A GENERALIZATION OF WALD'S IDENTITY WITH APPLICATIONS TO RANDOM WALKS

By H. D. Miller

Statistical Laboratory, University of Cambridge

0. Summary. Let $S_m = X_1 + \cdots + X_m$, where the X_j are independent random variables with common m.g.f. $\phi(t)$ which is assumed to exist in a real interval containing $t = 0$. Let the random variable n be defined as the smallest integer m for which either $S_m \geqq \alpha$ or $S_m \leqq -\beta$ ($\alpha > 0, \beta > 0$). Thus n can be regarded as the time to absorption for the random walk S_m with absorbing barriers at α and $-\beta$. Let $S = S_n$ and let

$$F_m(x) = P(-\beta < S_k < \alpha \quad \text{for} \quad k = 1, 2, \cdots m - 1 \quad \text{and} \quad S_m \leqq x).$$

The main result of the paper is the identity

$$E(e^{tS}z^n) = 1 + [z\phi(t) - 1]F(z, t), \tag{0.1}$$

where

$$F(z, t) = \sum_{m=0}^{\infty} z^m \int_{-\beta}^{\alpha} e^{tx}\, dF_m(x).$$

Wald's identity follows formally from (0.1) by setting $z = [\phi(t)]^{-1}$. Regions of validity of (0.1) and of Wald's identity are discussed, and it is shown that the latter holds for a larger range of values of t than is usually supposed.

In Section 5 there are three examples. In the first we consider the case where there is a single absorbing barrier and where the X_j are discrete and bounded. This is a gambler's ruin problem, and we obtain an expression for the probability of ruin. In the second we use the classical random walk to illustrate the region of validity of (0.1). In the third we obtain the Laplace transform of the distribution of the time to absorption in a random walk in which steps of $+1$ and -1 occur at random in continuous time.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with common distribution function $A(x)$ and moment generating function

$$\phi(t) = \int_{-\infty}^{\infty} e^{tx}\, dA(x). \tag{1.1}$$

Let $S_0 = 0$ and let S_m denote the cumulative sum

$$S_m = X_1 + X_2 + \cdots + X_m, \qquad m \geqq 1.$$

We ignore the trivial case where the X_j are constant with probability 1. The X_j can be regarded as the successive steps of a particle starting at the origin and S_m then represents the distance of the particle from the origin at the mth

Received May 25, 1960; revised November 4, 1960.

step. Suppose that there are two absorbing barriers, one at α and the other at $-\beta(\alpha > 0, \beta > 0)$, and that the regions $S_m \geqq \alpha$ and $S_m \leqq -\beta$ are absorbing regions. Let n be the integral-valued random variable denoting the step at which absorption occurs. Thus $n \geqq 1$ is defined by

$$-\beta < S_m < \alpha, \qquad m = 1, 2, \cdots n - 1,$$

$$S_n \leqq -\beta \quad \text{or} \quad S_n \geqq \alpha.$$

For convenience, write $S = S_n$. Then the fundamental identity of Wald [12] is

$$E[\{\phi(t)\}^{-n}e^{tS}] = 1. \tag{1.2}$$

If α and β are both finite and t is complex, then Wald showed (1.2) to be valid for all values of t for which $|\phi(t)| \geqq 1$.

In the single barrier case, say where $\alpha < \infty, \beta = \infty$, (1.2) has been used to determine $P(n < \infty)$. This is the probability of ultimate ruin in the gambler's ruin problem where X_j is the gambler's loss at the jth play and $E(X_j) < 0$. His initial capital is α. (Cf., Bahadur [1], Bartlett [2], p. 89, Wald [12].) Another application of Wald's identity is the determination of the characteristic function of n (Wald [12]). Some authors, in particular Bellman [3], Blackwell and Girshick [4], Ruben [9], and Tweedie [11] have generalized (1.2) in the direction of widening the class of processes for which such an identity is valid. Doob ([5], pp. 350–352) has shown that (1.2) may be derived from the theory of Martingales. In statistics the most important application of (1.2) has been in sequential analysis. However, the present paper has been written from the point of view of random walks rather than that of sequential analysis.

Generally, Wald's identity seems to have the character of an isolated result, unconnected with the Chapman-Kolmogorov relations which hold for a Markov process such as the random walk. In the present paper we stress the Chapman-Kolmogorov approach and show that (1.2) may be derived thereby. We restrict our attention to random variables whose distribution admits a moment generating function, since it is in this case that we are able to discuss regions of validity of (0.1) and (1.2). However, (0.1) is true even if $\phi(t)$ exists only on the imaginary axis.

