ON A COINCIDENCE PROBLEM CONCERNING PARTICLE COUNTERS

BY LAJOS TAKÁCS[1]

Columbia University

1. Introduction. A general model of particle counting will be considered. Suppose that particles arrive at a counting device at the instants $\tau_1, \tau_2, \cdots, \tau_n, \cdots$, where the inter-arrival times $\tau_n - \tau_{n-1}$ ($n = 1, 2, \cdots; \tau_0 = 0$) are identically distributed, independent, positive random variables with distribution function $\mathbf{P}\{\tau_n - \tau_{n-1} \leqq x\} = F(x)$, $n = 1, 2, \cdots$. Suppose that each particle, independently of the others, on its arrival gives rise to an impulse either with probability $p(0 < p \leqq 1)$ if at this instant there is at least one impulse present or with probability 1 if there is no impulse present. Let $q = 1 - p$. Denote by χ_n the duration of the impulse (if any) starting at τ_n. It is supposed that $\{\chi_n\}$ is a sequence of identically distributed, independent, positive random variables with distribution function

$$H(x) = \begin{cases} 1 - e^{-\mu x} & \text{if} \quad x \geqq 0, \\ 0 & \text{if} \quad x < 0, \end{cases} \tag{1}$$

and independent of $\{\tau_n\}$ and the events of realizations of the impulses.

Denote by $\eta(t)$ the number of impulses present at the instant t. Always $\eta(0) = 0$. We shall say that the system is in state E_k, $k = 0, 1, 2, \cdots$, at the instant t if $\eta(t) = k$. Write $\mathbf{P}\{\eta(t) = k\} = P_k(t)$. Furthermore, denote by $\nu_t^{(k)}$ the number of transitions $E_k \to E_{k+1}$ ($k + 1$ -fold coincidences, $k = 0, 1, 2, \cdots$) occurring in the time interval $(0, t]$. Write $\mathbf{E}\{\nu_t^{(k)}\} = M_k(t)$.

The stochastic behavior of the process $\{\eta(t); 0 \leqq t < \infty\}$ is characterized by two parameters, p and μ, and the distribution function $F(x)$. Throughout this paper μ will always be fixed and only p and $F(x)$ will vary. For the sake of brevity we shall say that the process $\{\eta(t); 0 \leqq t < \infty\}$ is of type $[F(x), p]$.

In what follows we shall give a method to determine the distributions of the random variables $\eta(t)$ and $\nu_t^{(k)}$ for finite t and the corresponding asymptotic distributions as $t \to \infty$. The above mentioned problems for process of type $[F(x), 1]$ were solved earlier by the author [13], [14]. The present model of particle counting in the particular case of Poisson input was introduced by G. E. Albert and L. Nelson [1] and generalizations have been given by the author [10], [12], R. Pyke [7], and W. L. Smith [9].

2. The structure of the process, $\{\eta(t)\}$. The stochastic behavior of the process of type $[F(x), 1]$ is already known [14]. Now we shall show that the investigation of the process of type $[F(x), p]$ can be reduced to that of the process of type

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$[F(x), 1]$. For this purpose let us associate a new process with the process of type $[F(x), p]$ by supposing that *each* particle independently of the others gives rise to an impulse with probability p, but otherwise every assumption remains unchanged. This new process can clearly be considered as a process of type $[\hat{F}(x), 1]$, where

$$\hat{F}(x) = p \sum_{n=1}^{\infty} q^{n-1} F_n(x) \tag{2}$$

and $F_n(x)$ denotes the nth iterated convolution of the distribution function $F(x)$ with itself. It is easy to see that the only difference between the processes of type $[F(x), p]$ and $[\hat{F}(x), 1]$ is that the latter contains an additional interval spent in state E_0 immediately before every transition $E_0 \to E_1$, where the lengths of these intervals are identically distributed, independent random variables with distribution function

$$Q(x) = p \sum_{n=0}^{\infty} q^n F_n(x) \tag{3}$$

and these random variables are independent of any other random variables in question. Here $F_0(x) = 1$ if $x \geqq 0$ and $F_0(x) = 0$ if $x < 0$. Thus, knowing the stochastic behavior of the process of type $[\hat{F}(x), 1]$ we can determine that of the process of type $[F(x), p]$.

It is to be remarked that the process of type $[F(x), p]$ is Markovian only in particular cases (e.g., $F(x) = 1 - e^{-\lambda x}$ for $x \geqq 0$; $F(x) = \sum_{j=0}^{[x]} (1 - \rho)\rho^j$ where $0 < \rho < 1$; $F(x) = 1$ if $x \geqq \alpha$ and $F(x) = 0$ if $x < \alpha$), but the instants τ_n, $n = 1, 2, \cdots$, always form the regeneration points of the process. Accordingly for fixed k, $k = 0, 1, 2, \cdots$, the instants of the successive transitions $E_k \to E_{k+1}$ form a recurrent (or renewal) process, i.e., the time differences between successive transitions $E_k \to E_{k+1}$ are identically distributed, independent, positive random variables. Let us denote by $R_k(x)$ their common distribution function. Furthermore it is clear that the time differences between successive transitions $E_{k-1} \to E_k$ and $E_k \to E_{k+1}$, $k = 0, 1, 2, \cdots$, are also independent random variables. Denote by $G_k(x)$, $k = 0, 1, 2, \cdots$, their distribution function. (We say that a transition $E_{-1} \to E_0$ takes place at time $t = 0$.)

3. Notation. We mention in advance that for the process of type $[\hat{F}(x), 1]$ we shall use the same symbols as for the process of type $[F(x), p]$ but with the circumflex added.

