QUEUES WITH BATCH DEPARTURES I

By F. G. Foster and K. M. Nyunt

London School of Economics

1. Introduction. This paper has a pattern closely similar to that of [4]. The following single-server queueing system is considered.

(i) Units arrive at the sequence of instants $\tau_1, \tau_2, \cdots$, such that the inter-arrival times, $\theta_n = \tau_{n+1} - \tau_n > 0$ $(n = 1, 2, \cdots)$, are identically distributed independent random variables with an exponential distribution function,

$$F(x) = P[\theta_n \leqq x] = 1 - e^{-\lambda x} \qquad (x \geqq 0).$$

Put $\alpha = \int_0^\infty x \, dF(x)$. Then $\lambda = 1/\alpha$.

(ii) Units are served in batches of exactly k units by a single server, in order of arrival. Denote by χ_n the service time of the nth batch to be served. We suppose that $\{\chi_n\}$ $(n = 1, 2, \cdots)$ is a sequence of identically distributed independent positive random variables, independent also of the sequence $\{\tau_n\}$, with common distribution function, $H(x) = P[\chi_n \leqq x]$. Put $\psi(s) = \int_0^\infty e^{-sx} \, dH(x)$, $\beta = \int_0^\infty x \, dH(x)$ and $\mu = 1/\beta$. Define $\rho = \lambda/\mu$.

In the terminology of Foster [3], this system can alternatively be described as having the 1-input (arrivals) untriggered with input quantity constantly unity, and an exponential distribution for the 1-input time. The 0-input (departures) is triggered with input quantity constantly, k and a general distribution for the 0-input time. The system has infinite capacity. For definitions of these concepts, the reader is referred to [3].

Such a batch-size model does not appear to have been treated explicitly in the literature, although it has obvious applications. A simple special case of it is, however, implicit in the work of Jackson and Nichols [5]. These authors suppose that an inter-arrival time devoted to one unit is composed of k consecutive phases, each exponentially distributed. If instead, we think of this unit as composed of k subunits (corresponding to the phases of arrival) then we have the idea of batch service: Jackson and Nichols treat the special case of exponential service times.

Justification for the explicit consideration of batch departure systems resides in the fact that the results one can obtain are elegant, and form a natural generalization of the case of unit departures, as treated, for example, in Kendall [6]. The analysis in this paper is similar to that in Bailey [1], but the model is in fact different, and the results obtained here are new. In the terminology of Foster [3], the model Bailey considered differs from the present one in that the 0-input in Bailey's model is untriggered with controlled input quantity of zero to k units, depending upon the state of the system: the input being, for example, virtual when the system contains no 1's. In other words, service begins from time to time

Received January 9, 1961.

whether or not there happen to be any units in the system. Bailey obtains for this system the equilibrium distribution of queue-size at instants just before service is due to begin.

Denote by $\xi(t)$, the number of units in the system, including the batch under service, at the instant t. Let σ_n be the instant at which the nth batch (of size k) departs from the system on receiving service. Put $\xi_n = \xi(\sigma_n + 0), n = 1, 2, \cdots$. We shall determine the probability generating function (p.g.f.) of the limiting distribution,

$$p_j^+ = \lim_{n\to\infty} P[\xi_n = j].$$

The distribution $\{p_j^+\}$ exists and is independent of the initial state of the system, if, and only if, $\rho/k < 1$. The proof of this statement follows the same lines as in the case, $k = 1$, as given in Foster [2].

Let us denote this batch departures model by $E_1/G^k/1$ where E_1 indicates an exponentially distributed inter-arrival time, and G^k indicates that the service time has a general distribution, and that service is in batches of k units. We shall consider its relation to the unit departures model, $E_k/G/1$, where E_k indicates an Erlang distribution with parameter k. We shall derive the equilibrium distribution of queue-size at instants just after departures for this latter system in terms of $\{p_j^+\}$. As a special case we shall consider the system, $E_k/E_r/1$. In our previous paper [4] we obtained the equilibrium distribution at instants just before arrivals for the same system. In this paper we shall establish the identity of the two formulae, thus verifying a special case of the general proposition that, for the system $G/G/1$, when these exist, the equilibrium distribution of queue-size at instants just after departures is identical with that at instants just before arrivals (*cf.* Khintchine [7]).

2. The system $E_1/G^k/1$. Let $\{\nu_n\}(n = 1, 2, \cdots)$ be a sequence of identically distributed independent random variables with distribution,

$$k_j = P[\nu_n = j], \qquad j = 0, 1, 2, \cdots,$$

where

$$k_j = \int_0^\infty \frac{e^{-\lambda x}(\lambda x)^j}{j!} dH(x).$$

Then ν_n is thought of as the number of units joining the queue during the service-time of the nth batch.