2. Notation and Definitions. We adopt the convention that a single absorbing barrier at, say, α is denoted by $\alpha < \infty, \beta = \infty$. We define

$$\begin{aligned} F_m(x) &= P(-\beta < S_k < \alpha \quad \text{for} \quad k = 1, 2, \cdots m - 1 \quad \text{and} \quad S_m \leqq x), \\ F_0(x) &= 1, \qquad x \geqq 0, \\ &= 0, \qquad x < 0. \end{aligned} \tag{2.1}$$

$F_m(x)$ is a distribution function in an extended sense simce $F_m(\infty) < 1$ in general, owing to the fact that probability has been "leaking" through the barriers (cf., Bartlett [2], p. 16). The relevant Chapman-Kolmogorov relations are the recurrence relations satisfied by the $F_m(x)$, namely

$$F_m(x) = \int_{-\beta}^{\alpha} A(x-y)\, dF_{m-1}(y), \qquad (m = 1, 2, \cdots). \tag{2.2}$$

We define the double generating function

$$F(z, t) = \sum_{m=0}^{\infty} z^m \int_{-\beta}^{\alpha} e^{tx}\, dF_m(x), \qquad (0 < \alpha \leqq \infty, 0 < \beta \leqq \infty), \tag{2.3}$$

where z and t are complex variables whose respective regions will be stated as the need arises. We define $G_m(x)$ to be the distribution function for the unrestricted sum S_m, i.e.,

$$G_m(x) = P(S_m \leqq x). \tag{2.4}$$

We shall assume in the sequel, except in Section 4(i), that the integral (1.1) defining $\phi(t)$ is convergent in a real interval surrounding $t = 0$, say $b < t < a$, where $-\infty \leqq b < 0 < a \leqq \infty$. This will be true, for example, if $A'(x)$ exists and decreases exponentially as $x \rightarrow \pm\infty$. It follows that $\phi(t)$ is an analytic function of t in the strip $b < \text{Re}\,(t) < a$, and for real t, $\phi''(t) > 0$. Thus $\phi(t)$ can have at most one minimum in $b < t < a$, and we assume that this minimum exists and that it occurs at the point t_0. Thus t_0 is the unique real root of $\phi'(t) = 0$ in $b < t < a$. (The point t_0 does not necessarily exist for an analytic moment generating function, for consider the probability generating function

$$M(z) = A[(1 - cz)^{3/2} + (\tfrac{3}{2})cz] + B[(1 - cz^{-1})^{3/2} + (\tfrac{3}{2})cz^{-1}],$$

where $0 < c < 1$ and A and B are chosen so that $M(1) = 1$. $M(z)$ has a Laurent expansion in the annulus $c < |z| < c^{-1}$, and A and B may be chosen so that $M'(z)$ has the same sign throughout the real interval $c < z < c^{-1}$.)

Let $\mu = E(X_j) = \phi'(0)$. Then $t_0 \gtreqless 0$ according as $\mu \lesseqgtr 0$, and if $\mu \neq 0$ then $0 < \phi(t_0) < 1$.

3. Main Results.

LEMMA 3.1. *Let u denote the real part of t. Then the series* (2.4) *defining $F(z, t)$ is convergent in the region*

(i)

$$\begin{aligned} &|z| < [\phi(t_0)]^{-1}, \quad t_0 \leqq u < \infty, \\ &|z| < [\phi(u)]^{-1}, \quad b < u < t_0, \end{aligned} \qquad \textit{for} \quad \alpha < \infty, \beta = \infty, \tag{3.1}$$

and correspondingly

$$\begin{aligned} &|z| < [\phi(t_0)]^{-1}, \quad -\infty < u \leqq t_0, \\ &|z| < [\phi(u)]^{-1}, \quad t_0 < u < a, \end{aligned} \qquad \textit{for} \quad \alpha = \infty, \beta < \infty; \tag{3.2}$$

(ii)

$$|z| < [\phi(t_0)]^{-1}, \quad \textit{all finite } t, \qquad \textit{for } \alpha < \infty, \beta < \infty. \tag{3.3}$$

PROOF.