Throughout this paper we shall use the following symbols:

$$\alpha = \int_0^\infty x \, dF(x),$$

$$\sigma^2 = \int_0^\infty (x - \alpha)^2 \, dF(x).$$

$$\phi(s) = \int_0^\infty e^{-sx} \, dF(x), \qquad \Re(s) \geqq 0,$$

$$\gamma_k(s) = \int_0^\infty e^{-sx}\, dG_k(x), \qquad \Re(s) \geqq 0,$$

$$\psi_k(s) = \int_0^\infty e^{-sx}\, dR_k(x), \qquad \Re(s) \geqq 0,$$

$$\mu_k(s) = \int_0^\infty e^{-st}\, dM_k(t), \qquad \Re(s) > 0,$$

$$\pi_k(s) = \int_0^\infty e^{-st} P_k(t)\, dt, \qquad \Re(s) > 0.$$

Furthermore

$$C_r = \prod_{j=1}^{r} \left(\frac{\phi_j}{1 - \phi_j} \right)$$

where $\phi_j = \phi(j\mu)$, $j = 0, 1, 2, \cdots$, and $C_0 = 1$.

Finally, we introduce a new random variable $\eta_n = \eta(\tau_n - 0)$ which is equal to the number of the impulses present at the arrival of the nth particle.

4. The determination of the distribution of $\eta(t)$. First we shall prove the following

LEMMA 1. (R. Pyke). *If $M_0(t)$ is the expectation of the number of transitions $E_0 \to E_1$ occurring in the time interval $(0,t]$ for the process of type $[F(x), p]$, then*

$$\mu_0(s) = \int_0^\infty e^{-st}\, dM_0(t) = \frac{\phi(s)}{1 - \psi_0(s)} \tag{4}$$

where

$$\psi_0(s) = \frac{1 - q\phi(s)}{p} - \frac{\phi(s)}{p} \left[\sum_{r=0}^{\infty} (-p)^r \prod_{i=0}^{r} \left(\frac{\phi(s + i\mu)}{1 - \phi(s + i\mu)} \right) \right]^{-1}. \tag{5}$$

PROOF. This lemma in two particular cases, when either $p = 1$ or $F(x) = 1 - e^{-\lambda x}$ if $x \geqq 0$, has been proved earlier by the author [10], [11], [12]. A proof for the general case has been given by R. Pyke [7]. Now we shall give another proof.

By using renewal theory we obtain

$$M_0(t) = G_0(t) + G_0(t) * R_0(t) + G_0(t) * R_0(t) * R_0(t) + \cdots \tag{6}$$

and here $G_0(x) = F(x)$. Forming the Laplace-Stieltjes transform of (6), we get (4). It remains only to determine $\psi_0(s)$. For this purpose consider the associated process $[\hat{F}(x), 1]$. Then we have

$$\int_0^\infty e^{-st}\, d\hat{M}_0(t) = \frac{\hat{\phi}(s)}{1 - \hat{\psi}_0(s)}, \tag{7}$$

where, by (2),

$$\hat{\phi}(s) = \frac{\phi(s)}{1 - q\phi(s)} \tag{8}$$

and, in this case,

$$\hat{\mu}_0(s) = \int_0^\infty e^{-st}\, d\hat{M}_0(t) = \sum_{r=0}^{\infty} (-1)^r \prod_{i=0}^{r} \left(\frac{p\phi(s+i\mu)}{1-\phi(s+i\mu)}\right). \tag{9}$$

Formula (9) follows by [14] where we showed that, if $M(t)$ denotes the expectation of the number of transitions $E_0 \rightarrow E_1$ occurring in the time interval $(0, t]$ for a process of type $[F(x), 1]$, then

$$\int_0^\infty e^{-st}\, dM(t) = \sum_{r=0}^{\infty} (-1)^r \prod_{i=0}^{r} \left(\frac{\phi(s+i\mu)}{1-\phi(s+i\mu)}\right). \tag{10}$$

If we replace $\phi(s)$ by $\hat{\phi}(s)$ in (10), we obtain (9). Comparing (7), (8) and (9) we obtain

$$\hat{\psi}_0(s) = 1 - \frac{p\phi(s)}{[1-q\phi(s)]}\left[\sum_{r=0}^{\infty} (-1)^r \prod_{i=0}^{r} \left(\frac{p\phi(s+i\mu)}{1-\phi(s+i\mu)}\right)\right]^{-1}. \tag{11}$$

On the other hand taking into consideration what we mentioned in Section 2 we have

$$\hat{R}_0(x) = Q(x) * R_0(x),$$

where $Q(x)$ is defined by (3). Thus

$$\hat{\psi}_0(s) = \frac{p}{1-q\phi(s)}\psi_0(s). \tag{12}$$

By (11) and (12) we get $\psi_0(s)$. This completes the proof of the lemma.

REMARK 1. The proof of (10) is simple. For a process of type $[F(x), 1]$ we have

$$\mathbf{E}\{\nu_t^{(0)} \mid \tau_1 = y, \chi_1 = z\} = \begin{cases} 1 + [M(t-y) - M(z)] & \text{if } y + z \leqq t, \\ 1 & \text{if } y \leqq t < y + z, \\ 0 & \text{if } y > t \end{cases}$$

and by the theorem of total expectation we get

$$\begin{aligned} M(t) = F(t) + \int_0^t M(t-y)[1 - e^{-\mu(t-y)}]\, dF(y) \\ - \int_0^t M(z)F(t-z)e^{-\mu z}\mu\, dz. \end{aligned} \tag{13}$$

Forming the Laplace-Stieltjes transform of (13) with

$$\mu(s) = \int_0^\infty e^{-st}\, dM(t),$$

we get the functional equation

$$\mu(s) = \frac{\phi(s)}{1-\phi(s)}[1 - \mu(s+\mu)],$$

whose solution is

$$\mu(s) = \sum_{r=1}^{\infty} (-1)^r \prod_{i=0}^{r} \left(\frac{\phi(s + i\mu)}{1 - \phi(s + i\mu)} \right). \tag{14}$$

