ANALYSIS OF A CLASS OF PBIB DESIGNS WITH MORE THAN TWO ASSOCIATE CLASSES

By P. V. Rao

College of Science, Nagpur, India[1]

1. Introduction and Summary. The use of 2-associate PBIB design is fairly common in experimental work. However, PBIB designs with more than two associate classes are not widely used because of the complicated nature of the analysis and construction involved. Recently, in an interesting paper [2], Shah constructed a number of 3-associate PBIB designs by what may be called the matrix substitution method. In this method, the incidence matrix of the 3-associate PBIB design is constructed by replacing the integers of a balanced matrix in S integers (for example, the matrix might be the incidence matrix of a BIB design, that is, a balanced matrix in two integers) by the incidence matrices of S associable BIB designs. The present author [1] and Shah [3] have shown that the above method may be used to construct a PBIB design with $2m + 1$ associate classes by replacing the integers of a PBIB design with m associate classes by the incidence matrices of two associable BIB designs. Shah [3] has also given a simple method of analysis for PBIB designs with $2^n - 1$ associate classes constructed by the matrix substitution method. Thus, Shah's method of analysis may be used to analyze PBIB designs with 3 and 7 associate classes (corresponding to $n = 2$ and 3) which are of practical interest. In the present paper, simple methods of analysis for a class of PBIB designs with 3 and 5 associate classes are given. The method given here for PBIB designs with three associate classes provides an alternate method, and it is hoped that this method is more simple and direct than that given by Shah.

2. Notation and some known results. The symbol '$\otimes$' will be used to denote the Kronecker product of two matrices. Thus, if $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are two $m \times n$ and $p \times q$ matrices we have,

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B}, & a_{12}\mathbf{B}, & \cdots & a_{1n}\mathbf{B} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1}\mathbf{B}, & a_{m2}\mathbf{B}, & \cdots & a_{mn}\mathbf{B} \end{bmatrix}.$$

The following square matrix:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1, & \mathbf{A}_2, & \cdots, & \mathbf{A}_2 \\ \mathbf{A}_2, & \mathbf{A}_1, & \cdots, & \mathbf{A}_2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{A}_2, & \mathbf{A}_2, & \cdots & \mathbf{A}_1 \end{bmatrix}$$

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[1] Present address: Department of Mathematics, University of Georgia, Athens, Ga.

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are scalars or square matrices of the same order, will be denoted by $(\mathbf{A}_1 \diagdown \mathbf{A}_2)_v$, where the subscript v stands for the order of $\mathbf{A}$ when $\mathbf{A}_1$ and $\mathbf{A}_2$ are considered as its elements. It can be easily verified that

$$(2.1) \qquad (c_1\mathbf{A}_1 \diagdown c_2\mathbf{A}_2)_v = \mathbf{I}(v) \otimes (c_1\mathbf{A}_1 - c_2\mathbf{A}_2) + \mathbf{E}(v) \otimes c_2\mathbf{A}_2,$$

where c_1 and c_2 are scalars, $\mathbf{I}(v)$ is the $v \times v$ identity matrix, and $\mathbf{E}(v)$ is the $v \times v$ matrix with all elements equal to unity.

Every design will be denoted by its incidence matrix. Two designs will be called associable if they satisfy the definition given by Shah [2]. It is known that, if $\mathbf{N}_0$ and $\mathbf{N}_1$ are associable BIB designs with parameters $v_1, b_1, r_0, k_0, \lambda_0$; $v_1, b_1, r_1, k_1, \lambda_1$; and $\mu_{01} = \mu$, $\eta_{01} = \eta$, and, if $\mathbf{A}$ is a PBIB design with parameters v_2, b_2, r_2, k_2; $\lambda_{12}, \lambda_{22}, \cdots, \lambda_{m2}$; $P^q_{uu'}(u, u', q = 1, 2, \cdots, m)$, then the design $\mathbf{N}$, obtained from $\mathbf{A}$ by replacing the integer i by $\mathbf{N}_i$ $(i = 0, 1)$, is a PBIB design with $2m + 1$ associate classes and parameters $v = v_1v_2$, $b = b_1b_2$, $r = r_1r_2 + (v_2 - r_2)r_0$, $k = k_1k_2 + (v_2 - k_2)k_0$.

