THE TRANSIENT BEHAVIOR OF A SINGLE SERVER QUEUING PROCESS WITH RECURRENT INPUT AND GAMMA SERVICE TIME[1]

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1. Introduction. Let us consider the following queuing process: Customers arrive at a counter at the instants $\tau_0, \tau_1, \cdots, \tau_n, \cdots$ where the interarrival times $\theta_n = \tau_{n+1} - \tau_n$ ($n = 0, 1, \cdots; \tau_0 = 0$) are identically distributed, mutually independent, positive random variables with distribution function $\mathbf{P}\{\theta_n \leqq x\} = F(x)$. We say that $\{\tau_n\}$ is a *recurrent process*. The customers will be served by a single server. The server is idle if and only if there is no customer waiting at the counter, otherwise the order of the services is irrelevant. The service times are identically distributed, mutually independent random variables with the distribution function

$$H_m(x) = \begin{cases} 1 - \sum_{j=0}^{m-1} e^{-\mu x} \frac{(\mu x)^j}{j!} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{1}$$

and independent of $\{\tau_n\}$.

We are interested in the investigation of the stochastic behavior of the queue size and the busy period of this process. We shall see, however, that if we know the stochastic behavior of the process defined below, then that of the above process can be deduced immediately.

To define the second process let us suppose that customers arrive at a counter in batches of size m at the instants $\tau_0, \tau_1, \cdots, \tau_n, \cdots$, where $\{\tau_n\}$ is the recurrent process defined above. There is a single server. The server is idle if and only if there is no customer waiting at the counter, otherwise the order of the services is irrelevant. The service times are identically distributed, mutually independent random variables with the distribution function

$$\mathrm{H}(x) = \begin{cases} 1 - e^{-\mu x} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{2}$$

and independent of $\{\tau_n\}$.

Denote by $\xi(t)$ the queue size at the instant t, i.e., $\xi(t)$ is the number of customers waiting or being served at the instant t. We say that the system is in state E_k at the instant t if $\xi(t) = k$. Further define $\xi_n = \xi(\tau_n - 0)$, i.e., ξ_n is the queue size immediately before the arrival of the nth batch ($n = 0, 1, \cdots$).

Received October 20, 1959; revised May 15, 1961.

[1] An invited paper presented on December 27, 1959 at the Washington meeting of the Institute of Mathematical Statistics. This research was sponsored by the Office of Naval Research under Contract Number Nonr-266 (33), Project Number NR 042-034. Reproduction in whole or in part is permitted for any purpose of the United States Government.

If we identify the arrivals of the batches of size m with the arrivals of individual customers and the total service time of a batch with the service time of an individual customer then the second process reduces to the first one. For, the distribution function of the total service time of a batch in the second process is equal to $H_m(x)$, the mth iterated convolution of $H(x)$ with itself.

If we consider the first process then the busy period follows the same probability law as in the second process, but the queue size will change to $[(\xi(t) + m - 1)/m]$.

Remark. If we suppose in particular that the batches will be served in the order of their arrival and if $\eta(t)$ denotes the virtual waiting time at the instant t, i.e., the time which the first customer in a batch would wait if the batch joined the queue at the instant t, then we have

$$\eta(t) = \sum_{i=1}^{\xi(t)} \chi_i, \tag{3}$$

where $\{\chi_i\}$ is a sequence of identically distributed, independent random variables with distribution function $H(x)$ and independent of $\xi(t)$. In this case the waiting time in the first process follows the same probability law as in the second process.

In what follows we shall consider only the second process and determine the stochastic behavior of the queue size and that of the busy period.

The asymptotic behavior of the queue size and that of the waiting time have been investigated already by F. Pollaczek [4], Chapter 7, D. M. G. Wishart [6], and F. G. Foster [2]. The stochastic law of the busy period has been given by B. W. Conolly [1] and it can be deduced from a general theorem of F. Pollaczek [3].

2. An auxiliary theorem. Denote by

$$\varphi(s) = \int_0^\infty e^{-sx}\, dF(x)$$

the Laplace-Stieltjes transform of $F(x)$ and let

$$\alpha = \int_0^\infty x\, dF(x).$$

Throughout this paper we use

Lemma 1. *If* (a) $\Re(s) \geqq 0$, $|w| < 1$ *or* (b) $\Re(s) > 0$, $|w| \leqq 1$ *or* (c) $\mu\alpha > m$ *and* $\Re(s) \geqq 0$, $|w| \leqq 1$ *then the equation*

$$z^m = w\varphi(s + \mu(1 - z)) \tag{4}$$

has exactly m roots $z = \gamma_r(s, w)$, $r = 1, 2, \cdots, m$, in the unit circle $|z| < 1$. We have

$$\gamma_r(s, w) = \sum_{j=1}^{\infty} \frac{(-\mu\epsilon_r)^j w^{j/m}}{j!} \left(\frac{d^{j-1}[\varphi(s + \mu)]^{j/m}}{ds^{j-1}}\right) \tag{5}$$

where $\epsilon_r = e^{(2\pi r i)/m}$, $r = 1, 2, \cdots, m$, are the mth roots of unity.

