ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Central Regional Meeting of the Institute, Urbana, Illinois, November 24–25, 1961. Additional abstracts will appear in the March, 1962 issue.)

1. Admissibility of the Optimal Invariant Estimate for a Translation Parameter Under Absolute Error Loss Function. Martin Fox and Herman Rubin, Michigan State University.

Let P satisfy the conditions given by Stein (*Ann. Math. Stat.*, Vol. 30, pp. 970–979) with the following changes: (i) replace Stein's condition (2.6) with the condition

$$\int d\nu(y) \int x^2 \, d_x P(x, y) < \infty$$

and (ii) add the condition that the unique median of $P(\cdot, y)$ be at 0 for each $y \varepsilon \mathcal{Y}$. Then, with absolute error loss function, x is an almost admissible estimate of the translation parameter θ where $(X - \theta, Y)$ has the joint distribution P. The proof is similar to Stein's but somewhat more intricate. Furthermore, under the assumption that $p(\cdot, y)$ is a density for each $y \varepsilon \mathcal{Y}$, Stein's proof of admissibility goes through. An example shows that almost admissibility is the best that can be obtained without a density. Farrell (Ithaca meeting, April 20–22, 1961) has shown that, if the median is nonunique, then there is no admissible estimate. The results stated above are still valid if the loss function is weighted by a if the estimate is to the left of θ and by b otherwise. The condition on the median is replaced by the same condition for the $(1 - \alpha)$th quantile where $\alpha = a/(a + b)$.

2. Unbiased Estimation of Probability Densities (Preliminary report). S. G. Ghurye, University of Minnesota.

Let $y = (y_1, \cdots, y_n)$ be a sample from a k-dimensional population P, which is an element of a family, $\mathcal{P}$, of distributions. Let $g(x)$ be a known numerical-valued function on R_k with finite expectation $\omega_P = \int_{R_k} g dP$, for all $P \varepsilon \mathcal{P}$. It is desired to find an unbiased estimate $\Phi(y)$ of ω_P. If $\mathcal{P}$ is a dominated family with respective probability densities $f_P(x)$ relative to a known measure μ on the k-dimensional Borel sets, then the problem is equivalent to that of finding $\varphi(x, y)$ satisfying $E_P^y \varphi(x, y) = f_P(x)$ for all $x \varepsilon R_k$, $P \varepsilon \mathcal{P}$. A number of special cases of the problem have been treated previously [Kolmogorov, *Izvest. Akad. Nauk SSSR, Ser. Mat.* (1950); Lieberman and Resnikoff, *J. Amer. Stat. Assoc.* (1955); Washio, Morimoto and Ikeda, *Bull. Math. Stat.* (1956); Schmetterer, *Ann. Math. Stat.* (1960)]. We give a detailed discussion for many families of densities, and also consider certain functions of densities.

3. On the Resolution of Statistical Hypotheses. Robert V. Hogg, University of Iowa.

Let ω_0 be the space of a parameter θ. Let ω_i be a subset of ω_{i-1}, $i = 1, 2, \cdots, k$. We test $\theta \varepsilon \omega_k$ against $\theta \varepsilon \omega_0 - \omega_k$ by testing iteratively the following hypotheses: $\theta \varepsilon \omega_i$ against $\theta \varepsilon \omega_{i-1} - \omega_i$, $i = 1, 2, \cdots, k$. The hypothesis $\theta \varepsilon \omega_k$ is accepted if and only if each intermediate hypothesis is accepted. If the test statistic for each intermediate hypothesis $\theta \varepsilon \omega_i$ is based on the corresponding likelihood ratio λ_i, we demonstrate why, under fairly general conditions, these test statistics are mutually stochastically independent. This argument is based on an independence theorem which deals with complete sufficient statistics.

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4. A $3(2^{8-4})$ Design of Resolution V. PETER W. M. JOHN, University of California, Davis. (By title)

The smallest 2^{k-p} fraction of the 2^8 design of resolution V (main effects and two factor interactions clear) is the quarter replicate, involving 64 points. A three-sixteenth replicate, 48 points, of resolution V is obtained, in which each of the main effects and two factor interactions is estimable from at least a combination of two of the sixteenths. In the three-quarter replicate of the 2^6 design, $3\,(2^{6-4})$, obtained by omitting the quarter replicated defined by $I = ABC = DEF = ABCDEF$, put $G = ABDE$ and $H = ACDF$ to form the $3\,(2^{8-4})$ design. The three sixteenths may be combined in pairs to give the following eighth replicates from which the desired estimates are obtained. They are the fractions whose defining contrast subgroups are generated by (i) *ABC, ABDEG, ACDFH*; (ii) *DEF, ABDEG, ACDFH*; (iii) *ABCDEF, ABDEG, ACDFH.*

5. The Bivariate Chi Distribution. P. R. KRISHNAIAH, PETER HAGIS, JR. AND LEON STEINBERG, Remington Rand Univac, Blue Bell, Penna.

