ESTIMATION OF THE SPECTRUM[1]

By V. K. Murthy

Stanford University and University of North Carolina

0. Summary. This paper extends some results of Grenander [1] relating to discrete real stationary normal processes with absolutely continuous spectrum to the case in which the spectrum also contains a step function with a finite number of saltuses.

It is shown by Grenander [1] that the periodogram is an asymptotically unbiased estimate of the spectral density $f(\lambda)$ and that its variance is $[f(\lambda)]^2$ or $2[f(\lambda)]^2$, according as $\lambda \neq 0$ or $\lambda = 0$. In the present paper the same results are established at a point of continuity.

The consistency of a suitably weighted periodogram for estimating $f(\lambda)$ is established by Grenander [1]. In this paper a weighted periodogram estimate similar to that of Grenander (except that the weight function is more restricted) is constructed which consistently estimates the spectral density at a point of continuity.

It appears that this extended result leads to a direct approach to the location of a single periodicity irrespective of the presence of others in the time series.

1. Introduction and preliminary lemmas. We shall now proceed to establish our results.

Let $x(n)$ be a discrete, real, stationary, normal process. It is known (Karhunen [2]) that the process can be decomposed into two mutually orthogonal stationary processes as $x(n) = x_1(n) + x_2(n)$, where $x_1(n)$ is a purely periodic process and $x_2(n)$ is a purely non-periodic process.

Let $[x(-N), x(-N+1), \cdots, x(-1), x(0), x(1), \cdots, x(N-1), x(N)]$ be a realization of size $2N+1$ from the process $x(n)$, and consider the statistic proposed by Grenander [1],

$$I_N(\lambda) = \frac{1}{2\pi(2N+1)} \left| \sum_{\nu=-N}^{N} x(\nu) e^{-i\nu\lambda} \right|^2. \tag{1.1}$$

This is the usual periodogram based on the realization. We have

$$\begin{aligned} I_N(\lambda) &= \frac{1}{2\pi(2N+1)} \left| \sum_{\nu=-N}^{N} x_1(\nu) e^{-i\nu\lambda} \right|^2 + \frac{1}{2\pi(2N+1)} \left| \sum_{\nu=-N}^{N} x_2(\nu) e^{-i\nu\lambda} \right|^2 \\ &\quad + \frac{1}{2\pi(2N+1)} \left(\sum_{\nu=-N}^{N} x_1(\nu) e^{-i\nu\lambda} \right) \left(\overline{\sum_{\nu=-N}^{N} x_2(\nu) e^{-i\nu\lambda}} \right) \\ &\quad + \frac{1}{2\pi(2N+1)} \left(\overline{\sum_{\nu=-N}^{N} x_1(\nu) e^{-i\nu\lambda}} \right) \left(\sum_{\nu=-N}^{N} x_2(\nu) e^{-i\nu\lambda} \right). \end{aligned} \tag{1.2}$$

Received July 25, 1960.

[1] This work was assisted in its final stages by the Office of Naval Research through the contract for research in probability and statistics at Chapel Hill, North Carolina, Project NR 042031 and National Science Foundation Grant 14336, Stanford University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

The two stationary parts, $x_1(n)$ and $x_2(n)$, have the spectral representations.

$$(1.3)\qquad \begin{cases} x_1(n) = \int_{-\pi}^{\pi} e^{in\lambda}\, dz_1(\lambda), \\ x_2(n) = \int_{-\pi}^{\pi} e^{in\lambda}\, dz_2(\lambda), \end{cases}$$

where $z_1(\lambda)$ and $z_2(\lambda)$ are orthogonal processes.

We shall use the following two lemmas.

LEMMA 1 (Karhunen [2]). *If $z(s)$ is an orthogonal process with the associated measure $\sigma(s)$ on the subsets s of the elements (λ) of W, and if $g_1(\lambda)$ and $g_2(\lambda)$ are complex valued functions of the real variable λ such that each of them is quadratically integrable on W with respect to the σ-measure, then we have*

$$(1.4)\qquad E\left[\int_W g_1(\lambda)\, dz(\lambda)\, \overline{\int_W g_2(\lambda)\, dz(\lambda)}\right] = \int_W g_1(\lambda)\overline{g_2(\lambda)}\, d\sigma(\lambda).$$

where $d\sigma(\lambda) = E\{dz(\lambda)\, \overline{dz(\lambda)}\}$.

