k} (-)^{r-k} \sum_{j=1}^{r} K_{j,r}\, a_j^m , \qquad n \geqq k + 1, k \geqq 1, \tag{25}$$

where

$$K_{j,r} = \prod_{i=1, i \neq j}^{r} a_i (a_j - a_i)^{-1}. \tag{26}$$

PROOF. From equation (23) we have

$$B_r(w) = w^r (1 - w)^{-1} \sum_{j=1}^{r} a_j^r K_{j,r} (1 - a_j w)^{-1},$$

where the $K_{j,r}$ are given by (26). From (19) we obtain

$$P_0(w) = (1 - w)^{-1} \left[1 + \sum_{r=1}^{\infty} (-)^r w^r \sum_{j=1}^{r} K_{j,r}\, a_j^r (1 - a_j w)^{-1} \right.$$

$$P_k(w) = (1 - w)^{-1} \left[\sum_{r=k}^{\infty} (-)^{r-k} \binom{r}{k} w^r \sum_{j=1}^{r} K_{j,r}\, a_j^r\, (1 - a_j w)^{-1} \right].$$

Equations (24), (25) follow by equating coefficients of powers of w in each side of the power series expansions of these equations.

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