Let y_{ij} denote the yield of the plot in the jth block of $\mathbf{N}$ to which the ith treatment is applied. For the purpose of the analysis we assume the model

$$(2.2) \qquad y_{ij} = \alpha + t_i + b_j + \epsilon_{ij},$$

where α, t_i, b_j are respectively the general effect, the effect of the ith treatment and the effect of the jth block. The ϵ_{ij} are independent normal variates with mean 0 and variance σ^2. Let T_i and B_j denote respectively the total yield due to the ith treatment and the jth block of $\mathbf{N}$. If the column vectors of $(T_1, T_2, \cdots, T_v)$, $(B_1, B_2, \cdots, B_b)$, $(t_1, t_2, \cdots, t_v)$ and $(\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_v)$ are denoted by $\mathbf{T}$, $\mathbf{B}$, $\mathbf{t}$, and $\hat{\mathbf{t}}$ respectively, then it is well known that the reduced normal equations for the intra-block estimation of treatment contrasts are

$$(2.3) \qquad \mathbf{Q} = \mathbf{C}\hat{\mathbf{t}},$$

where

$$(2.4) \qquad \mathbf{Q} = \mathbf{T} - (1/k)\mathbf{NB},$$

and

$$(2.5) \qquad \mathbf{C} = r\mathbf{I}(v) - (1/k)\mathbf{NN}'.$$

The adjusted treatment sum of squares is equal to

$$(2.6) \qquad \hat{\mathbf{t}}'\mathbf{Q} = \sum_i \hat{t}_i Q_i.$$

Further, if

$$(2.7) \qquad \hat{t}_i = d_{i1}Q_1 + d_{i2}Q_2 + \cdots + d_{iv}Q_v$$

is a solution of the normal equations, then the variance of the best estimate of any estimable parametric function $\sum_i c_i t_i$ is given by

$$(2.8) \qquad V(\sum_i c_i \hat{t}_i) = (\sum_i \sum_j c_i c_j d_{ij})\sigma^2.$$

Let $\mathbf{N}$ be a PBIB design with parameters $v, b, r, k; \lambda_1, \lambda_2, \cdots, \lambda_m$; $P^q_{uu'}(u, u', q = 1, 2, \cdots, m)$. Define $m + 1$ $v \times v$ matrices $\mathbf{B}_q$ $(q = 0, 1, \cdots, m)$ as

$$\mathbf{B}_q = (b^q_{ij}), \qquad i, j = 1, 2, \cdots, v, \qquad q = 0, 1, \cdots, m, \tag{2.9}$$

where

$$b^q_{ij} = 1 \text{ if the } i\text{th and } j\text{th treatments are } q\text{th associates in } \mathbf{N}$$
$$= 0 \text{ otherwise,}$$

with a convention that every treatment is its own 0th associate. Then it is easy to show that

$$\mathbf{B}_0 + \mathbf{B}_1 + \cdots + \mathbf{B}_m = \mathbf{E}(v), \tag{2.10}$$

and

$$\mathbf{NN}' = r\mathbf{B}_0 + \lambda_1\mathbf{B}_1 + \cdots + \lambda_m\mathbf{B}_m. \tag{2.11}$$

3. A Lemma.

LEMMA 3.1: *Let* $\mathbf{A}$ *be a 2-associate* PBIB *design with parameters* v_2, b_2, r_2, k_2; $\lambda_{12}, \lambda_{22}$; $P^q_{uu'}$ *and let* $\mathbf{N}_0$ *and* $\mathbf{N}_1$ *be two associable* BIB *designs with parameters* $v_1, b_1, r_0, k_0, \lambda_0$; $v_1, b_1, r_1, k_1, \lambda_1$; μ *and* η. *If* $\mathbf{N}$ *is a design obtained by replacing the integer* i *in* $\mathbf{A}$ *by the matrix* $\mathbf{N}_i$ $(i = 0, 1)$, *then*

$$\mathbf{NN}' = \mathbf{AA}' \otimes [c_1\mathbf{I}(v_1) + c_2\mathbf{E}(v_1)] + \mathbf{E}(v_2) \otimes [c_3\mathbf{I}(v_1) + c_4\mathbf{E}(v_1)], \tag{3.1}$$

where

$$\begin{aligned} c_1 &= (r_1 + r_0 - 2\mu) - (\lambda_1 + \lambda_0 - 2\eta), \\ c_2 &= \lambda_1 + \lambda_0 - 2\eta, \\ c_3 &= [b_2r_0 - 2r_2(r_0 - \mu)] - [b_2\lambda_0 - 2r_2(\lambda_0 - \eta)], \\ c_4 &= b_2\lambda_0 - 2r_2(\lambda_0 - \eta). \end{aligned}$$