LEMMA 2 (Grenander [1]). *For any discrete, real, stationary, normal process with absolutely continuous spectrum, it was shown that*

$$(1.5)\qquad E\,[I_N(\lambda)] = \frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2[(2N+1)(l-\lambda)/2]}{\sin^2[l-\lambda)/2]} f(l)\, dl.$$

$$(1.6)\qquad \begin{aligned} D^2[I_n(\lambda)] = &\left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2[(2N+1)(l-\lambda)/2]}{\sin^2[(l-\lambda)/2]} f(l)\, dl\right]^2 \\ &+ \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin[(2N+1)(l-\lambda)/2]}{\sin[(l-\lambda)/2]} \cdot \frac{\sin[(2N+1)(l+\lambda)/2]}{\sin[(l+\lambda)/2]} f(l)\, dl\right]^2, \end{aligned}$$

where $D^2[I_N(\lambda)]$ denotes the variance of $I_N(\lambda)$; also that

$$(1.7)\qquad \begin{aligned} \operatorname{cov}[I_N(\lambda), I_N(\mu)] &= R_N(\lambda, \mu) \\ &= \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin[(2N+1)(l-\lambda)/2]}{\sin[(l-\lambda)/2]} \cdot \frac{\sin[(2N+1)(l-\mu)/2]}{\sin[(l-\mu)/2]}\right] f(l)\, dl\Big]^2, \\ &+ \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin[(2N+1)(l-\lambda)/2]}{\sin[(l-\lambda)/2]} \frac{\sin[(2N+1)(l+\mu)/2}{\sin[(l+\mu)/2]} f(l) dl\right]^2, \end{aligned}$$

where $f(\lambda)$ is the spectral density.

2. Expectation and variance of $I_N(\lambda)$. Using Lemmas 1 and 2, it is easily seen that for our processes, i.e., for discrete, real, stationary, normal processes whose spectrum includes, besides the absolutely continuous part, a step part with a finite number of saltuses,

$$E[I_N(\lambda)] = \frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2[(2N+1)(l-\lambda)/2]}{\sin^2[(l-\lambda)/2]} d\sigma_1(l) + \frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2[(2N+1)(l-\lambda)/2]}{\sin^2[(l-\lambda)/2]} d\sigma_2(l), \tag{2.1}$$

$$D^2[I_N(\lambda)] = \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2[(2N+1)(l-\lambda)/2]}{\sin^2[(l-\lambda)/2]} d(\sigma_1(l)+\sigma_2(l))\right]^2 + \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin[(2N+1)(l-\lambda)/2]}{\sin[(l-\lambda)/2]} \cdot \frac{\sin[(2N+1)(l+\lambda)/2]}{\sin[(l+\lambda)/2]} d(\sigma_1(l)+\sigma_2(l))\right]^2, \tag{2.2}$$

$$\operatorname{cov}[I_N(\lambda), I_N(\mu)] = R_N(\lambda, \mu) = R_N^{(1)}(\lambda, \mu) + R_N^{(2)}(\lambda, \mu), \tag{2.3}$$

where

$$R_N^{(1)}(\lambda, \mu) = \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin[(2N+1)(l-\lambda)/2]}{\sin[(l-\lambda)/2]} \cdot \frac{\sin[(2N+1)(l-\mu)/2]}{\sin[(l-\mu)/2]} d(\sigma_1(l)+\sigma_2(l))\right]^2, \tag{2.4}$$

$$R_N^{(2)}(\lambda, \mu) = \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin[(2N+1)(l-\lambda)/2]}{\sin[(l-\lambda)/2]} \cdot \frac{\sin[(2N+1)(l+\mu)/2]}{\sin[(l+\mu)/2]} d(\sigma_1(l)+\sigma_2(l))\right]^2. \tag{2.5}$$

From the nature of the two parts $x_1(n)$ and $x_2(n)$ of the process $x(n)$, their spectra $\sigma_1(\lambda)$ and $\sigma_2(\lambda)$ are respectively a pure step function and an absolutely continuous bounded measure function. Also it is evident that the spectrum $\sigma(\lambda)$ of the process $x(n)$ is the sum of $\sigma_1(\lambda)$ and $\sigma_2(\lambda)$ the spectra of the two parts.

