PERCENTAGE POINTS AND MODES OF ORDER STATISTICS FROM THE NORMAL DISTRIBUTION

BY SHANTI S. GUPTA

Bell Telephone Laboratories

0. Summary. This paper deals with the order statistics from the normal distribution. Equations are obtained for the percentage points and the modal values of the kth order statistic in a sample of size n. Table I gives some percentage points for selected values of k and n. In Table II the modal values of the largest order statistic are given. Appropriate symmetry relations which enable one to obtain certain missing values in Tables I and II are mentioned.

1. Introduction. Let $x_1, x_2, \cdots, x_n$ be n independent observations from a normal distribution with probability density function

$$\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)$$

and cumulative distribution function $\Phi(x)$. Suppose the observations are arranged in order of increasing magnitude so that we have

$$x_{(1)} \leqq x_{(2)} \leqq \cdots \leqq x_{(k)} \leqq \cdots \leqq x_{(n)}; \tag{1}$$

then we shall denote the kth order statistic by $x_{(k)}$. In this paper we use the same symbol for both a random variable and an observation on it.

2. Percentage points of the order statistics. The probability density function (p.d.f.) of $y = x_{(k)}$ is given by

$$f(y) = \frac{n!}{(k-1)!(n-k)!} \Phi^{k-1}(y)\,[1 - \Phi(y)]^{n-k} \varphi(y), \tag{2}$$

and its cumulative distribution function (c.d.f.) $F(y)$ is

$$F(y) = \frac{n!}{(k-1)!(n-k)!} \int_{-\infty}^{y} \Phi^{k-1}(x)\,[1 - \Phi(x)]^{n-k} \varphi(x)\,dx, \tag{3}$$

which can be written as

$$F(y) = I_{\Phi(y)}(k, n-k+1), \tag{4}$$

where $I_x(p, q)$ denotes the ratio of the incomplete Beta function to the complete Beta function with arguments p and q and which is tabulated in [4]. Thus the α percentage point of the kth order statistic is given by

$$I_{\Phi(y)}(k, n-k+1) = \alpha. \tag{5}$$

The percentage points of the Beta distribution with parameters p and q are tabulated in [3] for selected values of p and q respectively.

Received October 20, 1960; revised April 10, 1961.

The following symmetry relation enables one to obtain other values of the percentage points of the Beta distribution

$$I_x(p, q) = 1 - I_{1-x}(q, p). \tag{6}$$

Other missing values can be obtained by inverse interpolation in the Table of the Incomplete Beta Function [1].

It should be noted that, in the particular cases when $k = 1$ and $k = n$, equation (4) reduces to simple forms, and hence the percentage point of the smallest order statistic is the solution of

$$\Phi(y) = 1 - (1 - \alpha)^{1/n} \tag{7}$$

and the percentage point of the largest order statistic is the solution of

$$\Phi(y) = \alpha^{1/n}. \tag{8}$$

By putting $y = -x$ and $\alpha = 1 - \gamma$ in (5) and using (6) we obtain

$$I_{\Phi(x)}(n - k + 1, k) = \gamma, \tag{9}$$

which implies that the α percentage point of the kth order statistic in a sample of size n is the negative of the $1 - \alpha$ percentage point of the $(n - k + 1)$th order statistic, and vice-versa.

Using (7), (8) and (5), the upper percentage points of these order statistics were computed for $n = 1(1)10$, $k = 1(1)n$ and $\alpha = .50, .75, .90, .95$ and $.99$. Also, the same upper percentage points for $n = 11(1)20$ for the smallest, largest and median order statistics were computed. All these values are given in Table I.

It should be pointed out that (1)-(8) are generally applicable to any continuous distributions when $\Phi(y)$ and $\varphi(y)$ are replaced appropriately; for example, these equations are used to obtain the percentage points of the order statistics from the gamma distributions in [2].