(i) Suppose that $\alpha < \infty$, $\beta = \infty$ and $u \geqq t_0$. Then

$$F(z, t) = \sum_{m=0}^{\infty} z^m \int_{-\infty}^{\alpha} e^{tx}\, dF_m(x). \tag{3.4}$$

Now we have

$$\begin{aligned}\left| \int_{-\infty}^{\alpha} e^{tx}\, dF_m(x) \right| &\leqq \int_{-\infty}^{\alpha} e^{ux}\, dF_m(x) \leqq \int_{-\infty}^{\alpha} e^{ux}\, dG_m(x) \\ &\leqq e^{\alpha(u-t_0)} \int_{-\infty}^{\infty} e^{t_0 x}\, dG_m(x) \\ &= e^{\alpha(u-t_0)} [\phi(t_0)]^m.\end{aligned}$$

Thus if $u \geqq t_0$, the series (3.4) is convergent for $|z| < [\phi(t_0)]^{-1}$.

If $u < t_0$, then

$$\begin{aligned}\left| z^m \int_{-\infty}^{\alpha} e^{tx}\, dF_m(x) \right| &\leqq | z^m | \int_{-\infty}^{\alpha} e^{ux} dF_m(x) \\ &\leqq | z |^m \int_{-\infty}^{\alpha} e^{ux}\, dG_m(x) \\ &\leqq | z |^m [\phi(u)]^m,\end{aligned}$$

and in this case the series (3.4) converges for $|z| < [\phi(u)]^{-1}$.

A similar argument gives the corresponding result for the case where $\alpha = \infty$ and $\beta < \infty$.

(ii) If α and β are both finite, then by a similar argument to that in (i) we have

$$\left| \int_{-\beta}^{\alpha} e^{tx}\, dF_m(x) \right| \leqq e^{\alpha(u-t_0)} [\phi(t_0)]^m \qquad \text{for } u \geqq t_0$$

and

$$\left| \int_{-\beta}^{\alpha} e^{tx}\, dF_m(x) \right| \leqq e^{\beta(t_0-u)} [\phi(t_0)]^m \qquad \text{for } u \leqq t_0.$$

Thus in this case the series (2.3) converges for all finite t and $|z| < [\phi(t_0)]^{-1}$. This completes the proof.

CONVENTION. In the sequel we adopt the convention that in cases where $P(n < \infty)$, the probability of absorption, is less than unity, the expectation symbol E is taken to mean $p_c E_c$, where $p_c = P(n < \infty)$ and E_c denotes expectation conditional on absorption.

THEOREM 3.1. (*A Generalization of Wald's identity*). *Let S and n denote the random variables defined in Section* 1. *Then we have the identity*

$$E(e^{tS} z^n) = 1 + [z\phi(t) - 1]F(z, t), \tag{3.5}$$

and provided we adopt the above convention, (3.5) *holds for all t for which $\phi(t)$ exists and for $|z| < [\phi(t_0)]^{-1}$.*

PROOF. In virtue of the definition of $F_m(x)$, (2.1), we have

$$E(e^{tS}z^n) = \sum_{m=1}^{\infty} z^m \left(\int_{-\infty}^{-\beta} + \int_{\alpha}^{\infty}\right) e^{tx}\, dF_m(x)$$
$$= \sum_{m=1}^{\infty} z^m \left(\int_{-\infty}^{\infty} - \int_{-\beta}^{\alpha}\right) e^{tx}\, dF_m(x)$$
$$= \sum_{m=1}^{\infty} z^m \int_{-\infty}^{\infty} e^{tx}\, dF_m(x) - [F(z, t) - 1],$$

provided that the series on the right hand side converges; it converges absolutely for $|z| < [\phi(u)]^{-1}$, $b < u < a$, where again $u = \text{Re}(t)$, since

$$\left| \int_{-\infty}^{\infty} e^{tx}\, dF_m(x) \right| \leqq \int_{-\infty}^{\infty} e^{ux}\, dG_m(x) = [\phi(u)]^m.$$

In virtue of the product theorem for the two-sided Laplace-Stieltjes transform (Widder [14], Ch. VI, Theorem 16a), we obtain from (2.2), on inverting the order of integration,

$$\int_{-\infty}^{\infty} e^{tx}\, dF_m(x) = \int_{-\infty}^{\infty} e^{ty}\, dA(y) \int_{-\beta}^{\alpha} e^{tx}\, dF_{m-1}(x),$$
$$= \phi(t) \int_{-\beta}^{\alpha} e^{tx}\, dF_{m-1}(x). \qquad (b < u < a)$$

Thus

$$E(e^{tS}z^n) = \sum_{m=1}^{\infty} z^m \phi(t) \int_{-\beta}^{\alpha} e^{tx}\, dF_{m-1}(x) - [F(z, t) - 1]$$
$$= 1 + [z\,\phi(t) - 1]\, F(z, t),$$

which is the identity (3.5).