PROOF: If we consider $\mathbf{N}$ as a partitioned matrix with elements $\mathbf{N}_0$ and $\mathbf{N}_1$, it is easy to verify that the (ij)th element in the product $\mathbf{NN}'$ is equal to

$$r_2\mathbf{N}_1\mathbf{N}_1' + (b_2 - r_2)\mathbf{N}_0\mathbf{N}_0' \qquad \text{if } i = j,$$

$$\lambda_{12}\mathbf{N}_1\mathbf{N}_1' + 2(r_2 - \lambda_{12})\mathbf{N}_0\mathbf{N}_1' + (b_2 - 2r_2 + \lambda_{12})\mathbf{N}_0\mathbf{N}_0'$$

if the treatments i and j are first associates in $\mathbf{A}$,

$$\lambda_{22}\mathbf{N}_1\mathbf{N}_1' + 2(r_2 - \lambda_{22})\mathbf{N}_0\mathbf{N}_1' + (b_2 - 2r_2 + \lambda_{22})\mathbf{N}_0\mathbf{N}_0'$$

if the treatments i and j are second associates in $\mathbf{A}$.

Defining $\mathbf{B}_q$ $(q = 0, 1, 2)$ as in (2.9), it follows that

$$\begin{aligned} \mathbf{NN}' &= \mathbf{B}_0 \otimes [r_2\mathbf{N}_1\mathbf{N}_1' + (b_2 - r_2)\mathbf{N}_0\mathbf{N}_0'] + \mathbf{B}_1 \otimes [\lambda_{12}\mathbf{N}_1\mathbf{N}_1' + 2(r_2 - \lambda_{12})\mathbf{N}_0\mathbf{N}_1' \\ &\quad + (b_2 - 2r_2 + \lambda_{12})\mathbf{N}_0\mathbf{N}_0'] + \mathbf{B}_2 \otimes [\lambda_{22}\mathbf{N}_1\mathbf{N}_1' + 2(r_2 - \lambda_{22})\mathbf{N}_0\mathbf{N}_1' \\ &\quad + (b_2 - 2r_2 + \lambda_{22})\mathbf{N}_0\mathbf{N}_0'] \\ &= [r_2\mathbf{B}_0 + \lambda_{12}\mathbf{B}_1 + \lambda_{22}\mathbf{B}_2] \otimes \mathbf{N}_1\mathbf{N}_1' + [(b_2 - 2r_2)(\mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2) + r_2\mathbf{B}_0 \\ &\quad + \lambda_{12}\mathbf{B}_1 + \lambda_{22}\mathbf{B}_2] \otimes \mathbf{N}_0\mathbf{N}_0' + 2[r_2(\mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2) - (r_2\mathbf{B}_0 + \lambda_{12}\mathbf{B}_1 + \lambda_{22}\mathbf{B}_2)] \\ &\quad \otimes \mathbf{N}_0\mathbf{N}_1'. \end{aligned} \tag{3.2}$$

From (2.10) and (2.11) we have

$$\mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2 = \mathbf{E}(v_2), \qquad r_2\mathbf{B}_0 + \lambda_{12}\mathbf{B}_1 + \lambda_{22}\mathbf{B}_2 = \mathbf{AA}'.$$

Hence from (3.2) it follows that

$$\begin{aligned}\mathbf{NN}' &= \mathbf{AA}' \otimes [\mathbf{N}_1\mathbf{N}_1' + \mathbf{N}_0\mathbf{N}_0' - 2\mathbf{N}_0\mathbf{N}_1'] \\ &\qquad + \mathbf{E}(v_2) \otimes [(b_2 - 2r_2)\mathbf{N}_0\mathbf{N}_0' + 2r_2\mathbf{N}_0\mathbf{N}_1'].\end{aligned} \tag{3.3}$$