3. Modal values of the order statistics. The mode or modal value x of the kth order statistic satisfies the equation

$$\frac{n!}{(k-1)!(n-k)!} \frac{d}{dx} \{\Phi^{k-1}(x)[1 - \Phi(x)]^{n-k}\varphi(x)\} = 0, \tag{10}$$

which simplifies to

$$(k - 1)[1 - \Phi(x)]\varphi(x) - (n - k)\Phi(x)\varphi(x) = \Phi(x)[1 - \Phi(x)]x. \tag{11}$$

The equation (11) remains unchanged if we substitute $-x$ for x and $n - k + 1$ for k. This shows that the modal values of the kth and $(n - k + 1)$th order statistics are equal and opposite in sign. In particular, the modal value of the median i.e., the $(m + 1)$th order statistic in a sample of size $n = 2m + 1$ is equal to zero. Also, from the above symmetrical relation, it follows that for any order statistic which is above the median the modal value is a positive number. For the particular case $k = n$, the equation (11) reduces to

$$(n - 1)\varphi(x) = \Phi(x)x, \tag{12}$$

TABLE I

Percentage Points of the kth Order Statistic in a Sample of Size n from the Normal Distribution

n	k \ α	.50	.75	.90	.95	.99
1	1	0.0000	0.6745	1.2816	1.6449	2.3263
2	1	-0.5450	0.0000	0.4783	0.7601	1.2816
	2	0.5450	1.1078	1.6322	1.9545	2.5750
3	1	-0.8193	-0.3317	0.0900	0.3361	0.7877
	2	0.0000	0.4500	0.8567	1.1015	1.5640
	3	0.8193	1.3319	1.8183	2.1212	2.7119
4	1	-0.9981	-0.5450	-0.1569	0.0681	0.4783
	2	-0.2905	0.1097	0.4664	0.6789	1.0764
	3	0.2905	0.6966	1.0689	1.2953	1.7279
	4	0.9981	1.4803	1.9432	2.2340	2.8058
5	1	-1.1290	-0.6994	-0.3344	-0.1238	0.2582
	2	-0.4851	-0.1151	0.2119	0.4054	0.7652
	3	0.0000	0.3600	0.6851	0.8806	1.2501
	4	0.4851	0.8641	1.2147	1.4294	1.8428
	5	1.1290	1.5900	2.0365	2.3187	2.8769
6	1	-1.2313	-0.8193	-0.4713	-0.2714	0.0900
	2	-0.6297	-0.2807	0.0259	0.2065	0.5408
	3	-0.1983	0.1338	0.4311	0.6088	0.9421
	4	0.1983	0.5333	0.8384	1.0230	1.3739
	5	0.6297	0.9897	1.3249	1.5313	1.9307
	6	1.2313	1.6765	2.1105	2.3862	2.9339
7	1	-1.3149	-0.9166	-0.5819	-0.3903	-0.0450
	2	-0.7438	-0.4105	-0.1192	0.0519	0.3675
	3	-0.3475	-0.0348	0.2435	0.4090	0.7182
	4	0.0000	0.3085	0.5870	0.7543	1.0702
	5	0.3475	0.6649	0.9556	1.1323	1.4698
	6	0.7438	1.0895	1.4130	1.6130	2.0016
	7	1.3149	1.7476	2.1717	2.4421	2.9814
8	1	-1.3852	-0.9982	-0.6741	-0.4892	-0.1569
	2	-0.8376	-0.5167	-0.2372	-0.0736	0.2274
	3	-0.4662	-0.1682	0.0960	0.2525	0.5440
	4	-0.1506	0.1395	0.3999	0.5556	0.8481
	5	0.1506	0.4425	0.7074	0.8673	1.1703
	6	0.4662	0.7702	1.0500	1.2206	1.5478
	7	0.8376	1.1719	1.4862	1.6809	2.0609
	8	1.3852	1.8078	2.2237	2.4898	3.0220

This table gives the values of y for which

$$\frac{n!}{(k-1)!\,(n-k)!} \int_{-\infty}^{y} \Phi^{k-1}(x) \left[1-\Phi(x)\right]^{n-k} \varphi(x)\,dx = \alpha ,$$

where $\varphi(x)$ and $\Phi(x)$ refer to the p.d.f. and c.d.f. of a standard normal chance variable, respectively.