Now we shall prove

THEOREM 1. *The distribution $\{P_k(t)\}$ is determined uniquely by the following Laplace transforms*

$$\int_0^\infty e^{-st} P_k(t)\, dt = \frac{\sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \frac{p^r}{s + r\mu} \prod_{i=0}^{r-1} \left(\frac{\phi(s + i\mu)}{1 - \phi(s + i\mu)} \right)}{1 - q \sum_{r=0}^{\infty} (-p)^r \prod_{i=1}^{r} \left(\frac{\phi(s + i\mu)}{1 - \phi(s + i\mu)} \right)} \tag{15}$$

if $k = 1, 2, \cdots$, and

$$\int_0^\infty e^{-st} P_0(t)\, dt = \frac{1}{s} - \frac{\sum_{r=1}^{\infty} (-1)^{r-1} \frac{p^r}{s + r\mu} \prod_{i=0}^{r-1} \left(\frac{\phi(s + i\mu)}{1 - \phi(s + i\mu)} \right)}{1 - q \sum_{r=0}^{\infty} (-p)^r \prod_{i=1}^{r} \left(\frac{\phi(s + i\mu)}{1 - \phi(s + i\mu)} \right)} \tag{16}$$

where the empty product means 1.

PROOF. Consider the process of type $[F(x), p]$ and denote by $C_k(t)$, $k = 1, 2, \cdots$, the probability that the system is in state E_k after a time t measured from a point of transition $E_0 \to E_1$ and during this time interval of length t there are no other transitions $E_0 \to E_1$. Clearly this probability is the same for the process of type $[\hat{F}(x), 1]$. Thus by the theorem of total probability we obtain

$$P_k(t) = \int_0^t C_k(t - u)\, dM_0(u), \qquad k = 1, 2, \cdots, \tag{17}$$

and similarly

$$\hat{P}_k(t) = \int_0^t C_k(t - u)\, d\hat{M}_0(u), \qquad k = 1, 2, \cdots, \tag{18}$$

if we take into consideration that the event that the system is in state E_k at the instant t can occur in several mutually exclusive ways: the last transition $E_0 \to E_1$ in the time interval $(0, t]$ is the 1st, 2nd, $\cdots$, nth, $\cdots$, and this transition takes place at the instant $u(0 \leqq u \leqq t)$.

Forming the Laplace-Stieltjes transforms of (17) and (18), we get

$$\pi_k(s) = \mu_0(s) \int_0^\infty e^{-st}\, dC_k(t) \tag{19}$$

and

$$\hat{\pi}_k(s) = \hat{\mu}_0(s) \int_0^\infty e^{-st}\, dC_k(t). \tag{20}$$

Comparing (19) and (20) we obtain

$$\pi_k(s) = \hat{\pi}_k(s)\,\frac{\mu_0(s)}{\hat{\mu}_0(s)}, \qquad k = 1, 2, \cdots, \tag{21}$$

and thus

$$\pi_0(s) = \hat{\pi}_0(s)\,\frac{\mu_0(s)}{\hat{\mu}_0(s)} + \frac{1}{s}\left(1 - \frac{\mu_0(s)}{\hat{\mu}_0(s)}\right) \tag{22}$$

also holds because

$$\sum_{k=0}^{\infty} \pi_k(s) = \sum_{k=0}^{\infty} \hat{\pi}_k(s) = 1/s.$$

In [14] we have determined $\pi_k(s)$ for the process of type $[F(x), 1]$. If we replace $\phi(s)$ by $\hat{\phi}(s)$ there, then we obtain $\hat{\pi}_k(s)$, namely

$$\hat{\pi}_k(s) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \frac{p^r}{s + r\mu} \prod_{i=0}^{r-1} \left(\frac{\phi(s+i\mu)}{1-\phi(s+i\mu)}\right), \quad k = 0, 1, 2, \cdots. \tag{23}$$

On the other hand $\mu_0(s)$ is defined by (4) and (5) and $\hat{\mu}_0(s)$ by (9) and thus

$$\begin{aligned} \frac{\mu_0(s)}{\hat{\mu}_0(s)} &= \left[1 - \frac{q[1-\phi(s)]}{p\phi(s)}\,\hat{\mu}_0(s)\right]^{-1} \\ &= \left[1 - q \sum_{r=0}^{\infty} (-p)^r \prod_{i=1}^{r} \left(\frac{\phi(s+i\mu)}{1-\phi(s+i\mu)}\right)\right]^{-1}. \end{aligned} \tag{24}$$

The formulas (21), (22), (23) and (24) prove the theorem.

REMARK 2. Using a well known Tauberian theorem we get that

$$P_k^* = \lim_{t\to\infty} \frac{1}{t} \int_0^t P_k(u)\,du, \qquad k = 0, 1, 2, \cdots, \tag{25}$$

exists and

$$P_k^* = \lim\nolimits_{s\to 0} s\pi_k(s). \tag{26}$$

If $\alpha < \infty$ then $\{P_k^*\}$ is a probability distribution for which

$$P_0^* = 1 - \frac{p \sum_{r=0}^{\infty} (-p)^r \dfrac{C_r}{r+1}}{\alpha\mu\left[1 - q \sum_{r=0}^{\infty} (-p)^r C_r\right]} \tag{27}$$

and

$$P_k^* = \frac{\sum_{r=k}^{\infty} (-1)^{r-k} p^r \binom{r-1}{k-1} C_{r-1}}{k\alpha\mu\left[1 - q \sum_{r=0}^{\infty} (-p)^r C_r\right]}, \qquad k = 1, 2, \cdots. \tag{28}$$

We shall show later that if $\alpha < \infty$ and $F(x)$ is not a lattice distribution then $\lim_{t\to\infty} P_k(t)$ exists and then obviously $\lim_{t\to\infty} P_k(t) = P_k^*$, $k = 0, 1, 2, \cdots$.