As far as the region of validity is concerned, we note that if $\alpha < \infty$, $\beta < \infty$, then $F(z, t)$ is an entire function of t and a regular function of z for

$$|z| < [\phi(t_0)]^{-1}.$$

The right hand side of (3.5) may thus be taken to define the left hand side for $|z| < [\phi(t_0)]^{-1}$ and for those values of t for which $\phi(t)$ exists.

In the case where $\alpha < \infty$ and $\beta = \infty$, we have

$$E(e^{tS}z^n) = \sum_{m=1}^{\infty} z^m \int_{\alpha}^{\infty} e^{tx}\, dF_m(x),$$

and an argument similar to that used in the proof of Lemma 3.1 shows that the series on the right is convergent for

$$(3.6)\qquad \begin{aligned} |z| &< [\phi(u)]^{-1}, \quad t_0 \leqq u < \infty, \\ |z| &< [\phi(t_0)]^{-1}, \quad b < u \leqq t_0 . \end{aligned} \qquad (u = \mathrm{Re}(t))$$

Thus for fixed t the left hand side of (3.5) is a regular function of z in the region given by (3.6), while the z-region of regularity of $F(z, t)$ is given by Lemma 3.1. Thus, as functions of z, each side of (3.5) may be regarded as the analytic continuation of the other beyond the common region of regularity, and the identity therefore holds for $|z| < [\phi(t_0)]^{-1}$ Each side is clearly an analytic function of t for $b < \mathrm{Re}(t) < a$.

A similar argument deals with the case where $\alpha = \infty$ and $\beta < \infty$, and the theorem is therefore proved.

For the special case $\alpha = 0, \beta = \infty$, the identity (3.5) was proved by Spitzer ([9a], Theorem 3.1) who quotes the result as being due to G. Baxter.

The identity (3.5) is a generalization of Wald's identity and the latter follows formally from (3.5) by putting $z = [\phi(t)]^{-1}$. In Theorem 3.2 we show the importance of the point t_0, the minimum point of $\phi(t)$, in determining the region of validity of Wald's identity.

THEOREM 3.2. (*Wald's identity*) *If we adopt the convention of the previous theorem regarding the expectation symbol* E, *then Wald's identity*

$$(3.7)\qquad E(e^{tS}[\phi(t)]^{-n}) = 1,$$

holds provided that $|\phi(t)| > \phi(t_0)$ *and in addition* t *satisfies*

$$(3.8)\qquad \begin{array}{lllll} \text{(i)} & a > \mathrm{Re}(t) > t_0 & \textit{if} & \alpha < \infty, & \beta = \infty; \\ \text{(ii)} & b < \mathrm{Re}(t) < t_0 & \textit{if} & \alpha < \infty, & \beta < \infty; \\ \text{(iii)} & b < \mathrm{Re}(t) < a & \textit{if} & \alpha < \infty, & \beta < \infty. \end{array}$$

PROOF. The result follows from the previous theorem by setting $z = [\phi(t)]^{-1}$ in (3.5), and noting that $F(z, t)$ is finite if $|z| < [\phi(t_0)]^{-1}$ and if t satisfies the conditions (3.8) (Lemma 3.1). In general, we have, for $b < \mathrm{Re}(t) < a$ and

$$|\phi(t)| > \phi(t_0),$$

$$E(e^{tS}[\phi(t)]^{-n}) = 1 + \lim_{z \to [\phi(t)]^{-1}} [z\phi(t) - 1]F(z, t).$$

It should be noted that the regions of validity of Wald's identity are sufficient for applications such as the determination of the probability of absorption and the characteristic function of n. For example, if $\alpha < \infty, \beta = \infty$ and $E(X_j) > 0$, then $t_0 < 0$, and using the root $t = t_1(z)$ of $z\phi(t) = 1$ which satisfies $t_1(z) > t_0$ for z real, we obtain the approximate relation (Wald [12]) $E\{\exp \alpha t_1(z) z^n\} = 1$ or $E(z^n) = \exp\{-\alpha t_1(z)\}$ approximately.