Consider a vector $\mathbf{X}_j = (X_{1j}, X_{2j})$ of two random variables whose joint distribution is the bivariate normal with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)$ and covariance matrix

$$\sum = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1 & \sigma_2\sigma_2^2 \end{bmatrix}$$

If we let $\chi_1 = [\sum_{j=1}^{n} (X_{1j}^2/\sigma_1^2)]^{\frac{1}{2}}$ and $\chi_2 = [\sum_{j=1}^{n} (X_{2j}^2/\sigma_2^2)]^{\frac{1}{2}}$, then the joint distribution of χ_1 and χ_2 is called the central or non-central bivariate chi distribution according to whether $\boldsymbol{\mu} = 0$ or $\boldsymbol{\mu} \neq 0$. In the present paper, some properties of this distribution are discussed. Also, a test is proposed to test for the homogeneity of mean lives of machines when the failure times follow a bivariate chi or chi-square distribution with known correlation. The monotonicity property of the power of this test is established and extensive tables are constructed for use in applications of the test. Applications of the test in areas other than life testing are also discussed.

6. On the Efficiency of Optimum Nonparametric Procedures in the Two Sample Case. P. W. MIKULSKI, University of California, Berkeley.

Consider the hypothesis that two samples are drawn from two populations with the same, continuous, completely specified distribution F. The alternative is that the distribution in the second population is shifted to the right. For testing this hypothesis consider: (a) the locally most powerful rank test, and (b) the locally asymptotically optimal test (J. Neyman: "Optimal Asymptotic Tests of Composite Statistical Hypotheses, *H. Cramer Volume*, 1959), or the asymptotically equivalent likelihood ratio test. The question is investigated, how the Pitman efficiency $e(F; \Psi)$ of the optimal rank test to the optimal parametric test behaves, if the true distribution Ψ departs from the assumed distribution F. Chernoff and Savage have shown that if F is normal, then $e(F; \Psi) \geqq 1$ for all Ψ. It turns out that under certain regularity restrictions for the assumed distribution, normality of F is a necessary condition for the inequality $e(F; \Psi) \geqq 1$ to hold for all Ψ. In particular if the logarithmic derivative of the density of F is bounded and satisfies additional regularity conditions, then for every $\epsilon > 0$ there exists a true distribution Ψ such that $e(F; \Psi) < \epsilon$. If however Ψ differs from F only by location and scale parameters then $e(F; \Psi) \geqq 1$ with strict inequality holding unless either $\Psi = F$ or F is normal.

7. Group-Screening with More Than Two Stages. M. S. PATEL, Research Triangle Institute.

The two stage group-screening procedure described earlier by W. S. Connor (Cf., The Proceedings of the Sixth Conference on the Design of Experiments in Army Research Development and Testing) and G. S. Watson (A study of the Group-screening method, August issue of *Technometrics*, 1961) is extended to multiple stages for the case when responses are observed with negligible error. Because of their potential usefulness, three and four stage procedures are treated in some detail. The general $n + 1$ stage procedure is defined, a formula is developed for the expected number of runs, and for a fixed number of factors is minimized with respect to the sizes of the group-factors at various stages. Finally the procedures at different stages are compared with respect to the minimum expected number of runs.

(Abstracts of papers presented at the Eastern Regional Meeting of the Institute, New York City, December 27–30, 1961. Additional abstracts will appear in the March, 1962 issue.)

1. Extensions of the Arc Sine Law. SIMEON M. BERMAN, Columbia University.

An arc sine law for the number of positive partial sums in a sequence of "symmetrically dependent" random variables is obtained by means of the de Finetti representation theorem; this arc sine distribution is more general than that obtained by E. S. Andersen (1954) and holds under wider conditions. A secondary result of the paper is a direct generalization of an arc sine law of D. A. Darling (1951) for sums of independent random variables.

2. Application of Simultaneous Confidence Intervals to Two Regression Problems (Preliminary report). ARTHUR COHEN, Columbia University.