Proof. In cases (a) and (b) we have $|w\varphi(s + \mu(1 - z))| < (1 - \epsilon)^m$ if $|z| = 1 - \epsilon$ and ϵ is a sufficiently small positive number. In case (c) we have $\varphi(\mu\epsilon) < (1 - \epsilon)^m$ if ϵ is a sufficiently small positive number. For, if $0 \leqq \epsilon \leqq 1$ then $\varphi(\mu\epsilon)$ and $(1 - \epsilon)^m$ are monotone decreasing functions of ϵ, they agree at $\epsilon = 0$ and their right-hand derivatives at $\epsilon = 0$ are $-\mu\alpha$ and $-m$ respectively. Hence $|w\varphi(s + \mu(1 - z))| \leqq \varphi(\mu\epsilon) < (1 - \epsilon)^m$ if $|z| = 1 - \epsilon$ and ϵ is small enough. That is in each of the three cases $|w\varphi(s + \mu(1 - z))| < (1 - \epsilon)^m$ if $|z| = 1 - \epsilon$ and $\epsilon > 0$ is small enough. Thus it follows by Rouché's theorem that (4) has exactly m roots $z = \gamma_r(s, w)$, $r = 1, 2, \cdots, m$, in the circle $|z| < 1 - \epsilon$ or

$$z = \epsilon_r[w\varphi(s + \mu(1 - z))]^{1/m}, \qquad r = 1, 2, \cdots, m,$$

has exactly one root $z = \gamma_r(s, w)$ in the circle $|z| < 1 - \epsilon$. The explicit form (5) of $\gamma_r(s, w)$ can be obtained by Lagrange's expansion. (Cf. e.g., E. T. Whittaker and G. N. Watson [5] p. 132.) This completes the proof of the lemma.

We note that the roots $z = \gamma_r(s, w)$, $r = 1, 2, \cdots, m$, of the equation (4) are regular functions of s and w and by analytical continuation they can be defined also in case $\mu\alpha \leqq m$ for $\Re(s) \geqq 0$ and $|w| \leqq 1$ without changing (5). We have always $|\gamma_r(s, w)| \leqq 1$, $r = 1, 2, \cdots, m$, if $\Re(s) \geqq 0$ and $|w| \leqq 1$. Note also that (4) has at most one root (possibly multiple) on the unit circle $|z| = 1$, namely $z = 1$ is a root if $w\varphi(s) = 1$. Furthermore $\gamma_r(s, w) = 0$ if and only if $w = 0$. If $w \neq 0$ then the roots $\gamma_r(s, w)$, $r = 1, 2, \cdots, m$, are distinct.

We remark further that by forming the Lagrange expansion of $[\gamma_r(s, w)]^k$, $r = 1, 2, \cdots, m$, we can prove that

$$(6) \qquad \sum_{r=1}^{m} [\gamma_r(s, w)]^k = k \sum_{j \geqq k/m} \frac{w^j \mu^{mj-k}}{j(mj - k)!} \int_0^\infty e^{-(\mu+s)x} x^{mj-k} \, dF_j(x),$$

where $F_j(x)$ denotes the jth iterated convolution of $F(x)$ with itself. By using (6) we can obtain explicit formulas for the probabilities considered in this paper.

Finally, we introduce the following abbreviations: $\gamma_r(s) = \gamma_r(s, 1)$, $g_r(w) = \gamma_r(0, w)$ and $\omega_r = \gamma_r(0, 1)$. They are the roots in z in the unit circle of the equations $z^m = \varphi(s + \mu(1 - z))$, $z^m = w\varphi(\mu(1 - z))$ and $z^m = \varphi(\mu(1 - z))$ respectively.

3. The transient behavior of $\{\xi_n\}$. Define $a^+ = \max(a, 0)$. It is easy to see that

$$(7) \qquad \xi_{n+1} = [\xi_n + m - \nu_n]^+,$$

where $\{\nu_n\}$ is a sequence of identically distributed, mutually independent random variables with the distribution

$$(8) \qquad \mathbf{P}\{\nu_n = j\} = \int_0^\infty e^{-\mu x} \frac{(\mu x)^j}{j!} \, dF(x), \qquad j = 0, 1, \cdots.$$

Accordingly the sequence of random variables $\{\xi_n\}$ forms a homogeneous Markov chain. We say that the system is in state E_k at the nth step if $\xi_n = k$.