5. The determination of the distribution of $\nu_t^{(k)}$. Knowing the distribution functions $G_0(x), G_1(x), \cdots, G_k(x)$ and $R_k(x)$, the distribution of $\nu_t^{(k)}$ can be determined easily. We have

$$\mathbf{P}\{\nu_t^{(k)} > n\} = G_0(t) * G_1(t) * \cdots * G_k(t) * R_k(t) * \cdots * R_k(t) \tag{29}$$

where the right hand side contains the nth iterated convolution of $R_k(t)$.

Define

$$\rho_k = \int_0^\infty x\, dR_k(x) \tag{30}$$

and

$$\sigma_k^2 = \int_0^\infty (x - \rho_k)^2\, dR_k(x). \tag{31}$$

If $\sigma_k^2 < \infty$, then we have

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\nu_t^{(k)} - \dfrac{t}{\rho_k}}{(\sigma_k^2\, t/\rho_k^3)^{\frac{1}{2}}} \leqq x\right\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-\frac{1}{2}y^2}\, dy \tag{32}$$

as is well known in renewal theory. (Cf., W. Feller [4], W. L. Smith [8], and the author [11].)

Thus the problem is reduced to the determination of the distribution functions $G_0(x), G_1(x), \cdots, G_k(x)$ and $R_k(x)$. We shall prove

THEOREM 2. *We have*

$$\gamma_r(s) = \int_0^\infty e^{-sx}\, dG_r(x) = \frac{D_r(s)}{D_{r+1}(s)}, \qquad r = 0, 1, 2, \cdots, \tag{33}$$

and

$$\begin{aligned}\psi_k(s) &= \int_0^\infty e^{-sx}\, dR_k(x)\\ &= 1 - \frac{\left\{1 - q \sum_{r=0}^\infty (-p)^r \prod_{i=1}^r \left(\dfrac{\phi(s+i\mu)}{1-\phi(s+i\mu)}\right)\right\}}{D_{k+1}(s) \left\{\sum_{r=k}^\infty (-1)^{r-k} \binom{r}{k} \prod_{i=0}^r \left(\dfrac{p\phi(s+i\mu)}{1-\phi(s+i\mu)}\right)\right\}},\end{aligned} \tag{34}$$

where $D_0(s) = 1$ *and*

$$\begin{aligned}D_r(s) = \Bigg\{&p \sum_{j=0}^r \binom{r}{j} \prod_{i=0}^{j-1} \left(\frac{1-\phi(s+i\mu)}{p\phi(s+i\mu)}\right)\\ &- \frac{q[1-\phi(s)]}{p\phi(s)} \sum_{j=0}^r \binom{r}{j} \sum_{l=1}^{j-1} (-1)^l \prod_{i=l+1}^{j-1} \left(\frac{1-\phi(s+i\mu)}{p\phi(s+i\mu)}\right)\Bigg\}\end{aligned} \tag{35}$$

if $r = 1, 2, \cdots$.

We shall prove Theorem 2 in two parts. First we shall determine $D_r(s)$, $r = 0, 1, 2, \cdots$, and then $\psi_r(s)$, $r = 0, 1, 2, \cdots$. But we should like to remark here that the mean ρ_k and the variance σ_k^2 of $R_k(x)$ can be calculated by (34) if we take into consideration that

$$\psi_k(s) = 1 - \rho_k s + \frac{\sigma_k^2 + \rho_k^2}{2} s^2 + o(s^2) \tag{36}$$

as $s \to 0$. Since

$$D_{k+1}(s) = p + s\alpha \sum_{j=1}^{k+1} \binom{k+1}{j} \frac{C_{j-1}}{p^{j-i}} - s\alpha \frac{q}{p} \sum_{j=0}^{k+1} \binom{k+1}{j} \sum_{l=1}^{j-1} (-1)^l \frac{C_{j-1}}{C_l} + o(s),$$

$$\sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \prod_{i=0}^{r} \left(\frac{p\phi(s+i\mu)}{1-\phi(s+i\mu)} \right) = \frac{p}{s} \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} p^r C_r + p \frac{\sigma^2 - \alpha^2}{2\alpha^2}$$

$$\sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} p^r C_r + \frac{p}{\alpha} \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} p^r C_r \sum_{i=1}^{r} \frac{\phi'(i\mu)}{\phi(i\mu)[1-\phi(i\mu)]} + o(s),$$

and

$$1 - q \sum_{r=0}^{\infty} (-p)^r \sum_{i=1}^{r} \left(\frac{\phi(s+i\mu)}{1-\phi(s+i\mu)} \right) = 1 - q \sum_{r=0}^{\infty} (-p)^r C_r$$

$$= qs \sum_{r=0}^{\infty} (-p)^r C_r \sum_{i=1}^{r} \frac{\phi'(i\mu)}{\phi(i\mu)[1-\phi(i\mu)]} + o(s)$$

as $s \to 0$, therefore

$$\rho_k = \frac{\alpha \left[1 - q \sum_{r=0}^{\infty} (-p)^r C_r \right]}{p \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} p^r C_r} \tag{37}$$

and

$$\begin{aligned} \sigma_k^2 = 2\rho_k \Bigg\{ & \frac{\alpha}{p} \sum_{j=1}^{k+1} \binom{k+1}{j} \frac{C_{j-1}}{p^{j-1}} - \frac{\alpha q}{p^2} \sum_{j=0}^{k+1} \binom{k+1}{j} \sum_{l=1}^{j-1} (-1)^l \frac{C_{j-1}}{C_l} \\ & - \frac{q \sum_{r=0}^{\infty} (-p)^r C_r \sum_{i=1}^{r} \frac{\phi'(i\mu)}{\phi(i\mu)[1-\phi(i\mu)]}}{1 - q \sum_{r=0}^{\infty} (-p)^r C_r} \Bigg\} \\ & - \rho_k^2 \Bigg\{ 1 - \frac{p^2 \frac{(\sigma^2 - \alpha^2)}{\alpha^2} \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} p^r C_r}{1 - q \sum_{r=0}^{\infty} (-p)^r C_r} \\ & + \frac{\frac{2p^2}{\alpha} \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} p^r C_r \sum_{i=1}^{r} \frac{\phi'(i\mu)}{\phi(i\mu)[1-\phi(i\mu)]}}{1 - q \sum_{r=0}^{\infty} (-p)^r C_r} \Bigg\}. \end{aligned} \tag{38}$$

6. The determination of $D_r(s)$. In this section we shall suppose more generally than formerly that each particle independently of the others on its arrival gives rise to an impulse with probability p_r if r impulses are present. Write $q_r = 1 - p_r$. The process of type $[F(x), p]$ corresponds to the particular case when $p_0 = 1$ and $p_r = p, r = 1, 2, \cdots$.