Since $\mathbf{N}_0$ and $\mathbf{N}_1$ are associable BIB designs with parameters of association μ and η, it follows that

$$\begin{aligned}\mathbf{N}_0\mathbf{N}_0' &= (r_0 - \lambda_0)\mathbf{I}(v_1) + \lambda_0\mathbf{E}(v_1), \\ \mathbf{N}_1\mathbf{N}_1' &= (r_1 - \lambda_1)\mathbf{I}(v_1) + \lambda_1\mathbf{E}(v_1), \\ \mathbf{N}_0\mathbf{N}_1' &= (\mu - \eta)\mathbf{I}(v_1) + \eta\mathbf{E}(v_1).\end{aligned}$$

Substituting for $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$ and $\mathbf{N}_0\mathbf{N}_1'$ in (3.3) it is easy to verify that

$$\mathbf{NN}' = \mathbf{AA}' \otimes [c_1\mathbf{I}(v_1) + c_2\mathbf{E}(v_1)] + \mathbf{E}(v_2) \otimes [c_3\mathbf{I}(v_1) + c_4\mathbf{E}(v_1)],$$

which completes the proof.

Most of the designs of practical interest will occur when $\mathbf{N}$ is taken as a null matrix or as the incidence matrix of a randomized block design. The expression for $\mathbf{NN}'$ may be further simplified in these two cases. The values of c_1, c_2, c_3, c_4 are given below for the two particular cases:

Case i:

$$\mathbf{N}_0 = \mathbf{0}(v_1 \times b_1),$$

where $\mathbf{0}(v_1 \times b_1)$ is the $v_1 \times b_1$ null matrix. Since $r_0 = k_0 = \lambda_0 = \mu = \eta = 0$, we have $c_1 = r_1 - \lambda_1$, $c_2 = \lambda_1$, $c_3 = 0$ and $c_4 = 0$.

Case ii:

$$\mathbf{N}_0 = \mathbf{E}(v_1 \times b_1)$$

where $\mathbf{E}(v_1 \times b_1)$ is a $v_1 \times b_1$ matrix with all elements equal to unity. Since $r_0 = \lambda_0 = b_1$, $k_0 = v_1$, $\mu = \eta = r_1$, we have $c_1 = r_1 - \lambda_1$, $c_2 = \lambda_1 + b_1 - 2r_1$, $c_3 = 0$ and $c_4 = b_2b_1 - 2r_2(b_1 - r_1)$.

It is known that $\mathbf{N}$ defined in Lemma 3.1 is the incidence matrix of a PBIB design with five associate classes. The reduced normal equations for the intrablock estimates of treatment comparisons are $\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}}$, where

$$\mathbf{C} = r\mathbf{I}(v_1v_2) - (1/k)\mathbf{NN}', \qquad r = r_1r_2 + (v_2 - r_2)r_0, \qquad k = k_1k_2 + (v_2 - k_2)k_0. \tag{3.4}$$

4. Analysis of the design N in particular cases.

Case (i): Let $\mathbf{A}$ be a BIB design with parameters $v_2, b_2, r_2, k_2, \lambda_{12} = \lambda_{22} = \lambda_2$. We then have $\mathbf{AA}' = (r_2 - \lambda_2)\mathbf{I}(v_2) + \lambda_2\mathbf{E}(v_2)$. Hence from Lemma 3.1

$$\begin{aligned}\mathbf{NN}' &= [(r_2 - \lambda_2)\mathbf{I}(v_2) + \lambda_2\mathbf{E}(v_2)] \otimes [c_1\mathbf{I}(v_1) + c_2\mathbf{E}(v_1)] \\ &\qquad + \mathbf{E}(v_2) \otimes [c_3\mathbf{I}(v_1) + c_4\mathbf{E}(v_1)].\end{aligned}$$

Hence

$$kC = [rk - c_1(r_2 - \lambda_2)]\mathbf{I}(v_2) \otimes \mathbf{I}(v_1) - (c_3 + c_1\lambda_1)\mathbf{E}(v_2) \otimes \mathbf{I}(v_1) \tag{4.1}$$
$$- c_2(r_2 - \lambda_2)\mathbf{I}(v_2) \otimes \mathbf{E}(v_1) - (c_4 + c_2\lambda_2)\mathbf{E}(v_2) \otimes \mathbf{E}(v_1).$$