Put $\kappa(z) = \sum_{j=0}^{\infty} k_j z^j$. We note that $\kappa(z) = \psi\{\lambda(1 - z)\}$. We assume that $\rho/k < 1$, and also that $\kappa(z)$ is regular within the circle $|z| = 1 + \delta$, where δ is some small positive number. This implies a slight restriction on the distribution, $H(x)$, which will always be satisfied in practice. It follows from Rouché's theorem that the equation,

$$\kappa(z) = z^k, \tag{1}$$

has exactly k roots inside or on the unit circle. For $\kappa'(1) = \rho < k$, so that for some small positive δ, $\kappa(1 + \delta) < (1 + \delta)^k$. Therefore, on the circle, $|z| = 1 + \delta$, $|\kappa(z)| = \sum k_j|z|^j < (1 + \delta)^k = |z^k|$. Clearly, $z = 1$ is one root, and it is a simple root. Denote the other $k - 1$ roots by $\delta_1, \delta_2, \cdots, \delta_{k-1}$.

Define $P^+(z) = \sum_{j=0}^{\infty} p_j^+ z^j$.

THEOREM 1.

$$P^+(z) = \frac{(k - \rho)(z - 1) \prod_{j=1}^{k-1} (z - \delta_j)/(1 - \delta_j)}{z^k/\kappa(z) - 1}. \tag{2}$$

PROOF. The process, $\{\xi_n\}$, is a Markov process with transition matrix described by the relations:

$$\xi_{n+1} = \max [\xi_n - k, 0] + \nu_n, \qquad n = 1, 2, \cdots.$$

The random variable, $\max [\xi_n - k, 0]$, has, in the limit as $n \to \infty$, the generating function,

$$P^+(z)z^{-k} - \sum_{j=0}^{k-1} p_j^+(z^{j-k} - 1).$$

Therefore, since ν_n and $\max [\xi_n - k, 0]$ are independent, we have the relation,

$$P^+(z) = \{P^+(z)z^{-k} - \sum_{j=0}^{k-1} p_j^+(z^{j-k} - 1)\}\kappa(z),$$

which, on simplifying, reduces to

$$P^+(z) = \frac{\sum_{j=0}^{k-1} p_j^+(z^k - z^j)}{z^k/\kappa(z) - 1}.$$

Since $P^+(z)$ is a probability generating function, it is absolutely convergent in the region, $|z| \leqq 1$. Therefore, the roots of the numerator in this region must coincide with those of the denominator, and the latter are

$$1, \delta_1, \delta_2, \cdots, \delta_{k-1}.$$

Therefore, since the numerator is a polynomial of degree k, we have

$$\sum_{j=0}^{k-1} p_j^+(z^k - z^j) \equiv C(z - 1)\prod_{j=1}^{k-1} (z - \delta_j),$$

where C is a constant to be determined. Using $P^+(1) = 1$, we find that

$$C = (k - \rho) \Big/ \prod_{j=1}^{k-1} (1 - \delta_j)$$

and (2) follows.

If (1) has any roots outside the unit circle, we shall denote them by

$$\beta_1, \beta_2, \cdots$$

and we define $\epsilon_j = 1/\beta_j$.

EXAMPLE 1. If the service-time distribution is Erlang, E_r, then

$$\kappa(z) = \{1 + [\rho(1 - z)/r]\}^{-r},$$

and the denominator of (2) becomes

$$z^k\{1 + [\rho(1 - z)/r]\}^r - 1,$$

which, being a polynomial of degree $k + r$ has precisely the $(k + r)$ zeros,

$$1, \delta_1, \delta_2, \cdots, \delta_{k-1}, \beta_1, \beta_2, \cdots, \beta_r,$$

and so can be expressed as

$$C(z - 1)\prod_{j=1}^{k-1}(z - \delta_j)\prod_{j=1}^{r}(z - \beta_j),$$

where C is a constant to be determined. Therefore, substituting in (2) and normalizing, we obtain

$$P^+(z) = \prod_{j=1}^{r}\left(\frac{1 - \beta_j}{z - \beta_j}\right) = \prod_{j=1}^{r}\left(\frac{1 - \epsilon_j}{1 - \epsilon_j z}\right), \tag{3}$$

where the ϵ_j's are the reciprocals of the roots outside the unit circle of the equation,

$$\{1 + [\rho(1 - z)/r]\}^{-r} = z^k. \tag{4}$$

We can show that these roots, and hence the ϵ_j's are distinct. For suppose, on the contrary, that (4) has a double root, say, a. Then for $z = a$, we should have, by differentiation of (4).