TABLE I (CONT'D)

n	k \ α	.50	.75	.90	.95	.99
9	1	-1.4457	-1.0680	-0.7530	0.5736	-0.2520
	2	-0.9168	-0.6060	-0.3362	-0.1786	0.1106
	3	-0.5644	-0.2779	-0.0249	0.1248	0.4024
	4	-0.2713	0.0050	0.2518	0.3990	0.6744
	5	0.0000	0.2742	0.5216	0.6702	0.9503
	6	0.2713	0.5505	0.8050	0.9591	1.2521
	7	0.5644	0.8579	1.1288	1.2946	1.6132
	8	0.9168	1.2418	1.5484	1.7389	2.1117
	9	1.4457	1.8598	2.2689	2.5312	3.0575
10	1	-1.4988	-1.1290	-0.8215	-0.6468	-0.3344
	2	-0.9852	-0.6828	-0.4211	-0.2685	0.0109
	3	-0.6477	-0.3707	-0.1267	0.0173	0.2839
	4	-0.3716	-0.1062	0.1300	0.2705	0.5327
	5	-0.1214	0.1395	0.3740	0.5142	0.7778
	6	0.1214	0.3835	0.6209	0.7640	1.0345
	7	0.3716	0.6407	0.8869	1.0363	1.3212
	8	0.6477	0.9325	1.1961	1.3578	1.6695
	9	0.9852	1.3024	1.6024	1.7894	2.1560
	10	1.4988	1.9055	2.3087	2.5679	3.0889
11	1	-1.5459	-1.1830	-0.8821	-0.7114	-0.4068
	6	0.0000	0.2493	0.4741	0.6090	0.8633
	11	1.5459	1.9462	2.3443	2.6007	3.1171
12	1	-1.5882	-1.2313	-0.9362	-0.7691	-0.4713
	6	-0.1017	0.1374	0.3524	0.4811	0.7228
	7	0.1017	0.3416	0.5587	0.6894	0.9361
	12	1.5882	1.9829	2.3764	2.6303	3.1427
13	1	-1.6265	-1.2750	-0.9850	-0.8210	-0.5293
	7	0.0000	0.2301	0.4375	0.5620	0.7965
	13	1.6265	2.0163	2.4056	2.6574	3.1660
14	1	-1.6615	-1.3149	-1.0294	-0.8682	-0.5819
	7	-0.0875	0.1345	0.3342	0.4537	0.6783
	8	0.0875	0.3101	0.5113	0.6323	0.8606
	14	1.6615	2.0468	2.4325	2.6822	3.1875
15	1	-1.6937	-1.3514	-1.0700	-0.9114	-0.6300
	8	0.0000	0.2147	0.4083	0.5244	0.7430
	15	1.6937	2.0749	2.4573	2.7052	3.2074
16	1	-1.7235	-1.3852	-1.1075	-0.9512	-0.6742
	8	-0.0768	0.1313	0.3185	0.4306	0.6412
	9	0.0768	0.2853	0.4738	0.5870	0.8005
	16	1.7235	2.1010	2.4803	2.7265	3.2259
17	1	-1.7512	-1.4165	-1.1423	-0.9880	-0.7150
	9	0.0000	0.2021	0.3843	0.4935	0.6991
	17	1.7512	2.1253	2.5018	2.7464	3.2432
18	1	-1.7771	-1.4457	-1.1746	-1.0223	-0.7530
	9	-0.0684	0.1282	0.3050	0.4108	0.6098
	10	0.0684	0.2653	0.4431	0.5499	0.7511
	18	1.7771	2.1480	2.5219	2.7651	3.2595
19	1	-1.8013	-1.4731	-1.2048	-1.0543	-0.7884
	10	0.0000	0.1914	0.3640	0.4675	0.6621
	19	1.8013	2.1694	2.5408	2.7826	3.2748
20	1	-1.8242	-1.4988	-1.2332	-1.0843	-0.8215
	10	-0.0617	0.1250	0.2930	0.3936	0.5826
	11	0.0617	0.2487	0.4175	0.5188	0.7097
	20	1.8242	2.1895	2.5586	2.7992	3.2892