Consider the general linear hypothesis of full rank; that is, let $y = Xb + u$ where y is an $n \times 1$ vector of observations, X is a fixed $n \times p$ matrix of rank p, b is a $p \times 1$ vector of parameters, and u is an $n \times 1$ vector which is multivariate normal with mean vector zero and covariance matrix $\sigma^2 I$. H. Scheffé has shown how to obtain simultaneous confidence intervals for any number of estimable functions. His result is used to show how to obtain simultaneous confidence intervals for any number of parameters which are the ratios of linear combinations of the parameters in b. This latter result is applied to the multiple bioassay problem.

J. Mandel (*Ann. Math. Stat.*, Vol. 29 (1958), pp. 903–907) has shown how one might obtain simultaneous confidence intervals for any number of any real functions of the parameters in b. His result, along with the above mentioned result on the ratios of estimable functions, is used to test whether one quadratic regression function lies uniformly above another quadratic regression function over any given interval of abscissae.

3. Contributions to the "Two-Armed Bandit" Problem. DORIAN FELDMAN, Michigan State University.

The Bayes sequential design is obtained for an optimization problem involving the choice of experiments. Given are experiments A, B, densities p_1, p_2, a positive integer N (which may be ∞) and a number $\xi \in [0, 1]$. A sequence of N observations is to be made such that at each stage either A or B is observed, the loss being 1 if the experiment with density p_2 is chosen, 0 otherwise. ξ is the prior probability that A has density p_1 and the risk of a procedure is the expected number of observations on the experiment with density p_2, given ξ. Let $R_N^A(\xi)$, $R_N^B(\xi)$ denote the risks of the procedures that choose A first, respectively B first, and follow the optimal procedure for the last $N - 1$ trials. It is shown inductively that for all N the difference between these risks is monotone in ξ and this is equivalent to optimality of the following procedure: At stage $i + 1$ (regardless of N) choose A or B according as $\xi_i \geqq \frac{1}{2}$ or $\xi_i \leqq \frac{1}{2}$. ξ_i is the posterior probability that A has p_1. For $N = \infty$, the risk of this procedure is shown to be finite (hence optimal) and some specific risk functions are computed for binomial experiments.

4. On the Axioms of Sample Formation and Their Bearing on the Construction of Linear Estimators in Sampling Theory for Finite Populations. J. C. Koop, North Carolina State College.

Consider a universe (or population) of N elements described by a frame which in this case will be a simple list. In drawing a sample according to any probability system defined for the selection of the elements one at a time, either with or without replacement, three features inherent in the nature of the process of selection are evident. They are as follows: (i) the order of appearance of the elements, (ii) the presence or absence of any given element in the sample which is a member of the universe or population, and (iii) the set of elements composing the sample considered as one of the total number possible (in repeated sampling) according to the given probability system. As the veracity of the statements at (i), (ii) and (iii) is self-evident, they may be designated as axioms. It will be noted that the statements are not mutually contradictory. These features, inherent in the actual process of selection and as a result sample formation, supply the bases for the construction of linear estimators on a deductive basis. The axioms, which are implicit in the work of Horvitz and Thompson (*Ann. Math. Stat.*, Vol. 22 (1951), p. 315), considered singly, two at a time and most generally all three together, result in seven very general classes of linear estimators. The extension of the application of these axioms to samples from a universe with subdivisions (strata, first-stage units, second-stage units, etc.) is almost immediate.

5. Maximum Likelihood Estimates for Certain Contagious Distributions Using High Speed Computers. Donald C. Martin and S. K. Katti, Florida State University.

Fortransit programs for fitting the Neyman Type A, the Poisson with Zeros and the simple Poisson, the Negative Binomial, and the Poisson Binomial by the method of maximum likelihood are available from the Statistics Department of the Florida State University. These programs have five operational modes allowing for combinations of the following: (i) Computing moment estimates. (ii) Using moment estimates or reading the initial parameter estimates and computing the maximum likelihood estimates by an iterative scheme. (iii) Reading in estimates computed by other means and using these to compute probabilities, expected frequencies and chi-square values, thus bypassing the maximum likelihood estimation process. (iv) Computing the probabilities, cumulative probabilities, expected frequencies, and term by term chi-square values and (v) Computing the chi-square value with some rudimentary grouping or the likelihood value. The chi-square section of the routine groups all expected frequencies less than a constant, usually 5, into a single cell. The Poisson is included as a special case of the Poisson with Zeros routine.

