ON THE QUEUEING PROCESS $M/G/1$

By C. R. Heathcote

Australian National University[1]

Introduction. The temporal development of the queueing process $M/G/1$ (Markov or Poisson input, general service time distribution, one server) has been studied by several authors using various methods. Takács [1] and others have centered their attention on distributions of waiting time; Gaver [2] utilized Kendall's method of the imbedded Markov Chain. More recently, Keilson and Kooharian [3] used a method first developed systematically by Cox [4]. This last approach consists of restating the original non-Markovian process as a Markov one by the inclusion of supplementary variables in the definition of states of the system. For the process $M/G/1$ only one supplementary variable is required, namely the elapsed service time of the customer currently in service.

The purpose of this note is to point out that both waiting time distributions and the distribution of queue length for the temporal process can be obtained in a simple way by the method of supplementary variables. We extend the results of Keilson and Kooharian [3] to find the temporal analogue of the classical Pollaczek-Khinchine formula [5], and from this obtain the distribution of waiting time of a customer arriving at time t. To avoid repetition, reference is made to [3] for a complete description of the problem, and we use the same notation, as follows.

Let the probability density of interarrival times be $\lambda e^{-\lambda t}$; $W_m(x, t)$, $m = 0, 1, 2, \cdots$, be the joint probability of m customers waiting (servee is excluded) at time t and the elapsed service time of the current customer x; $E(t)$ the null probability; and $D(x)$ the probability density of the service time. If $\eta(x)\delta x$ is the first order probability that a service completion occurs in the interval $(x, x + \delta x)$, conditional on a customer having reached the "age" x, then

$$D(x) = \eta(x) \exp\left\{-\int_0^x \eta(u)\,du\right\}.$$

Define the generating function

$$H(s, x, t) = \exp\left\{\int_0^x \eta(u)\,du\right\} \sum_{m=0}^{\infty} s^m W_m(x, t)$$

and assume that initially the system is empty,

$$E(0) = 1, \qquad W_m(x, 0) = 0. \tag{1}$$

Laplace transforms are denoted by lower case letters thus,

$$e(p) = \mathfrak{L}[E(t)] = \int_0^{\infty} e^{-pt} E(t)\,dt \qquad \operatorname{Re} p > 0.$$

Received August 18, 1960.

[1] Present address: Stanford University, Stanford, California.

Pollaczek-Khinchine Formula. It is shown in [3], equation (3.15), that

$$H(s, x, t) = \exp\{-\lambda(1 - s)x\}H_0(s, t - x)$$

and that for initial conditions (1) the following holds ([3], equation (3.21))

$$h_0(s, p) = \frac{(p + \lambda - \lambda s)e(p) - 1}{d(p + \lambda - \lambda s) - s}. \tag{2}$$

From these results we seek the generating function of the distribution of queue length,

$$\psi(s, t) = E(t) + \sum_{m=1}^{\infty} s^m P_m(t),$$

where $P_m(t) = \int_0^t W_{m-1}(x, t)\,dx$ is the probability of m customers irrespective of the elapsed service time x. Note that $x \leqq t$ because of the initial conditions chosen. The generating function

$$K(s, x, t) = E(t) + \sum_{m=0}^{\infty} s^{m+1} W_m(x, t)$$

is, from (2),

$$K(s, x, t) = E(t) + sH_0(s, t - x) \exp\left\{-\lambda(1 - s)x - \int_0^x \eta(u)\,du\right\}.$$

Using the relation

$$\exp\left\{-\int_0^x \eta(u)\,du\right\} = 1 - \int_0^x D(u)\,du$$

we find the Laplace transform of $\psi(s, t)$ as follows:

$$\begin{aligned} \psi^*(s, p) &= \int_0^\infty e^{-pt}\psi(s, t)\,dt = \int_0^\infty dt e^{-pt} \int_0^t K(s, x, t)\,dx \\ &= \frac{(p + \lambda - \lambda s)(1 - s)e(p) + s\{1 - d^{-1}(p + \lambda - \lambda s)\}}{(p + \lambda - \lambda s)\{1 - sd^{-1}(p + \lambda - \lambda s)\}}. \end{aligned} \tag{3}$$

The generating function of the equilibrium distribution, which exists for

$$1 > \lambda \int_0^\infty xD(x)\,dx,$$

is obtained from (3) by standard Tauberian arguments;

$$\lim_{p \to 0+} p\psi^*(s, p) = \lim_{t \to \infty} \psi(s, t) = \frac{(1 - s)E}{1 - sd^{-1}(\lambda - \lambda s)}. \tag{4}$$

E is the equilibrium null probability

$$E = \lim_{t \to \infty} E(t) = 1 - \lambda \int_0^\infty xD(x)\,dx.$$

Equation (4) is the Pollaczek-Khinchine formula for the distribution of queue length, and (3) is its temporal analogue for initial conditions (1).