As before denote by $G_k(x)$, $k = 0, 1, 2, \cdots$, the distribution function of the distance between two consecutive transitions $E_{k-1} \to E_k$ and $E_k \to E_{k+1}$. (We say that a transition $E_{-1} \to E_0$ takes place at time $t = 0$.) Define

$$\gamma_k(s) = \int_0^\infty e^{-sx}\, dG_k(x) = \frac{D_k(s)}{D_{k+1}(s)} \tag{39}$$

where $D_0(s) = 1$. Thus we must determine $D_r(s)$, $r = 1, 2, \cdots$.

We note that if we write $D_r(s)$ in the following form

$$D_r(s) = \sum_{j=0}^{r} \binom{r}{j} \Delta^j D_0(s) \tag{40}$$

where $\Delta^j D_0(s)$ is the jth difference of $D_r(s)$ at $r = 0$, i.e.,

$$\Delta^j D_0(s) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} D_i(s) \tag{41}$$

then $D_r(s)$ is uniquely determined by its differences.

Now we shall prove

THEOREM 3. *Starting from $D_0(s) = \Delta^0 D_0(s) = 1$, the functions $D_r(s)$, $r = 0, 1, 2, \cdots$, and the differences $\Delta^j D_0(s)$, $j = 0, 1, 2, \cdots$, can be obtained successively by the recurrence formulas*

$$\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} D_j(s) = \phi(s + j\mu) \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} [p_j\, D_{j+1}(s) + q_j\, D_j(s)] \tag{42}$$

and

$$\Delta^j D_0(s) = \frac{\phi(s + j\mu)}{1 - \phi(s + j\mu)} \sum_{i=0}^{j} \binom{j}{i} c_{ji} \Delta^{i+1} D_0(s) \tag{43}$$

respectively. Here

$$c_{ji} = \sum_{\nu=0}^{j-i} (-1)^{\nu} \binom{j-i}{\nu} p_{j-\nu}. \tag{44}$$

PROOF. By the theorem of total probability we can write that

$$G_r(x) = \int_0^x \sum_{j=0}^{r} \binom{r}{j} e^{-j\mu y}(1 - e^{-\mu y})^{r-j} [p_j\, G_{j+1}(x - y) * \cdots * G_r(x - y) + q_j\, G_j(x - y) * \cdots * G_r(x - y)]\, dF(y), \tag{45}$$

if $r = 0, 1, 2, \cdots$, where the empty convolution is taken to be 1. To prove (45) let us consider the instant of a transition $E_{r-1} \to E_r$, and measure time from this instant. Then $G_r(x)$ is the probability that the next transition $E_r \to E_{r+1}$ occurs in the time interval $(0, x]$. This event may occur in the following mutually exclusive ways: the first particle in the time interval $(0, x]$ arrives at the instant $y(0 < y \leqq x)$ and it finds state E_j, $j = 0, 1, \cdots, r$, the probability of which is

$$\binom{r}{j} e^{-j\mu y}(1 - e^{-\mu y})^{r-j}$$

further in the time interval $(y, x]$ a transition $E_r \to E_{r+1}$ occurs, the probability of which is

$$p_j G_{j+1}(x - y) * \cdots * G_r(x - y) + q_j G_j(x - y) * \cdots * G_r(x - y).$$

Introduce the notation

$$q_{r,j}(s) = \binom{r}{j} \int_0^\infty e^{-sx} e^{-j\mu x}(1 - e^{-\mu x})^{r-j}\, dF(x) \tag{46}$$

and form the Laplace-Stieltjes transform of (45); then

$$\gamma_r(s) = \sum_{j=0}^{r} q_{r,j}(s) \left[p_j \prod_{i=j+1}^{r} \gamma_i(s) + q_j \prod_{i=j}^{r} \gamma_i(s) \right] \tag{47}$$

$(r = 0, 1, 2, \cdots)$ where the empty product is 1. Now using (39) we find

$$D_r(s) = \sum_{i=0}^{r} q_{r,i}(s)[p_i D_{i+1}(s) + q_i D_i(s)], \tag{48}$$

$r = 0, 1, 2, \cdots$. This is already a recurrence formula for the determination of $D_r(s)$, $r = 0, 1, 2, \cdots$, but the coefficients can be simplified further.

If we form

$$\Delta^j D_0(s) = \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} D_l(s) \tag{49}$$

where $D_l(s)$ is replaced by (48) and take into consideration that

$$\sum_{l=i}^{j} (-1)^{j-l} \binom{j}{l} q_{l,i}(s) = (-1)^{j-i} \binom{j}{i} \phi(s + j\mu), \tag{50}$$

then we obtain

$$\Delta^j D_0(s) = \phi(s + j\mu) \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} [p_i D_{i+1}(s) + q_i D_i(s)]. \tag{51}$$

Now comparing (49) and (51) we obtain (42).