4. Further notes and generalizations.

(i) If $\phi(t)$ is not defined except on the imaginary axis, and if α and β are both finite, then the identity (3.5) is still valid although the region of validity is not the same since t_0 has no meaning. Stein [10] showed that as a consequence of the "leakage" of probability, $F_m(\alpha) - F_m(-\beta)$ tends to zero exponentially as $m \to \infty$. Thus for purely imaginary t, the series (2.3), regarded as a power series in z, has radius of convergence $1 + c$, where $c = c(\alpha, \beta) > 0$ and uniformly in t. Thus in (3.5) we may set $z = [\phi(t)]^{-1}$ provided that

$$|\phi(t)| > 1/(1 + c).$$

(ii) There is a connection between the identity (3.5) and the Wiener-Hopf integral equation. In the following formal argument we suppose that the random walk starts at $h > 0$ and that there is an absorbing barrier at the origin. We assume that $a(x) = A'(x)$ exists and it follows that $f_m(x) = F'_m(x)$ also exists. Let

$$f(z, x) = \sum_{m=0}^{\infty} z^m f_m(x),$$

where $f_0(x) = \delta(x - h)$, $\delta(x)$ being the Dirac delta function. Then the recurrence relation (2.2) becomes

$$f_m(x) = \int_0^\infty a(x - y) f_{m-1}(y)\, dy$$

and thus

$$f(z, x) - \delta(x - h) = z \int_0^\infty a(x - y) f(z, y)\, dy,$$

or

$$-\delta(x - h) = \int_0^\infty \{za(x - y) - \delta(x - y)\} f(z, y)\, dy.$$

This is an integral equation of the Wiener-Hopf type with the difference kernel $\{za(x - y) - \delta(x - y)\}$ and holds for $x > 0$. The method of solution is to assume that for $x < 0$, the left hand side is defined by some function $g(z, x)$ which vanishes for $x > 0$. Both sides of the equation are then transformed by multiplying by e^{tx} and integrating with respect to x from $-\infty$ to ∞, thus obtaining

$$\int_{-\infty}^0 e^{tx} g(z, x)\, dx - e^{th} = \{z\phi(t) - 1\} \int_0^\infty e^{ty} f(z, y)\, dy.$$

This is the identity (3.5) (in the generalized form (4.1) below) and it is seen that the transform of the unknown function $g(z, x)$ is identified with $E(e^{tS} z^n)$.

(iii) If the random walk starts at the point h, then (3.5) becomes

$$E(e^{tS} z^n) = e^{th} + \{z\phi(t) - 1\} F(z, t). \tag{4.1}$$

(iv) It was shown by Wald [13] that (3.7) may be differentiated any number of times under the expectation sign and then we may set $t = 0$. This procedure can be used to obtain moment relations such as $E(S) = E(X)E(n)$. The differentiation property is a simple consequence of the fact that $F(z, t)$ is a regular function of z at the point $z = 1$ when (a) $\alpha < \infty$, $\beta < \infty$ whether or not $\phi(t)$ is regular at $t = 0$, (b) $\alpha < \infty$, $\beta = \infty$, $E(X) > 0$ and $\phi(t)$ regular and (c) $\alpha = \infty$, $\beta < \infty$, $E(X) < 0$ and $\phi(t)$ again regular. We cannot use complex variable arguments to obtain moment relations in cases (b) and (c) when $\phi(t)$ is not regular.

(v) Suppose that the steps of a random walk occur in continuous time and are governed by several independent distributions $A_1(x), A_2(x), \cdots A_N(x)$ where steps with distribution $A_k(x)$ occur in a Poisson process with mean rate r_k per unit time ($k = 1, 2, \cdots N$). Let $S(\tau)$ be the displacement at time τ and let the function $F_\tau(x)$ correspond to $F_m(x)$ in the discrete-time case. Thus

$$F_\tau(x) = P\{-\beta < S(\tau') < \alpha \quad \text{for} \quad 0 < \tau' < \tau \quad \text{and} \quad S(\tau) \leqq x\}.$$

Let

$$J(v,\theta) = \int_0^\infty e^{-v\tau} \left\{ \int_{-\beta}^{\alpha} e^{\theta x} \, dF_\tau(x) \right\} d\tau$$

and $J(v, \theta)$ corresponds to $F(z, t)$ of (2.3). We now have N moment generating functions

$$\phi_k(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \, dA_k(x), \qquad k = 1, 2, \cdots N.$$

