$P_{st}(y, \cdot) = \hat{P}_{st}(y, \cdot)$ if $y \not\varepsilon M_s$ and $\hat{P}_{rs}(x, M_s) = 0$ for all x it follows that $\hat{P}_{rt} = \hat{P}_{rs} * \hat{P}_{st}$.

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A GENERALIZATION OF A THEOREM OF BALAKRISHNAN[1]

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1. Introduction. Given a stochastic process $\{X(t), t \, \varepsilon \, T\}$ on some probability space with second moment kernel

$$\mathcal{E}[X(s)\overline{X(t)}] = K(s, t),$$

a characterization is given of the function

$$m(t) = \mathcal{E}X(t).$$

This characterization includes the result of Balakrishnan [2] for the case of second order stationary, discrete or continuous parameter processes.

2. The characterization. Let T be an abstract set and let K be a positive definite kernel on $T \times T$. A function m on T is said to be an admissible mean value function for the kernel K if there exists a stochastic process $\{X(t), t \, \varepsilon \, T\}$ on some probability space with

$$\mathcal{E}[X(s)\overline{X(t)}] = K(s, t) \qquad \text{and} \qquad \mathcal{E}X(t) = m(t).$$

Lemma 1. *m is an admissible mean value function for the kernel K if and only if $K(s, t) - m(s)\overline{m(t)}$ is positive definite.*

Proof. if $K(s, t) - m(s)\overline{m(t)}$ is a positive definite kernel on $T \times T$, let $\{X(t), t \, \varepsilon \, T\}$ be a Gaussian process with mean function m and covariance kernel $K(s, t) - m(s)\overline{m(t)}$, ([3], p. 72). Then

$$\begin{aligned}\mathcal{E}[X(s)\overline{X(t)}] &= \mathcal{E}[X(s) - m(s)][\overline{X(t) - m(t)}] + m(s)\overline{m(t)} \\ &= K(s, t).\end{aligned}$$

Conversely, if m is admissible,

$$\mathcal{E}[X(s) - m(s)][\overline{X(t) - m(t)}] = K(s, t) - m(s)\overline{m(t)}$$

is positive definite.

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To characterize these functions m, we introduce, for a positive definite kernel R on $T \times T$, the corresponding reproducing kernel Hilbert space of functions on T, denoted by $H(R)$, the dependence on the set T having been suppressed. For a kernel R, $H(R)$ is specified by the conditions

(1) for every $t \varepsilon T$, $R(\cdot, t)$ $H(R)$,

(2) for every $t \varepsilon T$ and $f \varepsilon H(R)$, $(f, R(\cdot, t))_{H(R)} = f(t)$.

From these conditions, the following lemma is apparent.

LEMMA 2. *Given a function* m ($\not\equiv 0$) *on* T, $M(s, t) = m(s)\overline{m(t)}$ *is positive definite on* $T \times T$ *and* $H(M)$ *consists of all multiples of the function* m *with* $\|m\|_{H(M)} = 1$.

We appeal finally to the following general theorem given in [1].

THEOREM 1. *Let* R *and* R^* *be positive definite kernels on* $T \times T$. $R - R^*$ *is positive definite if and only if* $H(R^*) \subset H(R)$ *and for all* $f \varepsilon H(R^*)$,

$$\|f\|_{H(R^*)} \geqq \|f\|_{H(R)}.$$

Returning then to the determination of the functions m for which $K(s, t) - m(s)\overline{m(t)}$ is positive definite on $T \times T$, we have

THEOREM 2. *If* K *is a positive definite kernel on* $T \times T$, *then* $K(s, t) - m(s)\overline{m(t)}$ *is positive definite if and only if* $m \varepsilon H(K)$ *and* $\|m\|_{H(K)} \leqq 1$.

That is, the admissible mean value functions for a given second moment kernel K are those functions in the unit sphere of the reproducing kernel space $H(K)$.

Theorem 1 of Balakrishnan may be seen to coincide with Theorem 2 above when K has the representation

$$K(s, t) = k(s - t) = \int_{-\infty}^{+\infty} \exp [i(s - t)x] \, dG(x), \qquad -\infty < s, t < +\infty.$$

Then, according to Theorem 4D of [4], the unit sphere of $H(K)$ consists of functions of the form

$$m(t) = \int_{-\infty}^{+\infty} \exp (itx)u(x) \, dG(x)$$

with

$$\|m\|^2_{H(K)} = \int_{-\infty}^{+\infty} |u(x)|^2 \, dG(x) \leqq 1.$$

In particular stationary cases, alternative representations are known. Thus, if

$$K(s, t) = \exp [-(s - t)^2/2], \qquad -\infty < s, t < +\infty,$$

the unit sphere of $H(K)$ consists of analytic functions m for which

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left| \frac{d^n}{dt^n} [\exp (t^2/2)m(t)]_{t=0} \right|^2 \leqq 1.$$

It should be noted that Theorem 2 applies even to stationary kernels which do not possess the spectral representation.

Lastly, a nonstationary example is provided by the Brownian motion kernel. For

$$K(s, t) = \min(s, t), \qquad 0 \leqq s, t \leqq 1,$$

the unit sphere of $H(K)$ consists of absolutely continuous functions m for which $m(0) = 0$, and

$$\int_0^1 |m'(t)|^2 \, dt \leqq 1.$$

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THE OPINION POOL[1]

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1. Introduction and summary. When a group of k individuals is required to make a joint decision, it occasionally happens that there is agreement on a utility function for the problem but that opinions differ on the probabilities of the relevant states of nature. When the latter are indexed by a parameter θ, to which probability density functions on some measure $\mu(\theta)$ may be attributed, suppose the k opinions are given by probability density functions $p_{s1}(\theta), \cdots, p_{sk}(\theta)$. Suppose that D is the set of available decisions d and that the utility of d, when the state of nature is θ, is $u(d, \theta)$.

For a probability density function $p(\theta)$, write

$$u[d \mid p(\theta)] = \int u(d, \theta) p(\theta) \, d\mu(\theta).$$

The Group Minimax Rule of Savage [1] would have the group select that d minimising

$$\max_{i=1,\cdots,k} \{\max_{d' \epsilon D} u[d' \mid p_{si}(\theta)] - u[d \mid p_{si}(\theta)]\}.$$

As Savage remarks ([1], p. 175), this rule is undemocratic in that it depends only on the *different* distributions for θ represented in those put forward by the

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