TABLE II

Modal Values of the Largest Order Statistic in a Sample of Size n from the Standard Normal Distribution

n	Modal Value	n	Modal Value
1	0.0000	19	1.7173
2	0.5061	20	1.7398
3	0.7653	21	1.7611
4	0.9359	22	1.7812
5	1.0615	23	1.8004
6	1.1602	24	1.8186
7	1.2412	25	1.8359
8	1.3095	30	1.9123
9	1.3684	35	1.9754
10	1.4202	40	2.0290
11	1.4662	45	2.0756
12	1.5076	50	2.1167
13	1.5452	60	2.1866
14	1.5796	70	2.2446
15	1.6113	80	2.2940
16	1.6406	90	2.3369
17	1.6679	100	2.3749
18	1.6934		

This table gives the value of x for which

$$(n-1)\ \varphi(x) = x\ \Phi(x)$$

where $\varphi(x)$ and $\Phi(x)$ refer to the p.d.f. and the c.d.f. of the standard normal chance variable, respectively.

which was solved for $n = 1(1)\ 25(5)\ 50(10)\ 100$ to give the values of the mode of the largest order statistic in a sample of size n from the normal distribution. These values are given in Table II.

It should be noted that with obvious changes the above text, up to and including (12), can be applied to any continuous variable distributed symmetrically about zero.

4. Description of the tables. Table I gives the α percentage points of the kth order statistic in a sample of size n from a standard normal distribution. Values of α chosen are .50, .75, .90, .95, and .99, which correspond to the upper 50, 25, 10, 5 and 1 per cent points of the distribution. The percentage points corresponding to $\alpha = .25, .10, .05$ and $.01$, viz., the lower percentage points of the kth order statistic, can be obtained from Table I by changing the sign of the corresponding upper percentage point of the $(n - k + 1)$th order statistic. For values of $n = 1(1)10$, percentage points are given for all $k = 1(1)n$. For values of $n = 11(1)20$, the percentage points are given only for $k = 1, \frac{1}{2}(n + 1), n$ for odd values of n and for $k = 1, \frac{1}{2}n, \frac{1}{2}n + 1, n$ for even values of n.

The values given were computed by using Newton's method on an IBM 650. The values of the percentage points of the median then were checked against the available values in [1] and were found to be in agreement. Other independent checks indicate that the percentage points are correct to within one unit in the last decimal place.

Table II gives the values of the modes of the largest order statistic for $n = 1(1)\ 25(5)\ 50(10)\ 100$. The modal values of the smallest order statistic for a given n are obtained from this table by changing the sign of the corresponding value. These values were also obtained by using Newton's procedure and are correct to four decimal places.

5. Acknowledgments. The author wishes to thank Miss Phyllis A. Groll for assistance in computing Tables I and II and Mr. H. Leon Harter for helpful comments, including corrections to the entries in Table II.

REFERENCES

[1] Churchill Eisenhart, Lola S. Deming, and Celia S. Martin, "On the arithmetic mean and the median in small samples from the normal and certain nonnormal populations," working paper SEL-49-1, February 1949, Statistical Engineering Laboratory, NBS (unpublished).

[2] Shanti S. Gupta, "Order statistics from the Gamma distribution," *Technometrics*, Vol. 2 (1960), pp. 243–262.

[3] E. S. Pearson, and H. O. Hartley, *Biometrika Tables for Statisticians*, Vol. I, University Press, Cambridge, England, 1954.

[4] K. Pearson, *Tables of the Incomplete Beta Function*, University Press, Cambridge, England, 1948.