$$\rho\{1 + [\rho(1 - z)/r]\}^{-r-1} = kz^{k-1}. \tag{5}$$

Dividing (4) by (5) and simplifying, we obtain

$$a = (k/\rho)[(r + \rho)/(r + k)].$$

But this value must now satisfy (4); that is

$$\{(r + \rho)/(r + k)\}^{(r+k)/k} = \rho/k. \tag{6}$$

Now we are assuming that the traffic intensity, ρ/k, is less than unity, say $\rho/k = 1 - \delta$, where $0 < \delta < 1$. Substituting in (6) and putting $b = k/(r + k)$, we get after simplifying,

$$1 - b\delta = (1 - \delta)^b.$$

But this is impossible, unless $\delta = 0$. Therefore, (4) has no multiple roots, and so the ϵ_j's are distinct.

It follows that we can write

$$P^+(z) = \sum_{j=1}^{r} C_j/(1 - \epsilon_j z) \tag{7}$$

where

$$C_j = (1 - \epsilon_j) \prod_{i \neq j} (1 - \epsilon_i)/(1 - \epsilon_i/\epsilon_j). \tag{8}$$

Formula (3) gives the p.g.f. of the equilibrium distribution after departures for the system, $E_1/E_r^k/1$. The traffic intensity is ρ/k. We note here for later use that if the traffic intensity is changed to $r\rho/k$, the ϵ_j's become the reciprocals of the distinct roots outside the unit circle of the equation,

$$[1 + \rho(1 - z)]^{-r} = z^k. \tag{9}$$

EXAMPLE 2. If the service-time distribution is exponential, E_1, we have $\kappa(z) = \{1 + [\rho(1 - z)]\}^{-1}$, and (3) becomes

$$P^+(z) = (1 - \epsilon)/(1 - \epsilon z) \tag{10}$$

where ϵ is the reciprocal of the root outside the unit circle of

$$\{1 + [\rho(1 - z)]\}^{-1} = z^k. \tag{11}$$

The formula (10) is for the system $E_1/E_1^k/1$, with traffic intensity, ρ/k.

3. Relationship with the unit departures system $E_k/G/1$. If ξ is the number of units left by an arbitrary departing batch, then the number of complete batches of size k in the system at this instant will be

$$\varphi = [\xi/k],$$

where $[x]$ denotes the greatest integer not greater than x.

We now interpret the random variable φ as the number of units left by an arbitrary departing unit in the unit departures system $E_k/G/1$, which has mean service time $1/\mu$ and Erlang inter-arrival time distribution, E_k, with a mean of k/λ. The traffic intensity is thus ρ/k.

We consider the distribution of φ. Define

$$q_j^+ = P[\varphi = j],$$

and put

$$Q^+(z) = \sum_{j=0}^{\infty} q_j^+ z^j.$$

Now define

$$P_j^+ = \sum_{i=0}^{j} p_i^+$$

$$Q_j^+ = \sum_{i=0}^{j} q_i^+.$$

Then

$$\sum_{j=0}^{\infty} P_j^+ z^j = \frac{P^+(z)}{1 - z}, \tag{12}$$

and

$$\sum_{j=0}^{\infty} Q_j^+ z^j = \frac{Q^+(z)}{1 - z}. \tag{13}$$

We have

$$Q_0^+ = P_{k-1}^+$$
$$Q_1^+ = P_{2k-1}^+$$

and generally

$$Q_j^+ = P_{(j+1)k-1}^+ \qquad j = 0, 1, \cdots.$$

Therefore,

$$\frac{Q^+(z)}{1 - z} = \sum_{j=0}^{\infty} P_{(j+1)k-1}^+ z^j.$$

But from (12) we have

$$P_j^+ = \frac{1}{2\pi i} \int_{C} \frac{P^+(v)\, dv}{(1 - v)v^{j+1}},$$

where C is a contour around the origin excluding the poles of $P^+(z)/(1 - z)$. Therefore,

$$\frac{Q^+(z)}{1 - z} = \sum_{j=0}^{\infty} \frac{z^j}{2\pi i} \int_C \frac{P^+(v)\, dv}{(1 - v)v^{(j+1)k}},$$

so that

$$Q^+(z) = \frac{1 - z}{2\pi i} \int_C \frac{P^+(v)\, dv}{(1 - v)v^k(1 - v^{-k}z)}. \tag{14}$$