3. Asymptotic unbiassedness and inconsistency of $I_N(\lambda)$. We shall now prove

THEOREM 1. *For any real, discrete, stationary, normal process whose spectrum consists of an absolutely continuous part and a step function with a finite number of saltuses, $I_N(\lambda)$ is an asymptotically unbiased estimate of $f(\lambda)$ at every point of continuity of $\sigma(\lambda)$.*

PROOF. Let $S_1, S_2, \cdots, S_p$ be the steps of $\sigma_1(\lambda)$ corresponding to the values $\lambda_1, \lambda_2, \cdots, \lambda_p$ of λ in $(-\pi, \pi)$. We have from (2.1)

$$E[I_N(\lambda)] = \frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2[(2N+1)(l-\lambda)/2]}{\sin^2[(l-\lambda)/2]} d\sigma_1(l) + \frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2[(2N+1)(l-\lambda)/2]}{\sin^2[(l-\lambda)/2]} f(l)\, dl, \tag{3.1}$$

where

$$d\sigma_2(l) = f(l)\, dl. \tag{3.2}$$

The first term on the right-hand side (R.H.S.) of (3.1) can be written as

$$\frac{1}{2\pi(2N+1)} \sum_{k=1}^{p} S_k \frac{\sin^2[(2N+1)(\lambda_k - \lambda)/2]}{\sin^2[(\lambda_k - \lambda)/2]}. \tag{3.3}$$

If λ is a point of continuity of the spectrum $\sigma(\lambda)$, it does not coincide with any one of λ_k, $k = 1, 2, \cdots, p$, and hence all the p terms in the above expression are finite. As $N \to \infty$ the above expression tends to zero. By Fejér's theorem the second term on the R.H.S. of (3.1) tends to $f(\lambda)$ as $N \to \infty$. We have thus established that

$$\lim_{N\to\infty} E[I_N(\lambda)] = f(\lambda), \tag{3.4}$$

at a point of continuity.

THEOREM 2. *For any discrete, real, stationary, normal process whose spectrum consists of an absolutely continuous part and a step function with a finite number of saltuses, the variance $D^2[I_N(\lambda)]$ is equal to $[f(\lambda)]^2$ or $2[f(\lambda)]^2$ according as $\lambda \neq 0$ or $\lambda = 0$ at a point of continuity of the spectrum.*

PROOF. From (2.2)

$$\begin{aligned} D^2 I_N(\lambda) = &\left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2[(2N+1)(l-\lambda)/2]}{\sin^2[(l-\lambda)/2]}\, d(\sigma_1(l) + \sigma_2(l))\right]^2 \\ &+ \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin[(2N+1)(l-\lambda)/2]}{\sin[(l-\lambda)/2]}\right. \\ &\left.\cdot \frac{\sin[(2N+1)(l+\lambda)/2]}{\sin[(l+\lambda)/2]}\, d(\sigma_1(l) + \sigma_2(l))\right]^2 \end{aligned} \tag{3.5}$$

By an argument like that of the previous theorem, the first term on the R.H.S. of (3.5) tends to $[f(\lambda)]^2$ at point of continuity of $\sigma(\lambda)$. In the second term the contribution of the term containing $\sigma_1(l)$ tends to zero as $N \to \infty$, so that we have only to investigate the nature of

$$\begin{aligned} &\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin[(2N+1)(l-\lambda)/2]}{\sin[(l-\lambda)/2]} \\ &\qquad \cdot \frac{\sin[(2N+1)(l+\lambda)/2]}{\sin[(l+\lambda)/2]} f(l)\, dl. \end{aligned} \tag{3.6}$$

Case I: $\lambda = 0$. In view of Fejér's theorem it is easily seen that (3.6) tends to $[f(\lambda)]^2_{\lambda=0}$ as $N \to \infty$.