Now let the number pair (ij) denote the jth treatment which replaces the ith row of $\mathbf{A}$ $(i = 1, 2, \cdots, v_1, j = 1, 2, \cdots, v_2)$ in $\mathbf{N}$. This means that the rows of $\mathbf{N}$ are numbered as $(11), (12), \cdots, (1v_2), (21), (22), \cdots, (2v_2), \cdots, (v_11), (v_12), \cdots, (v_1v_2)$. Also, let t_{ij} and Q_{ij} denote the effect and the adjusted total yield of the treatment (ij). In view of (4.1) the normal equations can be written as

$$kQ_{ij} = u_1\hat{t}_{ij} - u_2\hat{t}_{i\cdot} - u_3\hat{t}_{\cdot j} - u\hat{t}_{\cdot\cdot},$$

where

$$\begin{aligned} u_1 &= rk - c_1(r_2 - \lambda_2), \\ u_2 &= c_3 + c_1\lambda_1, \\ u_3 &= c_2(r_2 - \lambda_2), \\ u_4 &= c_4 + c_2\lambda_2, \end{aligned} \tag{4.2}$$

and $\hat{t}_{i\cdot}$, $\hat{t}_{\cdot j}$ and $\hat{t}_{\cdot\cdot}$ have their usual meanings. Taking the additional equation $\hat{t}_{\cdot\cdot} = 0$, we can solve the normal equations uniquely and get the solution

$$\hat{t}_{ij} = d_1Q_{ij} + d_2Q_{i\cdot} + d_3Q_{\cdot j}, \tag{4.3}$$

where

$$d_1 = k/u_1, \qquad d_2 = ku_2/[u_1(u_1 - v_2u_2)], \qquad d_3 = ku_3/[u_1(u_1 - v_1u_3)],$$

and $Q_{i\cdot}$ and $Q_{\cdot j}$ have their usual meanings. Hence, from (2.6), the adjusted treatment sum of squares for testing the overall differences between the treatment effects is

$$\sum_i \sum_j t_{ij}Q_{ij} = d_1\sum_i \sum_j Q_{ij}^2 + d_2\sum_i Q_{i\cdot}^2 + d_3\sum_j Q_{\cdot j}^2. \tag{4.4}$$

(4.4) gives a very simple expression for the computation of adjusted treatment sum of squares from the adjusted treatment totals. Using (2.8) in the solution (4.3) we can get the variances of the estimates of various elementary treatment comparisons. These are given in Table I.

Case (ii): Let $\mathbf{A}$ be a group divisible design with parameters $v_2 = mn, b_2, r_2, k_2; \lambda_{12}, \lambda_{22}$. If the jth treatment in the ith group is numbered as $(i - 1)n + j$ $(i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$ it is easy to verify that

$$\mathbf{AA}' = (\mathbf{A}_1 \diagdown \mathbf{A}_2)_m = \mathbf{I}(m) \otimes (\mathbf{A}_1 - \mathbf{A}_2) + \mathbf{E}(m) \otimes \mathbf{A}_2, \tag{4.5}$$

where

$$\begin{aligned} \mathbf{A}_1 &= (r_2 - \lambda_{12})\mathbf{I}(n) + \lambda_{12}\mathbf{E}(n), \\ \mathbf{A}_2 &= \lambda_{22}\mathbf{E}(n). \end{aligned}$$

TABLE I

No.	Treatment Comparison.	Number of Comparisons.	Variance of the estimate.
1.	$t_{ij} - t_{ij'}(j \neq j')$	$v_2 v_1(v_1 - 1)$	$2(d_1 + d_3)\sigma^2$
2.	$t_{ij} - t_{i'j}(i \neq i')$	$v_1 v_2(v_2 - 1)$	$2(d_1 + d_2)\sigma^2$
3.	$t_{ij} - t_{i'j'}(i \neq i', j \neq j')$	$v_1 v_2(v_1 - 1)(v_2 - 1)$	$2(d_1 + d_2 + d_3)\sigma^2$

Hence

$$\mathbf{AA}' = (r_2 - \lambda_{12})\mathbf{I}(m) \otimes \mathbf{I}(n) + (\lambda_{12} - \lambda_{22})\mathbf{I}(m) \otimes \mathbf{E}(n) + \lambda_{22}\mathbf{E}(m) \otimes \mathbf{E}(n). \tag{4.6}$$