All programs are written in Fortransit II's for an IBM 650 computer with special characters. Running times vary widely between routines and data with the longest time on the order of 20 minutes and the shortest less than one minute. Typical times range from 1 to 10 minutes per distribution. An object program deck, a Fortransit statement deck, running instructions and test data will be provided on request. All inquiries should be addressed to: The Department of Statistics, Florida State University, Tallahassee, Florida, attention S. K. Katti.

6. Asymptotic Efficiency of a Class of c-Sample Tests. M. L. Puri, University of California, Berkeley. (By title)

For testing the equality of c continuous probability distributions on the basis of c independent random samples, the test statistics of the form $L = \sum_{j=1}^{c} m_j \mid (T_{N,j} - \mu_{N,j})/A_N \mid^2$ are considered. Here m_j is the size of the jth sample, $\mu_{N,j}$ and A_N are normalizing constants and $T_{N,j} = \sum_{i=1}^{N} E_{N,i} Z_{N,i}^{(j)}$ where $Z_{N,i}^{(j)} = 1$ if the ith smallest of $N = \sum_{j=1}^{c} m_j$ observations is from the jth sample and $Z_{N,i}^{(j)} = 0$ otherwise: and $E_{N,i}$ are given numbers. Generalizing results of Chernoff-Savage (*Ann. Math. Stat.*, Vol. 29 (1958), pp. 972–994), sufficient conditions are given for the joint asymptotic normality of $T_{N,j}$; $j = 1, \cdots, c$. Under suitable regularity conditions and the assumption that the ith distribution function is

$$F(x + (\theta_i/N^{\frac{1}{2}}))$$

it is shown that as $N \to \infty$, the statistic L has a limiting noncentral chi-squared distribution. The asymptotic relative efficiency in Pitman's sense of the L test and the Kruskal-Wallis H test (which is a particular case of the L test) is obtained and is shown to be independent of c.

7. Some Statistical and Operational Techniques in Reliability Studies (Preliminary report). A. S. Qureishi, IBM Service Bureau Corp., San Jose, California.

Given two production processes, the units from which fail in accordance with the Weibull Distribution, the problem is to select the particular process with the Smallest failure rate. It is assumed that there is a common guarantee period (known or unknown) during which no failures occur. It is desired to accomplish the above goal in as short a time as possible, thus maximizing the total gain without invalidating certain predetermined probability specifications. Three techniques, as suggested by Sobel (*Bell System*, Vol. 35 (1956) pp. 179–202) are considered and three procedures are constructed to show their advantages and disadvantages. Sobel's results on the assumption of exponential distribution turn out to be a particular case of the general solution presented in this paper. Expressions for average experiment time required to terminate the experiment have been obtained and the efficiencies of different procedures are compared. An alternative sequential procedure has been proposed and shown to be valuable in some situations. It has been shown with the aid of the tables (computed on Burroughs 220), that in some situations, one saves time and money if he assumes Weibull Distribution. Techniques have been developed to find the optimal sample size for each of the four procedures. Minimum Regret Principle has been applied to select the best procedure. The author is highly indebted to Professor N. L. Johnson, under whose Supervision this research is being carried.

8. Asymptotic Bounds for the Zero-Crossing Probability Distribution of Stationary Gaussian Processes. M. Rosenblatt, Brown University.

Let $X(t)$ be a separable stationary Gaussian process with mean zero and covariance function $r(t) = E[X(\tau)X(\tau + t)]$. Let $G(T) = P[X(t) > 0, 0 \leqq t \leqq T]$. Assume that $r(t) \to 0$ as $t \to \infty$. It is then shown that $G(T)$ approaches zero faster than any inverse power of T as $T \to \infty$. Stronger bounds are obtained for specific rates of decay of $r(t)$ such as $r(t) \sim t^{-\alpha}$, $\alpha > 0$, as $t \to \infty$. The basic tool is a powerful inequality of D. Slepian.

(Abstracts of papers not presented at any meeting of the Institute)

1. A Contribution to the Sphere-Packing Problem of Communication Theory. A. V. BALAKRISHNAN, University of California, Los Angeles. (By title)

It is shown that the "sphere packing" problem (optimal band-limited signal selection for coherent Gaussian channels) can be reduced to the following extremal problem: "Given an N-variate Gaussian, $\xi_1, \cdots, \xi_N$, with zero means and unit variances, maximize $E(e^{\lambda \max_i \xi_i})$, for $\lambda > 0$, with or without restriction on the rank m of the covariance matrix." It is shown that for a given m if there is a solution independent of λ, then this is given by maximizing the mean-width of the polyhedron generated by N unit vectors in Euclidean m-space. With no restriction on m, $m \leqq N$, it is shown that if there is a nonzero λ-interval for which the solution is independent of λ, this solution is given by the regular simplex in $(N - 1)$ dimensions. Additional results using generalized tetrachoric series are given for the general problem.