Case II: $\lambda \neq 0$. We divide the range of integration $(-\pi, \pi)$ into six parts as follows letting $\lambda > 0$: $(-\pi, -\lambda - \epsilon)$, $(-\lambda - \epsilon, -\lambda + \epsilon)$, $(-\lambda + \epsilon, 0)$, $(0, \lambda - \epsilon')$, $(\lambda - \epsilon', \lambda + \epsilon')$, and $(\lambda + \epsilon', \pi)$, where ϵ, ϵ' are small, arbitrary, positive constants. Denote the corresponding integrals by I_1, I_2, I_3, I_4, I_5 and I_6. Applying the first mean value theorem, it is easily seen that I_1, I_3, I_4 and I_6

tend to zero as $N \to \infty$. Consider

$$(3.7) \quad I_5 = \frac{1}{2\pi(2N+1)} \int_{\lambda-\epsilon'}^{\lambda+\epsilon'} \frac{\sin[(2N+1)(l-\lambda)/2]}{\sin[(l-\lambda)/2]} \cdot \frac{\sin[(2N+1)(l+\lambda)/2]}{\sin[(l+\lambda)/2]} f(l)\, dl.$$

Putting $l - \lambda = t$, we have

$$I_5 = \frac{1}{2\pi(2N+1)} \int_{-\epsilon'}^{\epsilon'} \frac{\sin[(2N+1)(t)/2]}{\sin[t/2]} \cdot \frac{\sin[(2N+1)(t+2\lambda)/2]}{\sin[(t+2\lambda)/2]} f(t+\lambda)\, dt$$

$$(3.8) \quad = \frac{1}{2\pi(2N+1)} \int_0^{\epsilon'} \frac{\sin[(2N+1)(t)/2]}{\sin[t/2]} \cdot \frac{\sin[(2N+1)(2\lambda-t)/2]}{\sin[(2\lambda-t)/2]} f(\lambda-t)\, dt + \frac{1}{2\pi(2N+1)} \int_0^{\epsilon} \frac{\sin[(2N+1)(t)/2]}{\sin[t/2]} \cdot \frac{\sin[(2N+1)(t+2\lambda)/2]}{\sin[(t+2\lambda)/2]} f(\lambda+t)\, dt.$$

Hence

$$I_5 \leqq \frac{k}{2\pi(2N+1)} \int_0^{\epsilon} \frac{|\sin[(2N+1)(t)/2]|}{\sin[t/2]}\, dt < \frac{k}{2\pi(2N+1)} \int_0^{\pi} \frac{|\sin[(2N+1)(t)/2]|}{\sin[t/2]}\, dt,$$

which can be written (Zygmund [3] p. 67) as

$$(3.9) \qquad I_5 < [k/(2\pi(2N+1))]O(\log N).$$

Hence $\lim_{N\to\infty} I_5 = 0$. Similarly $\lim_{N\to\infty} I_2 = 0$.

Therefore the expression (3.6), when $\lambda \neq 0$, tends to zero as $N \to \infty$. We thus established that, at a point of continuity

$$\lambda = \lambda_o \neq 0, \qquad \lim_{N\to\infty} D^2[I_N(\lambda)] = [f(\lambda_o)]^2;$$

while at $\lambda = 0$, $\lim_{N\to\infty} D^2[I_N(\lambda)] = 2[f(\lambda)]^2_{\lambda=0}$. Thus, except in the trivial case $f(\lambda) = 0$, $I_N(\lambda)$ is not a consistent estimate of the spectral density at a point of continuity of $\sigma(\lambda)$.

4. Consistency of the weighted periodogram estimator. We will now try to construct a weighted consistent estimator for the spectral density at a point of continuity.

Consider

$$(4.1)\qquad I_N(\lambda) = \frac{1}{2\pi(2N+1)} \sum_{\nu=-N}^{N} x(\nu)e^{-i\nu\lambda} \overline{\sum_{\nu=-N}^{N} x(\nu)e^{-i\nu\lambda}}.$$

Since $x(\nu)$ is real, it is easy to verify that $I_N(\lambda) = I_N(-\lambda)$, i.e., $I_N(\lambda)$ is an even function of λ in $(-\pi, \pi)$.

Let $w(\lambda)$ be an even function of λ such that, within $(0, \pi)$, $w(\lambda)$ vanishes outside $(\lambda_o \pm h)$ and h is so chosen that the h neighborhood of λ_o does not contain any saltus of $\sigma(\lambda)$.