The poles of the integrand within C are at

$$v = \omega^j z^{1/k}, \qquad j = 1, 2, \cdots, k,$$

where ω^j is a kth root of unity. The residue at $v = \omega^j z^{1/k}$ is

$$1/zkP^+(\omega^j z^{1/k})\omega^j z^{1/k}/(1 - \omega^j z^{1/k}).$$

Therefore, summing the residues, we obtain the alternative formula,

$$Q^+(z) = \frac{1 - z}{zk} \sum_{j=1}^{k} \frac{P^+(\omega^j z^{1/k})\omega^j z^{1/k}}{1 - \omega^j z^{1/k}}. \tag{15}$$

EXAMPLE 3. If the service-time distribution is Erlang, E_r, then $P^+(z)$ is given by (7), and so from (14),

$$Q^+(z) = \sum_{j=1}^{r} \frac{1}{2\pi i} \int_C \frac{C_j(1 - z)\, dv}{(1 - \epsilon_j v)(1 - v)v^k(1 - v^{-k}z)}. \tag{16}$$

This is the p.g.f. of the equilibrium distribution of queue-size after departures in the system $E_k/E_r/1$. For traffic intensity $r\rho/k$, the ϵ_j's are the reciprocals of the roots outside the unit circle of Equation (9).

Now we have

$$\frac{1}{2\pi i}\int_C \frac{C_j(1-z)\,dv}{(1-\epsilon_j v)(1-v)v^k(1-v^{-k}z)}$$

$$(17)\qquad = \frac{1}{2\pi i}\int_C \frac{C_j}{(1-\epsilon_j v)(1-v)}\left(1-\frac{1-v^{-k}}{1-v^{-k}z}\right)dv$$

$$= \frac{1}{2\pi i}\int_C \frac{C_j}{(1-\epsilon_j v)(v-1)}\frac{1-v^{-k}}{1-v^{-k}z}\,dv,$$

since the neglected part of the integral has no poles inside C. The integral (17) is most easily evaluated by considering the single pole of the integrand outside C at $v = 1/\epsilon_j$. The residue is

$$-C_j/(1-\epsilon_j)[(1-\epsilon_j^k)/(1-\epsilon_j^k z)].$$

Since the integrand is rational with denominator of degree at least 2 higher than that of the numerator, it follows that (17) is equal to minus the residue outside C. Therefore, we have

$$(18)\qquad Q^+(z) = \sum_{j=1}^{r} C_j/(1-\epsilon_j)(1-\epsilon_j^k)/(1-\epsilon_j^k z).$$

EXAMPLE 4. If the service-time distribution is exponential, E_1, we have from (14)

$$Q^+(z) = \frac{1-z}{2\pi i}\int_C \frac{1-\epsilon}{1-\epsilon v}\frac{dv}{(1-v)v^k(1-v^{-k}z)}$$

$$= (1-\epsilon^k)/(1-\epsilon^k z)$$

by consideration of the pole at $v = 1/\epsilon$ outside C. $1/\epsilon$ is the single root outside the unit circle of equation (11). This is the p.g.f. of the equilibrium distribution of queue-size after departures in the system $E_k/E_1/1$ when the traffic intensity is ρ/k. The formula is, however, found to be identical (apart from notation) with that obtained by Jackson and Nichols [5] for the distribution *before arrivals* in the same system. A more general case of this observation is considered in the next section.

4. Relationship between the queue-size distributions at departure and at arrival points for the system $E_k/E_r/1$. In our previous paper [4] we obtained the p.g.f., $Q(z)$, for the equilibrium distribution of queue size before arrivals for the system $E_k/E_r/1$,

$$(19)\qquad Q(z) = \frac{1-z}{2\pi i}\int_C \frac{P(v)\,dv}{v(1-v)(1-v^{-r}z)}.$$

C is a contour around the origin excluding the poles of $P(v)/(1 - v)$, and

$$P(v) = \prod_{j=1}^{r} \left(\frac{1 - \gamma_j}{1 - \gamma_j v} \right),$$

where for traffic intensity $r\rho/k$, the γ_j's are the roots inside the unit circle of

$$[\rho/(\rho + 1 - z)]^k = z^r. \tag{20}$$

We shall now establish that $Q(z)$ is identical with $Q^+(z)$ as given by formula (18) above. We first examine the relationship between the roots, $1/\epsilon_j$, of (9) and the roots, γ_j, of (20).

Now if $1/\epsilon_j$ is a root of Equation (9), then

$$\epsilon_j^k = [1 + \rho(1 - 1/\epsilon_j)]^r. \tag{21}$$

Let us define

$$\gamma_j = 1 + \rho(1 - 1/\epsilon_j). \tag{22}$$

It follows that

$$\epsilon_j = \rho/(\rho + 1 - \gamma_j) \tag{23}$$

and, from (21),

$$\gamma_j^r = \epsilon_j^k. \tag{24}$$

Now from (23) and (24),

$$\gamma_j^r = [\rho/(\rho + 1 - \gamma_j)]^k.$$

But this shows that γ_j is a root of Equation (20), and moreover, from (24), $|\gamma_j| < 1$.