The only unknown occurring in (3) is $e(p)$, the Laplace transform of the null probability. An explicit expression for $e(p)$, at least in principle, can be obtained by the following standard argument. Since $\psi^*(s, p)$ is a generating function it converges for at least $|s| \leqq 1$, so that within the unit circle zeros in s of numerator and denominator coincide. By Rouché's Theorem the only zero of the denominator within the unit circle is the smallest zero of the equation

$$d(p + \lambda - \lambda s) = s.$$

If this zero is $s_0 = \xi(p)$, then

$$e(p) = [p + \lambda - \lambda\xi(p)]^{-1}. \tag{5}$$

The Lagrange Inversion formulae ([6], page 132) can now be applied to obtain $e(p)$ and hence the null probability $\mathcal{L}^{-1}[e(p)]$. This procedure is often too complicated to carry out, but in many cases of practical importance, for example with χ^2 type service distributions, an explicit expression for $E(t)$ can be found.

Distribution of Waiting Times. Let $\eta(t)$ be the waiting time of a customer arriving at the instant t. If the system is empty at t then $\eta(t) = 0$. Otherwise $\eta(t)$ is the sum of the service times of those customers already waiting and the remaining service time of the current customer. Cox [4] previously used this argument to obtain the waiting time distribution in the equilibrium state. If

$$F(x, t) = \Pr\{\eta(t) \leqq x\}$$

and

$$\Phi(\alpha, t) = \int_0^\infty e^{-\alpha x}\, dF(t, x),$$

then, since the service times are distributed independently,

$$\begin{aligned} \Phi(\alpha, t) &= E(t) + \int_0^t du \sum_{m=0}^{\infty} \frac{W_m(u, t)[d(\alpha)]^m}{1 - \int_0^u D(y)\, dy} \int_0^\infty e^{-\alpha v} D(u + v)\, dv \\ &= E(t) + \int_0^t du H_0(d(\alpha), t - u) e^{\lambda\{d(\alpha)-1\}u} \int_0^\infty e^{-\alpha v} D(u + v)\, dv. \end{aligned} \tag{6}$$

Let $\theta(p, \alpha) = \int_0^\infty e^{-pt}\Phi(\alpha, t)\, dt$. Using elementary properties of the Laplace transform we find that

$$\theta(p, \alpha) = \{\alpha e(p) - 1\}\{\alpha - p - \lambda(1 - d(\alpha))\}^{-1}$$

and hence that

$$\Phi(\alpha, t) = e^{\alpha t - \lambda t[1-d(\alpha)]} \left\{1 - \alpha \int_0^t e^{-\alpha\tau + \lambda\tau[1-d(\alpha)]} E(\tau)\, d\tau\right]. \tag{7}$$

This result has been obtained previously by Takács ([1], Theorem 2) using a different method.

REFERENCES

[1] LAJOS TAKÁCS, "Investigation of waiting time problems by reduction to Markov processes," *Acta Math. Acad. Sci. Hung.*, Vol. 6 (1955), pp. 101–129.

[2] DONALD P. GAVER, JR., "Imbedded Markov Chain analysis of a waiting line process in continuous time," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 698–720.

[3] J. KEILSON AND A. KOOHARIAN, "On time dependent queueing processes," *Ann. Math. Stat.*, Vol. 31 (1960), pp. 104–112.

[4] D. R. COX, "The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables," *Proc. Camb. Phil. Soc.*, Vol. 51 (1955), pp. 433–441.

[5] D. G. KENDALL, "Some problems in the theory of queues," *J. Roy. Stat. Soc.*, Ser. B, Vol. 13 (1951), pp. 151–185.

[6] E. T. WHITTAKER AND G. N. WATSON, *Modern Analysis*, Cambridge University Press, 1946.