Consider

$$(4.2)\qquad f_N^*(\lambda_o) = \int_{-\pi}^{\pi} I_N(l)w(l)\,dl = 2\int_{\lambda_o-h}^{\lambda_o+h} I_N(l)w(l)\,dl.$$

Taking expectations on both sides of (4.2), we have

$$(4.3)\qquad E[f_N^*(\lambda_o)] = 2\int_{\lambda_o-h}^{\lambda_o+h} E[I_N(\lambda)]w(\lambda)\,d\lambda.$$

Taking limits as $N \to \infty$ we have, at a point of continuity,

$$(4.4)\qquad \lim_{N\to\infty} E[f_N^*(\lambda_o)] = 2\int_{\lambda_o-h}^{\lambda_o+h} f(l)w(l)\,dl.$$

Adding the condition for $f_N^*(\lambda)$ to estimate asymptotically unbiasedly $f(\lambda)$ at a point of continuity λ_o of $\sigma(\lambda)$, we have

$$(4.5)\qquad 2\int_{\lambda_o-h}^{\lambda_o+h} f(l)w(l)\,dl = f(\lambda_0).$$

If $f(\lambda)$ does not vary too much in the neighborhood of λ_o, the approximate condition for asymptotic unbiassedness, is

$$(4.6)\qquad \int_{\lambda_o-h}^{\lambda_o+h} w(\lambda)\,d\lambda = \tfrac{1}{2}.$$

THEOREM 3. *Let $w(\lambda)$ be a continuous weight function satisfying the conditions imposed in Section 4 and (4.6). Let the spectral density $f(\lambda)$ be continuous. Then, at a point of continuity λ_o of $\sigma(\lambda)$, the variance of the weighted estimator $f_N^*(\lambda_0)$ goes to zero as $N \to \infty$.*

PROOF. We have from Grenander [1] that

$$(4.7)\qquad \begin{aligned} 4\pi^2(2N+1)^2D^2f_N^*(\lambda_0) &= \sum_{n,m,k,l\,=\,-N}^{N} r(n+m)r(k+l)W(n+k)W(m+l) \\ &\quad + \sum_{n,m,k,l\,=\,-N}^{N} r(n+m)r(k+l)W(m+l)\overline{W(n+k)}, \end{aligned}$$

where

$$(4.8)\qquad r(n) = \int_{-\pi}^{\pi} e^{in\lambda}\, d\sigma(\lambda) = \int_{-\pi}^{\pi} \cos n\lambda\, d\sigma(\lambda),$$

$$(4.9)\qquad W(n) = \int_{-\pi}^{\pi} e^{in\lambda} w(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \cos n\lambda w(\lambda)\, d\lambda.$$

Since $w(\lambda)$ is an even function we have

$$(4.10)\qquad \begin{aligned} &4\pi^2(2N+1)^2 D^2(f_N^*(\lambda_o)) \\ &\qquad = 2 \sum_{\substack{n,m,k,l \\ -N}}^{N} r(n+m)r(k+l)W(n+k)W(m+l). \end{aligned}$$

Again following Grenander [1] we have

$$(4.11)\qquad \begin{aligned} 2\pi^2(2N+1)D^2[f_N^*(\lambda_0)] &< \sum_{\alpha,\beta,\gamma} r(\alpha)r(\beta)W(\gamma)W(\alpha+\beta-\gamma) \\ &= \sum_{\nu=-2N}^{2N} \left[\sum_{n=-2N}^{2N} r(n)W(n+\nu)\right]\left[\sum_{n=-2N}^{2N} r(n)W(n-\nu)\right]. \end{aligned}$$

Case 1. $\qquad d\sigma(\lambda) = f(\lambda)d\lambda$

where $f(\lambda)$ is an even function, being the spectral density of a real process. We have

$$(4.12)\qquad \begin{cases} f(\lambda) = \lim_{N\to\infty} \sum_{-N}^{N} r(n)e^{-in\lambda} = \sum_{-\infty}^{\infty} r(n) \cos n\lambda, \\ w(\lambda) = \lim_{N\to\infty} \sum_{-N}^{N} W(n)e^{-in\lambda} = \sum_{-\infty}^{\infty} W(n) \cos n\lambda, \\ f(\lambda)w(\lambda) = \lim_{N\to\infty} \sum_{-N}^{N} d(n)e^{-in\lambda}, \end{cases}$$

where

$$(4.13)\qquad d(\nu) = \sum_{-\infty}^{\infty} r(n)W(n+\nu) = \sum_{-\infty}^{\infty} W(n)r(n+\nu).$$