On the other hand by (51) it follows

$$\Delta^j D_0(s) = \phi(s + j\mu)\Delta^j D_0(s) + \phi(s + j\mu) \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} p_i\, \Delta D_i(s)$$

whence

$$\Delta^j D_0(s) = \frac{\phi(s + j\mu)}{1 - \phi(s + j\mu)} \Delta^j [p_0\, \Delta D_0(s)],$$

where

$$\Delta^j [p_0 \, \Delta D_0(s)] = \sum_{i=0}^{j} \binom{j}{i} c_{ji} \, \Delta^{i+1} D_0(s)$$

and

$$c_{ji} = \Delta^{j-i} p_i = \sum_{\nu=0}^{j-i} (-1)^\nu \binom{j-i}{\nu} p_{j-\nu} \, .$$

This proves (43).

THE PROOF OF (35). In the case of a process of type $[F(x), p]$ we have $p_0 = 1$ and $p_r = p$, $r = 1, 2, \cdots$. In this particular case (43) reduces to the following difference equation

$$(52) \qquad \Delta^{j+1} D_0(s) - \frac{1 - \phi(s + j\mu)}{\phi(s + j\mu)} \Delta^j D_0(s) + (-1)^j \frac{q \, [1 - \phi(s)]}{p\phi(s)} = 0,$$

$j = 0, 1, 2, \cdots$. A simple calculation shows that the solution of the difference equation (52) is

$$(53) \qquad \Delta^j D_0(s) = \left\{ p \prod_{i=0}^{j-1} \left(\frac{1 - \phi(s + i\mu)}{\phi(s + i\mu)} \right) - \frac{q \, [1 - \phi(s)]}{p\phi(s)} \sum_{l=1}^{j-1} (-1)^l \prod_{i=l+1}^{j-1} \left(\frac{1 - \phi(s + i\mu)}{p\phi(s + i\mu)} \right) \right\}$$

(Cf., Ch. Jordan [6]) and finally

$$(54) \qquad D_r(s) = \sum_{j=0}^{r} \binom{r}{j} \Delta^j D_0(s)$$

which completes the proof of (35).

REMARK 3. If specifically we consider the process of type $[F(x), 1]$ when $p_r = 1$, $r = 0, 1, 2, \cdots$, then (35) has the following simple form

$$\Delta^{j+1} D_0(s) = \frac{1 - \phi(s + j\mu)}{\phi(s + j\mu)} \Delta^j D_0(s), \qquad j = 0, 1, 2, \cdots,$$

whence

$$\Delta^j D_0(s) = \prod_{i=0}^{j-1} \left(\frac{1 - \phi(s + i\mu)}{\phi(s + i\mu)} \right)$$

and

$$(55) \qquad D_r(s) = \sum_{j=0}^{r} \binom{r}{j} \prod_{i=0}^{j-1} \left(\frac{1 - \phi(s + i\mu)}{\phi(s + i\mu)} \right)$$

in agreement with our previous result [14].

7. The determination of $\psi_k(s)$. First we shall prove the following

THEOREM 4. *If $M_k(t)$ denotes the expectation of the number of transitions $E_k \to$*

E_{k+1} occurring in the time interval $(0, t]$ at the process of type $[F(x), p]$ then we have

$$(56)\qquad \mu_k(s) = \int_0^\infty e^{-st} dM_k(t) = \frac{\sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \prod_{i=0}^{r} \left(\frac{p\phi(s+i\mu)}{1-\phi(s+i\mu)} \right)}{1 - q \sum_{r=0}^{\infty} (-1)^r \prod_{i=1}^{r} \left(\frac{p\phi(s+i\mu)}{1-\phi(s+i\mu)} \right)}.$$

PROOF. Evidently the difference of the number of transitions $E_k \to E_{k+1}$ and $E_{k+1} \to E_k$ occurring in the time interval $(0, t]$ is 0 or 1 according to whether at the instant t the system is in one of the states $E_0, E_1, \cdots, E_k$ or in one of the states $E_{k+1}, E_{k+2}, \cdots$ respectively. Accordingly if we denote by $N_{k+1}(t)$ the expectation of the number of transitions $E_{k+1} \to E_k$ occurring in the time interval $(0, t]$ then we have

$$(57)\qquad M_k(t) - N_{k+1}(t) = \sum_{j=k+1}^{\infty} P_j(t), \qquad k = 0, 1, 2, \cdots.$$

On the other hand

$$N_{k+1}(t) = (k+1)\mu \int_0^t P_{k+1}(u)\, du.$$

For, if we consider the process $\{\eta(t)\}$ only at those instants when there is a state E_{k+1} then the transitions $E_{k+1} \to E_k$ form a Poisson process with density $(k+1)\mu$. Hence

$$(58)\qquad M_k(t) = (k+1)\mu \int_0^t P_{k+1}(u)\, du + \sum_{j=k+1}^{\infty} P_j(t).$$

Forming the Laplace-Stieltjes transform of (58) we obtain

$$(59)\qquad \mu_k(s) = (k+1)\mu\pi_{k+1}(s) + s \sum_{j=k+1}^{\infty} \pi_j(s), \qquad k = 0, 1, 2, \cdots.$$

Similarly if we consider the process of type $[\hat{F}(x), 1]$ then we have

$$(60)\qquad \hat{\mu}_k(s) = (k+1)\mu\hat{\pi}_{k+1}(s) + s \sum_{j=k+1}^{\infty} \hat{\pi}_j(s), \qquad k = 0, 1, 2, \cdots.$$

Now comparing (59) and (60) and using the relation (21) we get

$$(61)\qquad \mu_k(s) = \frac{\mu_0(s)}{\hat{\mu}_0(s)} \hat{\mu}_k(s), \qquad k = 0, 1, 2, \cdots.$$

In [14] we have showed that

$$(62)\qquad \hat{\mu}_k(s) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \prod_{i=0}^{r} \left(\frac{p\phi(s+i\mu)}{1-\phi(s+i\mu)} \right), \quad k = 0, 1, 2, \cdots,$$

and we have seen earlier that

$$(63)\qquad \frac{\mu_0(s)}{\hat{\mu}_0(s)} = \left[1 - q \sum_{r=0}^{\infty} (-1)^r \prod_{i=1}^{r} \left(\frac{p\phi(s+i\mu)}{1-\phi(s-i\mu)} \right) \right]^{-1}.$$

Thus (61), (62) and (63) prove (56).