2. Some Aspects of Statistical Invariance. DAVID R. BRILLINGER, Bell Telephone Labs, Murray Hill, N. J. (By title)

Necessary and sufficient conditions are presented for a statistical problem to be invariant under a Lie transformation group. For the case of a (multi-dimensional) real-valued random variable x with c.d.f. $F(x, \theta)$ the conditions reduce to:

(i) there must exist analytic functions $\psi_i^\alpha(\theta)$, or $\sigma_i^\gamma(x)$ such that,

$$X_i F = \sum_\alpha \psi_i^\alpha(\theta) \frac{\partial F}{\partial \theta^\alpha} + \sum_\gamma \sigma_i^\gamma(x) \frac{\partial F}{\partial x^\gamma} = 0, \qquad \text{for all } i$$

and

(ii) the infinitesimal generators X_i generate a group. The paper continues with a theorem concerning the distribution of statistics that are the maximal invariants of a compact topological transformation group. The theorem generalizes the technique that James has been using to derive some multi-dimensional distributions. The paper concludes with the following theorem justifying the "re-use" of samples: consider a random variable x with probability measure P. Let G be a set of measure preserving transformations of P. Let $\varphi(x)$ be an unbiased estimator of α, then $\int_G \varphi(gx)\, d\mu(g)$ is also an unbiased estimator of α and has smaller loss for any real valued convex loss function where $\mu(g)$ is any measure of total mass 1 on G.

3. The δ-Method for Banach Valued Random Variables. DAVID R. BRILLINGER, Bell Telephone Labs, Murray Hill, N. J. (By title)

The "δ-method" or method of "propagation of error" is extended so that it may be used in deriving the asymptotic distribution of Banach valued functions of Banach valued random variables. Define plim $x_n = \theta$, θ the 0 element of the Banach space, if for every $\epsilon > 0 \lim P(\|x_n\| \leqq \epsilon) = 1$. Define $\pi \lim x_n = \theta$, if for every $\epsilon > 0$ there exists an A_ϵ such that $P(\|x_n\| \leqq A_\epsilon) \geqq 1 - \epsilon$ for all n. *Theorems* proved include:

(1) Let x_n, y_n be sequences of Banach valued random variables with associated measures μ_n, ν_n on X. Let $\{\nu_n\}$ converge weakly to a probability measure μ. If plim $(x_n - y_n) = \theta$, then $\{\mu_n\}$ converges weakly to μ.

(2) Let x_n, x induce measures μ_n, μ on X. Let μ_n converge weakly to μ. Let $g: X \to Y$ be a continuous map of X onto the Banach space Y. This map induces measures $\{\nu_n\}$, ν such that ν_n converges weakly to ν.

(3) Let $\{\lambda_n\}$ be a sequence of scalars such that $|\lambda_n| \to \infty$. Let $g:X \to Y$ possess a Frechet differential everywhere in X. If $\lim \lambda_n(x_n - y_n) = \theta$ then $\text{plim } \lambda_n[g(x_n) - g(y_n) - dG(y_n, h_n)] = \theta$ where $h_n = x_n - y_n$.

4. Some Fiducial Examples. DAVID R. BRILLINGER, Bell Telephone Labs, Murray Hill, N. J. (By title)

This paper presents three examples with fiducial relevance. The first example concerns the definition of joint fiducial distributions. Quotes are given from works of Fisher concerning the genuine fiducial argument. An example is given that appears to satisfy all of Fisher's requirements, but yet that doesn't lead to a unique fiducial distribution. The second example demonstrates that recent results of Lindley concerning the identity of a fiducial distribution and a Bayes' *posteriori* distribution cannot be extended to higher dimensions in the obvious manner. The third example is the following: let x be $N(\mu, 1)$. The fiducial distribution of μ^2 may be derived in two manners, firstly from the fiducial distribution of μ by means of a Jacobian multiplier, secondly from the fact that x^2 is non-central χ^2 with parameter μ^2. The results are different.