Let us write $d^{2N}(\nu) = \sum_{-2N}^{2N} r(n)W(n+\nu)$. We have from (4.11) that

$$(4.14)\qquad 2\pi^2(2N+1)D^2 f_N^*(\lambda_o) < \sum_{-2N}^{2N} \{d^{2N}(\nu)\}^2.$$

Taking the limit as $N \to \infty$ we have, since

$$\sum_{-\infty}^{\infty} d^2(\nu) < \infty,$$

that $\qquad \lim_{N\to\infty} D^2[f_N^*(\lambda_0)] = 0.$

Case 2. $\qquad d\sigma(\lambda) = f(\lambda)d\lambda + d\sigma_1(\lambda),$

where $\sigma_1(\lambda)$ is a step function with a finite number of saltuses $S_1, S_2, \cdots, S_p$ at $\lambda_1, \lambda_2, \cdots, \lambda_p$ respectively.

We have from (4.8)

$$r(n) = r_1(n) + \sum_{i=1}^{p} S_i \cos n\lambda_i . \tag{4.15}$$

We have from (4.11) in another form

$$2\pi^2(2N+1)D^2(f_N^*(\lambda_o)) < \sum_{\nu=-2N}^{2N} \left[\sum_{n=-2N}^{2N} W(n)r(n+\nu) \right] \cdot \left[\sum_{n=-2N}^{2N} W(n)r(n-\nu) \right]$$

$$= \sum_{\nu=-2N}^{2N} \left\{ \left[\sum_{n=-2N}^{2N} W(n)r_1(n+\nu) + \sum_{i=1}^{p} S_i \sum_{n=-2N}^{2N} W(n) \cos \overline{n+\nu}\lambda_i \right] \cdot \left[\sum_{n=-2N}^{2N} w(n)r_1(n-\nu) + \sum_{i=1}^{p} S_i \sum_{n=-2N}^{2N} W(n) \cos \overline{n-\nu}\lambda_i \right] \right\}$$

$$= \sum_{\nu=-2N}^{2N} \left\{ (d^{2N}(\nu))^2 + d^{2N}(\nu) \sum_{i=1}^{p} S_i \cdot \sum_{n=-2N}^{2N} W(n)[\cos n\lambda_i \cos \nu\lambda_i + \sin n\lambda_i \sin \nu\lambda_i] + d^{2N}(\nu) \sum_{i=1}^{p} S_i \sum_{n=-2N}^{2N} W(n)[\cos n\lambda_i \cos \nu\lambda_i - \sin n\lambda_i \sin \nu\lambda_i] + \sum_{i,j=1}^{p} S_i S_j \sum_{n=-2N}^{2N} W(n)[\cos n\lambda_i \cos \nu\lambda_i + \sin n\lambda_i \sin \nu\lambda_i] \cdot \sum_{n=-2N}^{2N} W(n)[\cos n\lambda_j \cos \nu\lambda_j - \sin n\lambda_j \sin \nu\lambda_j] \right\}. \tag{4.16}$$

But we have, in view of the conditions imposed on the weight function, that

$$\begin{cases} \sum_{-\infty}^{\infty} W(n) \cos n\lambda_i = W(\lambda_i) = 0, \\ \sum_{-\infty}^{\infty} W(n) \sin n\lambda_i = 0, \quad \text{and} \\ \sum_{-\infty}^{\infty} d^2(\nu) < \infty. \end{cases} \tag{4.17}$$

Taking limits on both sides of (4.16) as $N \to \infty$, and taking into account (4.17), we have, at a point of continuity of $\sigma(\lambda)$, that

$$\lim_{N\to\infty} D^2[f_N^*(\lambda_o)] = 0,$$

which proves the theorem.

Acknowledgment. I wish to express my indebtedness to Professor K. Nagabhushanam, for suggesting this problem to me and for his guidance during the preparation of this paper, and I thank Professor Harold Hotelling for going through the manuscript and for his comments and criticism.

REFERENCES

[1] ULF GRENANDER, "On empirical spectral analysis of stochastric processes," *Arkiv for Mathmatik*, Vol. 1 (1951), pp. 503–531.

[2] KARI KARHUNEN, "Über lineare Methoden in der Wahrscheinlichkeitsrechung," *Ann. Acad. Sc. Fenn. Sco. AI*, Maths-Phys No. 37, 1947.

[3] ANTONI ZYGMUND, *Trigonometrical Series*, Vol. 1, Cambridge, University Press, 1959.