REMARK 4. By a well known Tauberian theorem it follows that

$$\lim_{t\to\infty} \frac{M_k(t)}{t} = \lim_{s\to 0} s\mu_k(s) \tag{64}$$

and thus by (56) we obtain

$$\lim_{t\to\infty} \frac{M_k(t)}{t} = \frac{p \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} p^r C_r}{\alpha\left[1 - q \sum_{r=0}^{\infty} (-p)^r C_r\right]}. \tag{65}$$

This result can be obtained also by former results of this paper. Thus by (58) we obtain

$$\lim_{t\to\infty} \frac{M_k(t)}{t} = \lim_{t\to\infty} \frac{(k+1)\mu}{t} \int_0^t P_{k+1}(u)\, du = (k+1)\, \mu P_{k+1}^* \tag{66}$$

where P_k^*, $k = 1, 2, \cdots$, is defined by (28). Further we can conclude by renewal theory that

$$\lim_{t\to\infty} \frac{M_k(t)}{t} = \frac{1}{\rho_k} \tag{67}$$

where ρ_k is defined by (37). For, the time differences between consecutive transitions $E_k \to E_{k+1}$ are identically distributed, independent random variables with expectation ρ_k.

THE PROOF OF (34). By using renewal theory we have

$$M_k(t) = G_0(t) * G_1(t) * \cdots * G_k(t) * [I(t) + R_k(t) + R_k(t) * R_k(t) + \cdots] \tag{68}$$

where $I(t) = 1$ if $t \geqq 0$ and $I(t) = 0$ if $t < 0$. Forming the Laplace transform of (68) we obtain

$$\mu_k(s) = \frac{\gamma_0(s)\gamma_1(s) \cdots \gamma_k(s)}{1 - \psi_k(s)} = \frac{1}{D_{k+1}(s)[1 - \psi_k(s)]} \tag{69}$$

whence

$$\psi_k(s) = 1 - [D_{k+1}(s)\mu_k(s)]^{-1} \tag{70}$$

where $\mu_k(s)$ is defined by (56) and $D_{k+1}(s)$ by (35). This proves (34).

8. The limiting distribution of $\eta(t)$. We shall prove

THEOREM 5. *If $\alpha < \infty$ and $F(x)$ is not a lattice distribution then the limiting distribution* $\lim_{t\to\infty} P_k(t) = P_k^*$, $k = 0, 1, 2, \cdots$, *exists and is defined by* (27) *and* (28).

PROOF. At Remark 2 we showed that if $\alpha < \infty$ then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P_k(u)\, du = P_k^*, \qquad k = 0, 1, 2, \cdots. \tag{71}$$

Now if we show that $\lim_{t\to\infty} P_k(t)$ exists then by (71) we get that $\lim_{t\to\infty} P_k(t) = P_k^*$. To prove the existence we need the following auxiliary theorem: If $F(x)$ is not a lattice distribution then

$$\lim_{t\to\infty} \frac{M_k(t+h) - M_k(t)}{h} \tag{72}$$

exists for every $h > 0$ and is independent of h. This statement follows from a theorem of D. Blackwell [2], since if $F(x)$ is not a lattice distribution then the distribution of the distance between successive transitions $E_k \to E_{k+1}$ is also a non-lattice distribution. If (72) exists then it clearly agrees with (66), i.e., if $\alpha < \infty$ and $F(x)$ is not a lattice distribution then for every $h > 0$

$$\lim_{t\to\infty} \frac{M_k(t+h) - M_k(t)}{h} = (k+1)\mu P_{k+1}^*, \qquad k = 0, 1, 2, \cdots, \tag{73}$$

where P_k^*, $k = 1, 2, \cdots$, is defined by (28).

Now by the theorem of total probability we can write that

$$P_k(t) = \sum_{j=k}^{\infty} \int_0^t \binom{j}{k} e^{-k\mu(t-u)}(1 - e^{-\mu(t-u)})^{j-k}[1 - \hat{F}(t-u)]\, dM_{j-1}(u), \tag{74}$$

$k = 1, 2, \cdots$, where the distribution function $\hat{F}(x)$ is defined by (2). To prove (74) let us note that the event that the system is in state E_k at the instant t can occur in several mutually exclusive ways: the last transition in the time interval $(0, t]$ is $E_{j-1} \to E_j$, $j = k, k+1, \cdots$; this is the nth $(n = 1, 2, \cdots)$ among the transitions $E_{j-1} \to E_j$; this transition takes place at the instant $u (0 \leqq u \leqq t)$; and in the time interval $(u, t]$ no new impulses are starting, but $j - k$ impulses terminate.

The function

$$e^{-k\mu x}(1 - e^{-\mu x})^{j-k}[1 - \hat{F}(x)]$$

is of bounded variation in the interval $(0, \infty)$ and so it follows from (73) that the limit of (74) exists and we have

$$\lim_{t\to\infty} P_k(t) = \mu \sum_{j=k}^{\infty} P_j^* j \binom{j}{k} \int_0^\infty e^{-k\mu x}(1 - e^{-\mu x})^{j-k}[1 - \hat{F}(x)]\, dx, \tag{75}$$

$k = 1, 2, \cdots$. The limit may be formed term by term, the series being uniformly convergent. Finally, $\lim_{t\to\infty} P_0(t)$ also exists, because

$$P_0(t) = 1 - \sum_{k=1}^{\infty} P_k(t).$$

This completes the proof of the theorem.