The relation (22) thus establishes a one-to-one correspondence between the r roots, $1/\epsilon_j$, of (9) outside the unit circle and the r roots, γ_j, of (20) inside the unit circle. Since we have proved that the roots, $1/\epsilon_j$, are simple, it follows that the poles of the integrand in (19) outside C are also simple.

From (22), we have

$$\rho(1/\epsilon_i - 1) = 1 - \gamma_i,$$

and

$$\rho(1/\epsilon_i - 1/\epsilon_j) = \gamma_j - \gamma_i.$$

Therefore,

$$(1 - \epsilon_i)/(1 - \epsilon_i/\epsilon_j) = (1 - \gamma_i)/(\gamma_j - \gamma_i).$$

Now define

$$D_j = (1 - \gamma_j) \prod_{i \neq j} (1 - \gamma_i)/(1 - \gamma_i/\gamma_j).$$

It follows that

$$(25) \qquad C_j/(1 - \epsilon_j) = \gamma_j^{1-r} D_j/(1 - \gamma_j).$$

Analogously to formulae (16) and (17), we can now write

$$(26) \qquad Q(z) = \sum_{j=1}^{r} \frac{1}{2\pi i} \int_C \frac{D_j(1 - z)\, dv}{(1 - \gamma_j v)(1 - v)v(1 - v^{-r}z)}$$

$$(27) \qquad = \sum_{j=1}^{r} \frac{1}{2\pi i} \int_C \frac{D_j}{(1 - \gamma_j v)(v - 1)} \frac{v^{r-1}(1 - v^{-r})}{1 - v^{-r}z}\, dv.$$

Each integrand in (27) has a single pole outside C at $v = 1/\gamma_j$, and the residue is

$$-\gamma_j^{1-r} D_j/(1 - \gamma_j)(1 - \gamma_j^r)/(1 - \gamma_j^r z)$$

which, by using (24) and (25), we can transform to

$$-C_j/(1 - \epsilon_j)(1 - \epsilon_j^k)/(1 - \epsilon_j^k z).$$

Since in (27) the sum of the integrands has its denominator of degree higher by 2 than that of the numerator, it follows that $Q(z) = \sum_{j=1}^{r} C_j/(1 - \epsilon_j)(1 - \epsilon_j^k)/(1 - \epsilon_j^k z)$, which is formula (18) above, and we have proved that, in the system $E_k/E_r/1$, the equilibrium distribution of queue size at instants just before arrivals is identical with that at instants just after departures.

5. Further work. Let $\tau_1, \tau_2, \cdots$ denote the sequence of instants at which units join the batch departures system. In a sequel we shall consider the existence of the limiting distributions, $\{p_j^*\}$ and $\{p_j\}$, defined, respectively, by

$$p_j^* = \lim_{t\to\infty} P[\xi(t) = j]$$

and

$$p_j = \lim_{n\to\infty} P[\xi(\tau_n - 0) = j].$$

We shall also examine the relationships existing between the three distributions

$$\{p_j\}, \{p_j^*\} \quad \text{and} \quad \{p_j^+\}.$$

REFERENCES

[1] N. T. J. BAILEY, "On queueing processes with bulk service," *J. Roy. Stat. Soc.*, Ser. B, Vol. 16 (1954), pp. 80–87.

[2] F. G. FOSTER, "On the stochastic matrices associated with certain queueing processes," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 355–360.

[3] F. G. FOSTER, "A unified theory for stock, storage and queue control," *Operat. Res. Quart.*, Vol. 10 (1959), pp. 121–130.

[4] F. G. FOSTER, "Queues with batch arrivals I," *Acta Math. Acad. Sci. Hung.*, Vol. 12 (1961), pp. 1–10.

[5] R. R. P. JACKSON AND D. G. NICHOLS, "Some equilibrium results for the queueing process $E_k/M/1$," *J. Roy. Stat. Soc.*, Ser. B, Vol. 18 (1956), pp. 275–279.

[6] D. G. KENDALL, "Some problems in the theory of queues," *J. Roy. Stat. Soc.*, Ser. B, Vol. 13 (1951), pp. 151–185.

[7] A. KHINTCHINE, "Mathematical theory of a stationary queue," *Matem. Sbornik*, Vol. 39 (1932), pp. 73–84. (Russian; German summary).