Let T be the time at which absorption takes place (corresponding to n in the discrete-time case), and let $S = S(T)$. Then the continuous-time analogue of (3.5) is

$$(4.2) \qquad E(e^{\theta S - vT}) = 1 + [\sum_k r_k\{\phi_k(\theta) - 1\} - v] \; J(v, \theta)$$

and the manner of derivation is similar. If we now set

$$v = \sum_k r_k\{\phi_k(\theta) - 1\},$$

then we obtain the corresponding form of Wald's identity

$$E(\exp [\theta S - T \sum_k r_k\{\phi_k(\theta) - 1\}]) = 1.$$

Dvoretzky, Kiefer and Wolfowitz [5a] have shown that Wald's identity holds for processes in continuous time and have given applications to sequential tests for such processes. In the third example of Section 5, we show how (4.2) may be applied to a simple random walk in continuous time.

5. Three examples.

(i) We consider a discrete time random walk starting at the origin. The steps $X_j (j = 1, 2, \cdots)$ are discrete and bounded. There is a single absorbing barrier at α, ($\alpha > 0$). This is a gambler's ruin problem, in which α is the gambler's initial capital and X_j is the adversary's gain at the jth play. Ruin corresponds to absorption. It is well known that ruin is certain if $E(X_j) \geqq 0$. We shall use the identity (3.5) to obtain an expression for the probability of ruin in the case where $E(X_j) < 0$. We now take α to be a positive integer.

The probability generating function for the X_j is given by

$$M(w) = \sum_{k=-b}^{a} p_k w^k \qquad (0 < a < \infty, 0 < b < \infty).$$

We assume that

$$\text{g.c.d.}(k - k') = 1, \tag{5.1}$$

where k and k' run through all integers which satisfy $p_k \neq 0$ and $p_{k'} \neq 0$. The identity (3.5) now assumes the form

$$E(w^S z^n) = 1 + \{zM(w) - 1\}H(z, w), \tag{5.2}$$

where

$$H(z, w) = \sum_{m=0}^{\infty} z^m \sum_{j=-\infty}^{\alpha-1} p_j^{(m)} w^j \tag{5.3}$$

and

$$p_j^{(m)} = P(S_k < \alpha \quad \text{for} \quad k = 1, 2, \cdots m - 1 \quad \text{and} \quad S_m = j)$$

In this case, $w_0 > 0$ is defined by $M'(w_0) = 0$, and $w_0 \gtreqless 1$ according as $E(X) \lesseqgtr 0$. The series (5.3) converges for

$$|z| < [M(w_0)]^{-1}, \qquad |w| \geqq w_0,$$
$$|z| < [M(|w|)]^{-1}, \qquad |w| < w_0.$$

Since S can only take the values $\alpha, \alpha + 1, \cdots \alpha + a - 1$, the left hand side of (5.2) can be written as

$$E(w^S z^n) = \sum_{k=0}^{a-1} w^{\alpha+k} R_k(z), \tag{5.4}$$

where $R_k(z) = P(S = \alpha + k)E(z^n | S = \alpha + k)$. If $w(z)$ is a root of the equation

$$zM(w) = 1 \tag{5.5}$$

then from (5.2) and (5.4) we obtain

$$\sum_{k=0}^{a-1} [w(z)]^{\alpha+k} R_k(z) = 1. \tag{5.6}$$

There will in general be $a + b$ roots of (5.5) and P. Whittle has indicated that, by a modification of an argument used by Lindley [8], we may show that for $z < [M(w_0)]^{-1}$, b of these roots lie inside the circle $|w| = w_0$ and a of them lie outside. For complex z and $|z| < [M(w_0)]^{-1}$, we denote the roots which, when

z is real lie outside the circle $|w| = w_0$, by $w_1(z), w_2(z), \cdots w_a(z)$. For real z we assume that these roots are distinct and satisfy

$$w_1(z) \leqq |w_k(z)|, \qquad k = 2, 3, \cdots a.$$

It follows from the assumption (5.1) that the above inequality is strict in a neighbourhood of $z = 1$.