9. The limiting distribution of η_n. We shall prove

THEOREM 6. *The limiting distribution* $\lim_{n\to\infty} \mathbf{P}\{\eta_n = k\} = P_k$, $k = 0, 1, 2, \cdots$, *always exists and*

$$P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r \tag{76}$$

where B_r is the rth *binomial moment of $\{P_k\}$. We have $B_0 = 1$ and*

$$B_r = \frac{p^r C_r}{1 - q \sum_{j=0}^{\infty} (-1)^j p^j C_j}, \qquad r = 1, 2, \cdots. \tag{77}$$

Specifically

$$P_0 = \frac{p \sum_{r=0}^{\infty} (-1)^r p^r C_r}{1 - q \sum_{r=0}^{\infty} (-1)^r p^r C_r}. \tag{78}$$

Proof. Define

$$\pi_{jk}(x) = p \binom{j+1}{k} e^{-k\mu x}(1 - e^{-\mu x})^{j+1-k} + q \binom{j}{k} e^{-k\mu x}(1 - e^{-\mu x})^{j-k}$$

if $j = 1, 2, 3, \cdots$, and

$$\pi_{00}(x) = 1 - e^{-\mu x}, \qquad \pi_{01}(x) = e^{-\mu x}, \qquad \pi_{0k}(x) = 0 \quad \text{if} \quad k > 1.$$

It is easy to see that the sequence of random variables $\{\eta_n\}$, $n = 1, 2, \cdots$, forms a Markov chain with transition probabilities $\mathbf{P}\{\eta_{n+1} = k \mid \eta_n = j\} = p_{jk}$ where

$$p_{jk} = \int_0^{\infty} \pi_{jk}(x)\, dF(x). \tag{79}$$

The Markov chain $\{\eta_n\}$ is evidently irreducible and aperiodic. By a theorem of F. G. Foster [5] we can prove that the states are also ergodic. Consequently the limiting distribution $\lim_{n\to\infty} \mathbf{P}\{\eta_n = k\} = P_k$, $k = 0, 1, 2, \cdots$, exists and is independent of the initial distribution. The limiting distribution $\{P_k\}$ is uniquely determined by the following system of linear equations

$$P_k = \sum_{j=k-1}^{\infty} p_{jk} P_j, \qquad k = 0, 1, 2, \cdots, \tag{80}$$

and

$$\sum_{k=0}^{\infty} P_k = 1 \tag{81}$$

(Cf., W. Feller [3]). In (80) $P_{-1} = 0$.

To solve this system of linear equations let us introduce the generating function

$$U(z) = \sum_{k=0}^{\infty} P_k z^k. \tag{82}$$

By (80) we obtain

$$\begin{aligned} U(z) = p \int_0^\infty (1 - e^{-\mu x} + ze^{-\mu x}) U(1 - e^{-\mu x} + ze^{-\mu x})\, dF(x) \\ - qP_0(1 - z)\phi_1 + q \int_0^\infty U(1 - e^{-\mu x} + ze^{-\mu x})\, dF(x). \end{aligned} \tag{83}$$

Now let us introduce the binomial moments

$$B_r = \sum_{k=r}^{\infty} \binom{k}{r} P_k, \qquad r = 0, 1, 2, \cdots, \tag{84}$$

of the distribution $\{P_k\}$. If B_r exists, then by (82) we have

$$B_r = \frac{1}{r!}\left(\frac{d^r U(z)}{dz^r}\right)_{z=1}, \qquad r = 1, 2, \cdots. \tag{85}$$

By (81), $B_0 = 1$. Forming the rth derivative of (83) at $z = 1$ we obtain

$$B_1 = p\phi_1(B_1 + B_0) + pP_0\phi_1 + qB_1\phi_1$$

if $r = 1$ and

$$B_r = p\phi_r(B_r + B_{r-1}) + q\phi_r B_r \tag{86}$$

if $r = 2, 3, \cdots$. Hence

$$B_r = p^r C_r(1 + (qP_0/p)), \qquad r = 1, 2, \cdots, \tag{87}$$

where P_0 is still to be determined. The probability distribution $\{P_k\}$ is uniquely determined by its binomial moments, namely by (82) and (85)

$$P_k = \frac{1}{k!}\left(\frac{d^k U(z)}{dz^k}\right)_{z=0} = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r. \tag{88}$$

Since

$$P_0 = \sum_{r=0}^{\infty} (-1)^r B_r = 1 + \left(1 + \frac{q}{p} P_0\right) \sum_{r=1}^{\infty} (-p)^r C_r, \tag{89}$$

consequently

$$P_0 = \frac{p \sum_{r=0}^{\infty} (-p)^r C_r}{1 - q \sum_{r=0}^{\infty} (-p)^r C_r} \tag{90}$$

and by (87)

$$B_r = \frac{p^r C_r}{1 - q \sum_{j=0}^{\infty} (-p)^j C_j} . \tag{91}$$

The theorem is proved by (88) and (91).

REMARK 5. By (67) and the theory of Markov chains we can conclude

$$\lim_{t\to\infty} \frac{M_k(t)}{t} = \begin{cases} P_0/\alpha & \text{if } k = 0, \\ pP_k/\alpha & \text{if } k = 1, 2, \cdots \end{cases} \tag{92}$$

Further we have seen earlier that

$$\lim_{t\to\infty} \frac{N_{k+1}(t)}{t} = (k+1)\mu P_{k+1}^*, \qquad k = 0, 1, 2, \cdots , \tag{93}$$

where P_k^* is defined by (28). Obviously $0 \leqq M_k(t) - N_{k+1}(t) \leqq 1$ for every $t \geqq 0$ and thus (92) and (93) agree. Accordingly a simple relationship exists between the distributions $\{P_k\}$ and $\{P_k^*\}$, namely

$$P_k^* = \frac{pP_{k-1}}{k\alpha\mu} \qquad \text{if} \quad k = 2, 3, \cdots ,$$

$$P_1^* = \frac{P_0}{\alpha\mu}$$

and

$$P_0^* = 1 - \sum_{k=1}^{\infty} P_k^* .$$

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