Therefore, from Lemma 3.1,

$$\begin{aligned} \mathbf{NN}' = (rk - u_1)\mathbf{I}(m) \otimes \mathbf{I}(n) \otimes \mathbf{I}(v_1) + u_2\mathbf{I}(m) \otimes \mathbf{E}(n) \otimes \mathbf{I}(v_1) \\ + u_3\mathbf{E}(m) \otimes \mathbf{E}(n) \otimes \mathbf{I}(v_1) + u_4\mathbf{I}(m) \otimes \mathbf{I}(n) \otimes \mathbf{E}(v_1) \\ + u_5\mathbf{I}(m) \otimes \mathbf{E}(n) \otimes \mathbf{E}(v_1) + u_6\mathbf{E}(m) \otimes \mathbf{E}(n) \otimes \mathbf{E}(v_1), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} u_1 &= rk - c_1(r_2 - \lambda_{12}), \\ u_2 &= c_1(\lambda_{12} - \lambda_{22}), \\ u_3 &= c_3 + c_1\lambda_{22}, \\ u_4 &= c_2(r_2 - \lambda_{12}), \\ u_5 &= c_2(\lambda_{12} - \lambda_{22}), \\ u_6 &= c_4 + c_2\lambda_{22}. \end{aligned} \tag{4.8}$$

Hence

$$\begin{aligned} k\mathbf{C} = u_1\mathbf{I}(m) \otimes \mathbf{I}(n) \otimes \mathbf{I}(v_1) - u_2\mathbf{I}(m) \otimes \mathbf{E}(n) \otimes \mathbf{I}(v_1) \\ - u_3\mathbf{E}(m) \otimes \mathbf{E}(n) \otimes \mathbf{I}(v_1) - u_4\mathbf{I}(m) \otimes \mathbf{I}(n) \otimes \mathbf{E}(v_1) \\ - u_5\mathbf{I}(m) \otimes \mathbf{E}(n) \otimes \mathbf{E}(v_1) - u_6\mathbf{E}(m) \otimes \mathbf{E}(n) \otimes \mathbf{E}(v_1). \end{aligned} \tag{4.9}$$

Let the number triplet (ijq) denote the qth treatment in $\mathbf{N}$ which replaces the treatment numbered $(i - 1)n + j$ of $\mathbf{A}$ $(i = 1, 2, \cdots, m, j = 1, 2, \cdots, n, q = 1, 2, \cdots, v_1)$. Also, let t_{ijq} and Q_{ijq} denote the effect and the adjusted total for the treatment (ijq). In view of (4.9), it can be verified that the reduced normal equations may be written as

$$kQ_{ijq} = u_1\hat{t}_{ijq} - u_2\hat{t}_{i\cdot q} - u_3\hat{t}_{\cdot\cdot q} - u_4\hat{t}_{ij\cdot} - u_5\hat{t}_{i\cdot\cdot} - u_6\hat{t}_{\cdot\cdot\cdot}, \tag{4.10}$$

where $\hat{t}_{i\cdot q}$ etc., have their usual meanings. Taking the additional equation $\hat{t}_{\cdot\cdot\cdot} = 0$, we can solve (4.10) uniquely and get the solution as

$$\hat{t}_{ijq} = d_1Q_{ijq} + d_2Q_{i\cdot q} + d_3Q_{ij\cdot} + d_4Q_{i\cdot\cdot} + d_5Q_{\cdot\cdot q}, \tag{4.11}$$

where

$$d_1 = k/u_1, \quad d_2 = ku_2/[u_1(u_1 - nu_2)], \quad d_3 = ku_4/[u_1(u_1 - v_1u_4)],$$

$$d_4 = \{k/[u_1(u_1 - nu_2 - v_1u_1 - nv_1u_5)]\} \cdot\{u_2(u_4 + nu_5)/(u_1 - nu_2) + u_4(u_2 + v_1u_5)/(u_1 - v_1u_4)\},$$

$$d_5 = ku_3/[u_1(u_1 - nu_2)(u_1 - nu_2 - mnu_3)],$$

and $u_1, u_2, \cdots$, etc., are defined as in (4.8) and $Q_{i\cdot q} \cdots$, etc., have their usual meanings. Hence the adjusted treatment sum of squares for testing the overall differences between the treatment effects is

$$(4.12) \qquad d_1\sum_i \sum_j \sum_q Q_{ijq}^2 + d_2\sum_i \sum_q Q_{i\cdot q}^2 + d_3\sum_i \sum_j Q_{ij\cdot}^2 + d_4\sum_i Q_{i\cdot\cdot}^2 + d_5\sum_j Q_{\cdot\cdot q}^2 .$$

