GENERATING EXPONENTIAL RANDOM VARIABLES

By G. Marsaglia

Boeing Scientific Research Laboratories

Introduction. The use of exponential random variables is mentioned in numerous papers on Monte Carlo techniques, in connection with particle or radiation studies, reliability, life testing, etc., but there seems to be little discussion of fast techniques for their generation. Von Neumann discussed a method in [1], where he remarked that in spite of the method's appeal, in that it produced the desired result by performing only discriminations on the relative magnitude of numbers in (0, 1), it was a sad fact of life that it was slightly quicker to use a power series expansion to compute the logarithm of a uniform random variable.

We offer here a simpler device for producing exponential random variables by performing discriminations on the relative magnitudes of uniform (0, 1) random variables. The method is easy to understand, easy to program, requires little storage, and is quite fast, although not quite as fast as one of the versions of the general method given in [2].

The idea is to choose the minimum of a random number of uniform random variables, then add a random integer—roughly, let n and m be random integers taking values according to this schedule:

Value of n	probability	total	Value of m	probability	total
1	.58	.58	0	.63	.63
2	.29	.87	1	.23	.86
3	.10	.97	2	.09	.95
4	.02	.99	3	.03	.98
⋮			⋮		

Then, if $u_1, u_2, \cdots$ is a sequence of independent uniform random variables on $(0, 1)$, the random variable

$$(m + \min(u_1, u_2, \cdots, u_n))$$

has the exponential distribution. The expected value of n is about 1.58, so that we need on the average only 1.58 u's from which the minimum must be selected, and 1.58 discriminations to assign a value to n. Assigning a value to m also requires an average of 1.58 discriminations.

Details. We are concerned with a method for expressing an exponentially distributed random variable x in terms of the members of a sequence

$$u_1, u_2, \cdots \tag{1}$$

Received November 18, 1960.

of independent random variables, each uniformly distributed over (0, 1). Since there are available a wide variety of arithmetic procedures for producing a sequence of numbers which the users are willing to view as determinations of the sequence (1), we assume the availability of that sequence for our starting point.

Let n be a random variable taking values 1, 2, 3, $\cdots$ with probabilities $p_1, p_2, \cdots$. If

$$y = \min(u_1, u_2, \cdots, u_n),$$

then it is easy to see that the distribution of y is, for $0 \leqq \theta \leqq 1$,

$$P[y \leqq \theta] = 1 - p_1(1-\theta) - p_2(1-\theta)^2 - \cdots. \tag{2}$$

In particular, if $c = 1/(e-1) = .5819767 \cdots$, and

$$p_1 = c, \qquad p_2 = c/2!, \qquad p_3 = c/3!, \cdots,$$

then

$$p[y \leqq \theta] = ce(1 - e^{-\theta}), \qquad 0 \leqq \theta \leqq 1.$$

We express our principal result in this way:

THEOREM. *If* $c = 1/(e-1)$ *and if the random variable* n *takes values* 1, 2, 3, $\cdots$ *with probabilities* $c, c/2!, c/3!, \cdots$ *and if, independently, the random variable* m *takes values* 0, 1, 2, $\cdots$ *with probabilities* $1/(ce), 1/(ce^2), 1/(ce^3), \cdots$, *then the random variable*

$$x = m + \min(u_1, u_2, \cdots, u_n)$$

has the exponential distribution,

$$P[x \leqq a] = 1 - e^{-a}, \qquad 0 \leqq a.$$

The proof is a simple matter of verification—if $a = k + \theta$ where k is a non-negative integer and $0 \leqq \theta \leqq 1$, then

$$\begin{aligned} P[x \leqq k + \theta] &= P[x \leqq k - 1] + P[m = k, y \leqq \theta] \\ &= 1 - e^{-k} + [ce/(ce^{k+1})](1 - e^{-\theta}) = 1 - e^{-(k+\theta)}. \end{aligned}$$

Remarks. The trick of taking the minimum of a random number of uniform random variables may be applied to any density function which can be put in the form of (2). Unfortunately, the normal density doesn't fit this pattern, although generalizations of the method, such as taking the minimax of a doubly indexed set of u's seem to be feasible without entailing too much programming detail. But these tricks cannot get much more complicated and still stay simpler than the methods suggested in [2], which lead to very fast programs, surpassed only by colossal table look-ups.

REFERENCES

[1] JOHN VON NEUMANN, "Various techniques in connection with random digits," *Monte Carlo Methods*, National Bureau of Standards, AMS 12 (1951), pp. 36–38.

[2] G. MARSAGLIA, "Expressing a random variable in terms of uniform random variables," *Ann. Math. Stat.*, Vol. 32 (1961), pp. 894–898.