A NOTE ON SIMPLE BINOMIAL SAMPLING PLANS

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Introduction. This note gives two equivalent characterizations of simple sampling plans (s.p.'s) of size n, both of which prove the following Theorem: *The number of simple sampling plans of size n is* $n^{-1}\binom{3n}{n-1}$. The definitions and notations used will be those of M. H. DeGroot [1].

Proof of the Theorem. We indicate only the main steps in the proof, as the details are straightforward and can be filled in by reference to [1].

1. A simple s.p. of size n is characterized by the set C of its continuation points in the lattice quadrant.

2. A set C of lattice points in the quadrant is the set of continuation points of a simple s.p. S of size n if and only if

(i) the intersection C_k of C with each diagonal

$$A_k = \{x + y = k; x \geqq 0, y \geqq 0\}$$

is connected.

(ii) C_k is non-empty if and only if $k < n$.

(iii) No point of C_{k+1} is to the left of the leftmost point of C_k or below the lowest point of C_k. (If A, B are any two points in the lattice plane, A is to the left of B if and only if the x coordinate of A is less than that of B and A is below B if and only if the y coordinate of A is less than that of B).

3. Each non-empty C_k is characterized by how far southeast, t_k, its top is from $(0, k)$ and how far northwest, b_k, its bottom is from $(k, 0)$. t_k, b_k are non-negative integers.

4. The only restrictions on $\{t_k, b_k\}$ of a simple s.p. of size n are

$$t_k + b_k \leqq k, \qquad k = 0, 1, \cdots, n-1,$$
$$0 \leqq t_k \leqq t_{k+1}, \quad 0 \leqq b_k \leqq b_{k+1}, \qquad k = 0, 1, \cdots, n-2.$$

5. The number of different solutions of the above set of inequalities is the number of different simple s.p.'s of size n.

The combinatorial problem posed in 4, 5 may be solved thus. (A more general treatment of such problems is contained in [2].)

If $(x, y)_n$ denotes the number of simple s.p.'s of size n with $t_{n-1} = x$ and $b_{n-1} = y$, then plainly

$$(1) \qquad (x, y)_n = \sum_{a=0}^{x} \sum_{b=0}^{y} (a, b)_{n-1} \qquad \text{for } x + y < n$$
$$= 0 \qquad \text{for } x + y > n.$$

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The condition $(0, 0)_1 = 1$ together with (1) determines $(x, y)_n$ recursively for all non-negative integers x, y and positive integers n.

The number of different simple s.p.'s with $t_{n-1} + b_{n-1} = k$ is $k_{(n)}$, where

$$
(2) \qquad \begin{aligned} k_{(n)} &= \sum_{x+y=k} (x, y)_n = \sum_{x+y=k} \sum_{a=0}^{x} \sum_{b=0}^{y} (a, b)_{n-1} \\ &= \sum_{c=0}^{k} (k - c + 1) c_{(n-1)} \quad \text{for} \quad k < n, \qquad k_{(n)} = 0 \text{ for } k \geqq n, \end{aligned}
$$

and $0_{(1)} = 1$. These conditions determine $k_{(n)}$ recursively. Experiment leads to the conjectured solution

$$
(3) \qquad \begin{aligned} k_{(n)} &= \frac{2n - 2k}{2n + k} \binom{2n + k}{2n} = \frac{2n - 2k}{2n + k} \binom{2n + k}{k} \\ &= \binom{2n + k}{k} - 3 \binom{2n + k - 1}{k - 1} \qquad \text{for } k \leqq n \\ &= 0 \qquad \text{for } k \geqq n. \end{aligned}
$$

Recalling the simple general formula

$$
(4) \qquad \sum_{b=0}^{c} \binom{a + b}{b} = \binom{a + c + 1}{c},
$$

it is straightforward to verify that (3) does determine $k_{(n)}$. Then (3) and (4) together show that the total number of simple s.p.'s of size n is

$$
(5) \qquad \sum_{k=0}^{n-1} k_{(n)} = \binom{3n}{n - 1} - 3 \binom{3n - 1}{n - 2} = \frac{1}{n} \binom{3n}{n - 1},
$$

and the problem is solved.

Characterization by boundary points. Let us define a symmetric s.p. of size n as one symmetric about the line $x = y$. From the above, a simple symmetric s.p. of size $2n$ is characterized by the vector of non-negative integers

$$
(6) \qquad (t_2, t_3, \cdots, t_{2n-1}) \qquad (t_1 = t_0 = 0).
$$

where $t_2 \leqq t_3 \leqq \cdots \leqq t_{2n-1}$ and $t_i \leqq [i/2]$, $i = 1, \cdots, n - 1$. Consider the vectors of non-negative integers $(a_1, \cdots, a_{n-1})$ satisfying

$$
(7) \qquad a_1 \geqq \cdots \geqq a_{n-1} \geqq 0, \qquad a_i \leqq 2n - 2i, \qquad i = 1, \cdots, n - 1.
$$

From a vector $(a_1, \cdots, a_{n-1})$ satisfying (6) we obtain a vector $(t_2, \cdots, t_{2n-1})$ satisfying (7) (and conversely) by the following 1:1 correspondence:

Given $(a_1, \cdots, a_{n-1})$ construct a vector $(t_2, \cdots, t_{2n-1})$ in which

(8)

The first $(2n - 2 - a_1)t$'s are zero.
The next $(a_1 - a_2)t$'s are one.
⋮
The next $(a_{n-2} - a_{n-1})t$'s are $n - 2$.
The last a_{n-1} t's are $n - 1$.

Thus the simple symmetric s.p.'s of size $2n$ can also be characterized by the vectors $(a_1, \cdots, a_{n-1})$ satisfying (7); or more precisely the vectors $(a_1, \cdots, a_{n-1}, 0, 0, 0, a_{n-1}, \cdots a_1)$ satisfying (7) characterize simple symmetric s.p.'s of size $2n$. The a's in fact represent the "distances" of its boundary points from the points on the line $x + y = 2n$. From known results [3. p. 170], the number of simple symmetric s.p.'s of size $2n$ is $n^{-1}\binom{3n}{n-1}$.

Evidently, a 1:1 correspondence similar to (8) yields a characterization of any simple s.p. of size n in terms of the "distances" of its boundary points from the line $x + y = n$. The vectors $(a_1, \cdots, a_{n+1})$ depend on both $(t_1, \cdots, t_{n-1})$ and $(b_1, \cdots, b_{n-1})$ in this case, but the method as well as the conditions satisfied by $(a_1, \cdots, a_{n+1})$ can be easily derived. Since the lattice-theoretic ideas developed in [2, 3] yield a simple 1:1 correspondence between the vectorial representations (using boundary points) of simple s.p.'s of size n and simple symmetric s.p.'s of size $2n$, we obtain without further calculations another proof of our theorem. The characterization (7) of s.p.'s and their interpretation as a distributive lattice applies with little change to other problems in probability theory, and yields a unified approach for rederiving and extending many results. [cf., 2].

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AN INEQUALITY FOR BALANCED INCOMPLETE BLOCK DESIGNS

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1. Summary. For a resolvable balanced incomplete block design, R. C. Bose [1] obtained the inequality $b \geqq v + r - 1$, and P. M. Roy [2] and W. F. Mikhail [3] proved this inequality without the assumption of resolvability, but with the weaker assumption that v is a multiple of k. In this note an alternative and simpler proof of Roy's theorem is given.

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