The variances of the estimates of various elementary treatment comparisons are given in Table II.

Case (iii): Let **A** be a Latin Square type of design with parameters $v_2 = n^2, b_2$, $r_2, k_2, \lambda_{12}, \lambda_{22}$. The association scheme of the design is determined by the rows and columns of the $n \times n$ array in which the treatments of the design are arranged. If the jth treatment in the ith row of this array is numbered as $(i - 1)n + j$, it is easy to verify that

$$(4.13) \qquad \mathbf{AA}' = (\mathbf{A}_1\diagdown\mathbf{A}_2)_n = \mathbf{I}(n) \otimes (\mathbf{A}_1 - \mathbf{A}_2) + \mathbf{E}(n) \otimes \mathbf{A}_2,$$

where

$$\mathbf{A}_1 = (r_2 - \lambda_{12})\mathbf{I}(n) + \lambda_{12}\mathbf{E}(n),$$

$$\mathbf{A}_2 = (\lambda_{12} - \lambda_{22})\mathbf{I}(n) + \lambda_{22}\mathbf{E}(n).$$

Hence

$$\mathbf{AA}' = \mathbf{I}(n) \otimes [(r_2 - 2\lambda_{12} + \lambda_{22})\mathbf{I}(n) + (\lambda_{12} - \lambda_{22})\mathbf{E}(n)] + \mathbf{E}(n) \otimes [(\lambda_{12} - \lambda_{22})\mathbf{I}(n) + \lambda_{22}\mathbf{E}(n)].$$

As in case (ii), after some simplification, it can be shown that the coefficient

TABLE II

No.	Treatment comparison.	Number of comparisons.	Variance of the estimate.
1.	$t_{ijq} - t_{ijq'}(q \neq q')$	$mnv_1(v_1 - 1)$	$2(d_1 + d_2 + d_5)\sigma^2$
2.	$t_{ijq} - t_{ij'q}(j \neq j')$	$mn(n - 1)v_1$	$2(d_1 + d_3)\sigma^2$
3.	$t_{ijq} - t_{ij'q'}(j \neq j', q \neq q')$	$mn(n - 1)v_1(v_1 - 1)$	$2(d_1 + d_2 + d_3 + d_5)\sigma^2$
4.	$t_{ijq} - t_{i'j'q}(i \neq i')$	$m(m - 1)n^2v_1$	$2(d_1 + d_2 + d_3 + d_4)\sigma^2$
5.	$t_{ijq} - t_{i'j'q'}(i \neq i', q \neq q')$	$m(m - 1)n^2v_1(v_1 - 1)$	$2(d_1 + d_2 + d_3 + d_4 + d_5)\sigma^2$

matrix **C** of the reduced normal equations is given by

$$\begin{aligned} kC = {} & u_1 I(n) \otimes I(n) \otimes I(n) - u_2[I(n) \otimes E(n) \otimes I(v_1) \\ & + E(n) \otimes I(n) \otimes I(v_1)] - u_3 E(n) \otimes E(n) \otimes I(v_1) \\ & - u_4 I(n) \otimes I(n) \otimes E(v_1) - u_5[I(n) \otimes E(n) \otimes E(v_1) \\ & + E(n) \otimes I(n) \otimes E(v_1)] - u_6 E(n) \otimes E(n) \otimes E(v_1), \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} u_1 &= rk - c_1(r_2 - 2\lambda_{12} + \lambda_{22}), \\ u_2 &= c_1(\lambda_{12} - \lambda_{22}), \\ u_3 &= c_3 + c_1\lambda_{22}, \\ u_4 &= c_2(r_2 - 2\lambda_{12} + \lambda_{22}), \\ u_5 &= c_2(\lambda_{12} - \lambda_{22}), \\ u_6 &= c_4 + c_2\lambda_{22}. \end{aligned} \tag{4.15}$$