Thus we obtain a equations of the type (5.6), one from each of the roots $w_k(z)$ $(k = 1, 2, \cdots a)$, and these may be solved for the $R_k(z)$. The solution may be expressed in the form of a polynomial in w which takes the value 1 at the points $w = w_k(z)$ $(k = 1, 2, \cdots a)$. Hence

$$\sum_{k=0}^{a-1} w^{\alpha+k} R_k(z) = w^\alpha \sum_{k=1}^{a} [w_k(z)]^{-\alpha} \prod_{j \neq k} [\{w - w_j(z)\}/\{w_k(z) - w_j(z)\}].$$

We put $w = 1$ and $z = 1$ and write $w_k(1) = w_k$, thus obtaining

$$(5.7) \quad P(n < \infty) = \sum_{k=0}^{a-1} R_k(1) = \sum_{k=1}^{a} w_k^{-\alpha} \prod_{j \neq k} \{(1 - w_j)/(w_k - w_j)\}.$$

If $E(X) > 0$, then $w_1 = 1$ and hence $P(n < \infty) = 1$. This is well known. If $E(X) < 0$ then $w_1 > 1$, and (5.7) is an explicit expression for the probability of ruin or absorption. Also, as $\alpha \to \infty$, we have the asymptotic relation

$$P(n < \infty) = w_1^{-\alpha} \left[\prod_{j=2}^{a} \{(w_j - 1)/(w_j - w_1)\} \right] [1 + O(\delta^\alpha)],$$

where δ satisfies $0 < \delta < 1$. The usual approximation for $P(n < \infty)$ is

$$P(n < \infty) = w_1^{-\alpha},$$

approximately. See, for example, Wald [12] and Bartlett ([2], p. 20).

(ii) In this example we consider the classical random walk for which

$$P(X = 1) = p, \qquad P(X = -1) = q = 1 - p,$$

and we use the explicit results known for this case to illustrate the regions of convergence discussed in Lemma 3.1. It will be seen that in the single barrier case the regions given by Lemma 3.1 are sharper than in the two barrier case. Feller ([6] pp. 318–323) discusses this random walk in detail. Again, α and β are positive integers.

Using the notation of the previous example we have $M(w) = pw + qw^{-1}$. The roots of $zM(w) = 1$ are

$$w_{1,2}(z) = \{1 \pm (1 - 4pqz^2)^{\frac{1}{2}}\}(2pz)^{-1}.$$

If $\alpha < \infty$ and $\beta = \infty$, then $H(z, w)$ (5.3) is given by

$$H(z, w) = \{w^\alpha [w_1(z)]^{-\alpha} - 1\}/[pz\{1 - w_1(z)w^{-1}\}\{w - w_2(z)\}],$$

the method of derivation being the same as in the previous example. The factor $w - w_1(z)$ divides both the numerator and denominator of $H(z, w)$, and therefore $w = w_1(z)$ is not a singularity of $H(z, w)$. On the other hand $w = w_2(z)$ is a singularity of $H(z, w)$. Now $w = w_1(z)$ corresponds to $z = [M(w)]^{-1}$ in the region $|w| > w_0$, while $w = w_2(z)$ corresponds to $z = [M(w)]^{-1}$ in the region $|w| < w_0$, where $w_0 = (q/p)^{\frac{1}{2}}$ in this case. Thus in the region $|w| < w_0$, the series for $H(z, w)$ is convergent for $|z| < |M(w)|^{-1}$, while in the region $|w| > w_0$, the series is convergent for $|z| < [M(w_0)]^{-1} = \frac{1}{2}(pq)^{-\frac{1}{2}}$. Lemma 3.1 gives these z-regions as $|z| < [M(|w|)]^{-1}$ and $|z| < [M(w_0)]^{-1}$ respectively.

If we consider the two barrier case, then from Feller's results it is not difficult to show that the z-singularity of $H(z, w)$ nearest the origin is

$$z = [2(pq)^{\frac{1}{2}} \cos \{\pi/(\alpha + \beta)\}]^{-1} > [M(w_0)]^{-1},$$

which means that the series for $H(z, w)$ actually converges in a region larger than that given by Lemma 3.1.

(iii) As an application of the identity (4.2) we consider a random walk in continuous time, in which steps of $+1$ occur in a Poisson process with mean rate r_1, and steps of -1 occur in a Poisson process with mean rate r_2. The barriers are again at α and $-\beta$ (α, β positive integers). The use of (4.2) in this case is made simple by the fact that the walk will terminate exactly on a barrier. In the notation of Section 4(v), we have $\phi_1(\theta) = e^{\theta}$ and $\phi_2(\theta) = e^{-\theta}$. If we write $w = e^{\theta}$, then (4.2) takes the form

$$E(w^S e^{-vT}) = 1 + \{r_1(w - 1) + r_2(w^{-1} - 1) - v\} K(v, w), \tag{5.8}$$

where $K(v, w) = J(v, \log w)$. Let $w_1(v)$ and $w_2(v)$ be the roots of

$$r_1(w - 1) + r_2(w^{-1} - 1) = v.$$

Then

$$w_{1,2}(v) = \tfrac{1}{2}[r_1 + r_2 - v \pm \{(r_1 + r_2 - v)^2 - 4r_1r_2\}^{\frac{1}{2}}].$$