Let the number triplet (ijq) denote the ith treatment, which replaces the treatment numbered $(i - 1)n + j$ of **A** in **N**. Also, let t_{ijq} and Q_{ijq} denote the effect and adjusted total of the treatment (ijq). Using (4.14), the reduced normal equations can be written as

$$kQ_{ijq} = u_1\hat{t}_{ijq} - u_2(\hat{t}_{i\cdot q} + \hat{t}_{\cdot jq}) - u_3\hat{t}_{\cdot\cdot q} - u_4\hat{t}_{ij\cdot} - u_5(\hat{t}_{i\cdot\cdot} + \hat{t}_{\cdot j\cdot}) - u_6\hat{t}_{\cdot\cdot\cdot}, \tag{4.16}$$

where $u_1, u_2, \cdots$, etc., are as in (4.15) and $\hat{t}_{i\cdot q}, \cdots$, etc., have their usual meanings. Taking the additional equations $\hat{t}_{\cdot\cdot\cdot} = 0$, we can solve the equations (4.16) uniquely and get the solution as

$$\hat{t}_{ijq} = d_1Q_{ijq} + d_2(Q_{i\cdot q} + Q_{\cdot jq}) + d_3Q_{ij\cdot} + d_4(Q_{i\cdot\cdot} + Q_{\cdot j\cdot}) + d_5Q_{\cdot\cdot q}, \tag{4.17}$$

where

$d_1 = k/u_1, \quad d_2 = ku_2/[u_1(u_1 - nu_2)], \quad d_3 = ku_4/[u_1(u_1 - v_1u_4)],$

TABLE III

No.	Treatment Comparison.	Number of comparisons.	Variance of the estimate.
1.	$t_{ijq} - t_{ijq'} (q \neq q')$	$n^2v_1(v_1 - 1)$	$2(d_1 + 2d_2 + d_5)\sigma^2$
2.	$t_{ijq} - t_{ij'q}$ or $t_{ijq} - t_{i'jq}$ $(i \neq i', j \neq j')$	$2n^2(n - 1)v_1$	$2(d_1 + d_2 + d_3 + d_4)\sigma^2$
3.	$t_{ijq} - t_{ij'q'}$ or $t_{ijq} - t_{i'jq'}$ $(i \neq i', j \neq j', q \neq q')$	$2n^2(n - 1)v_1(v_1 - 1)$	$2(d_1 + 2d_2 + d_3 + d_4 + 2d_5)\sigma^2$
4.	$t_{ijq} - t_{i'j'q} (i \neq i', j \neq j')$	$n^2(n - 1)^2v_1$	$2(d_1 + 2d_2 + d_3 + 2d_4)\sigma^2$
5.	$t_{ijq} - t_{i'j'q'} (i \neq i', j \neq j', q \neq q')$	$n(n - 1)v_1(v_1 - 1)$	$2(d_1 + 2d_2 + d_3 + 2d_4 + d_5)\sigma^2$

$$d_4 = \{k/(v_1 - nu_2 - v_1u_4 - nv_1u_5)\}\{[u_2(u_2 + nu_3)]/[u_1(u_1 - nu_3)] + [u_4(u_2 + v_1u_5)]/[u_1(u_1 - v_1u_4)] + u_5/u_1\},$$

$$d_5 = \{k/(u_1 - 2nu_2 - n^2u_3)\}\{[2u_2(u_2 + nu_3)]/[u_1(u_1 - nu_2)] + u_3/u_1\}.$$

Hence the adjusted treatment sum of squares is given by

$$d_1 \sum_i \sum_j \sum_q Q_{ijq}^2 + d_2(\sum_i \sum_q Q_{i\cdot q}^2 + \sum_j \sum_q Q_{\cdot jq}^2) + d_3 \sum_i \sum_j Q_{ij\cdot}^2 + d_4(\sum_i Q_{i\cdot\cdot}^2 + \sum_j Q_{\cdot j\cdot}^2) + d_5 \sum_q Q_{\cdot\cdot q}^2 . \tag{4.18}$$

The variances of the estimates of various elementary treatment comparisons are given in Table III.

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