Let $p_1 = P(S = \alpha)$, $p_2 = P(S = -\beta)$ and let E_1, E_2 denote expectations conditional on $S = \alpha$, $S = -\beta$ respectively. Then the left hand side of (5.8) may be written $w^{\alpha} p_1 E_1(e^{-vT}) + w^{-\beta} p_2 E_2(e^{-vT})$ and on setting $w = w_i(v)$, $(i = 1, 2)$, in (5.8), we obtain two equations

$$\{w_i(v)\}^{\alpha} p_1 E_1(e^{-vT}) + \{w_i(v)\}^{-\beta} p_2 E_2(e^{-vT}) = 1, \qquad i = 1, 2,$$

whence, if we write $w_1 = w_1(v)$ and $w_2 = w_2(v)$,

$$p_1 E_1(e^{-vT}) = (w_2^{-\beta} - w_1^{-\beta})/(w_2^{-\beta} w_1^{\alpha} - w_2^{\alpha} w_1^{-\beta})$$

and

$$p_2 E_2(e^{-vT}) = (w_2^{\alpha} - w_1^{\alpha})/(w_1^{-\beta} w_2^{\alpha} - w_1^{\alpha} w_2^{-\beta}).$$

The above expressions are Laplace transforms of the two conditional distributions of T, the time to absorption. For further details we refer to the paper of Heathcote and Moyal [7], in which the above result is obtained using difference equations and in which this random walk is discussed in detail.

6. Acknowledgments. I should like thank Dr. P. Whittle for many helpful suggestions. Also, I should like to acknowledge a grant from the South African Council for Scientific and Industrial Research.

REFERENCES

[1] R. R. BAHADUR, "A note on the fundamental identity of sequential analysis," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 534-543.

[2] M. S. BARTLETT, *Stochastic Processes*, Cambridge University Press, London, 1955.

[3] RICHARD BELLMAN, "On a generalization of the fundamental identity of Wald," *Proc. Camb. Phil. Soc.*, Vol. 53 (1957), pp. 257–259.

[4] D. BLACKWELL AND M. A. GIRSHICK, "On functions of sequences of independent chance vectors with applications to the problem of the 'random walk' in k dimensions," *Ann. Math. Stat.*, Vol. 17 (1946), pp. 310–317.

[5] J. L. DOOB, *Stochastic Processes*, John Wiley and Sons, London, 1953.

[5a] A. DVORETZKY, J. KIEFER AND J. WOLFOWITZ, "Sequential decision problems for processes with continuous time parameter. Testing hypotheses," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 254–264.

[6] WILLIAM FELLER, *An Introduction to Probability Theory and its Applications*, 2nd ed., John Wiley and Sons, London, 1957.

[7] C. R. HEATHCOTE AND J. E. MOYAL, "The random walk (in continuous time) and its application to the theory of queues," *Biometrika*, Vol. 46 (1959), pp. 400–411.

[8] D. V. Lindley, "The theory of queues with a single server," *Proc. Camb. Phil. Soc.*, Vol. 48 (1952), pp. 277–289.

[9] HAROLD RUBEN, "A theorem on the cumulative product of independent random variables," *Proc. Camb. Phil. Soc.*, Vol. 55 (1959), pp. 333–337.

[9a] FRANK SPITZER, "A Tauberian theorem and its probability interpretation," *Trans. Amer. Math. Soc.*, Vol. 94 (1960), pp. 150–169.

[10] CHARLES STEIN, "A note on cumulative sums," *Ann. Math. Stat.*, Vol. 17 (1946), pp. 498-499.

[11] M. C. K. TWEEDIE, "Generalizations of Wald's fundamental identity of sequential analysis to Markov chains," *Proc. Camb. Phil. Soc.*, Vol. 56 (1960), pp. 205–214.

[12] ABRAHAM WALD, *Sequential Analysis*, John Wiley and Sons, London, 1947.

[13] ABRAHAM WALD, "Differentiation under the expectation sign in the fundamental identity of sequential analysis," *Ann. Math. Stat.*, Vol. 17 (1946), pp. 493–497.

[14] D. V. WIDDER, *The Laplace Transform*, Princeton University Press, London, 1946.