The higher transition probabilities

$$p_{ik}^{(n)} = \mathbf{P}\{\xi_n = k \mid \xi_0 = i\}$$

can be obtained by the following

THEOREM 1. *If* $|z| \leqq 1$, $|w| < 1$, *and* $|y| < 1$ *then we have*

$$(9)\quad \begin{aligned}(1 - y)(1 - w) \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p_{ik}^{(n)} y^i z^k w^n &= \prod_{r=1}^{m} \left(\frac{1 - g_r(w)}{1 - z g_r(w)}\right) \\ &- \frac{y(1 - z)(1 - w)}{(1 - zy)[y^m - w\varphi(\mu(1 - y))]} \prod_{r=1}^{m} \left(\frac{y - g_r(w)}{1 - z g_r(w)}\right),\end{aligned}$$

where $g_r(w)$, $r = 1, 2, \cdots, m$ *are the* m *roots in* z *of the equation*

$$(10)\quad z^m = w\varphi(\mu(1 - z))$$

in the unit circle $|z| < 1$.

Instead of proving this theorem we shall prove the more general Theorem 2 from which Theorem 1 can be deduced as a particular case. Theorem 2 determines the joint distribution of τ_n and ξ_n which we need at the investigation of the stochastic law of the busy period. Theorem 2 can be proved in exactly the same way as the more special Theorem 1.

The joint distribution of the random variables τ_n and ξ_n is determined by the probabilities

$$P_{ik}^{(n)}(x) = \mathbf{P}\{\tau_n \leqq x, \xi_n = k \mid \xi_0 = i\}$$

and these probabilities can be uniquely determined by the Laplace-Stieltjes transforms

$$\pi_{ik}^{(n)}(s) = \int_0^{\infty} e^{-sx}\, dP_{ik}^{(n)}(x).$$

These Laplace-Stieltjes transforms are given by

THEOREM 2. *If* $\Re(s) \geqq 0$, $|z| \leqq 1$, $|w| < 1$, *and* $|y| < 1$ *then we have*

$$(11)\quad \begin{aligned}(1 - y)[1 - w\varphi(s)] \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \pi_{ik}^{(n)}(s) y^i z^k w^n &= \prod_{r=1}^{m} \left(\frac{1 - \gamma_r(s, w)}{1 - z\gamma_r(s, w)}\right) \\ &- \frac{y(1 - z)[1 - w\varphi(s)]}{(1 - zy)[y^m - w\varphi(s + \mu(1 - y))]} \prod_{r=1}^{m} \left(\frac{y - \gamma_r(s, w)}{1 - z\gamma_r(s, w)}\right),\end{aligned}$$

where $\gamma_r(s, w)$, $r = 1, 2, \cdots, m$, *are the* m *roots in* z *of the equation*

$$(12)\quad z^m = w\varphi(s + \mu(1 - z))$$

in the unit circle $|z| < 1$.

PROOF. If $w = 0$ then the theorem is obviously true, therefore we suppose that $w \neq 0$. We shall use only the following theorem of the theory of functions of a complex variable: If $f(z)$ is regular for all finite values of z and

$$\lim_{|z| \to \infty} [f(z)/|z|^k] = 0,$$

then $f(z)$ is a polynomial of degree $< k$. If $k = 1$ then $f(z)$ is constant.

Let us introduce the generating function

$$\Pi_i^{(n)}(s, z) = \sum_{j=0}^{\infty} \pi_{ij}^{(n)}(s) z^j$$

which is convergent if $|z| \leqq 1$ and $\Re(s) \geqq 0$. We shall show that if $|z| = 1$ then $\Pi_i^{(n)}(s, z)$, $n = 0, 1, \cdots$, satisfies the following recurrence formula

$$\Pi_i^{(n+1)}(s, z) = z^m \varphi\left(s + \mu\left(1 - \frac{1}{z}\right)\right) \Pi_i^{(n)}(s, z) + \sum_{j=0}^{\infty} C_{ij}^{(n+1)}(s) \left(1 - \frac{1}{z^j}\right), \tag{13}$$

where for every i and n $\sum_{j=0}^{\infty} |C_{ij}^{(n)}(s)| < 1$.

We have

$$\Pi_i^{(n)}(s, z) = \mathbf{E}\{e^{-s\tau_n} z^{\xi_n} \mid \xi_0 = i\}$$

and further

$$\tau_{n+1} = \tau_n + \theta_n \tag{14}$$

and

$$\xi_{n+1} = [\xi_n + m - \nu_n]^+ \tag{15}$$

where $\{\theta_n, \nu_n\}$ is a sequence of independent vector random variables with distributions $\mathbf{P}\{\theta_n \leqq x\} = F(x)$ and

$$\mathbf{P}\{\nu_n = j \mid \theta_n = x\} = e^{-\mu x}[(\mu x)^j / j!], \qquad j = 0, 1, \cdots .$$

By (14) and (15) we obtain (13). The first term on the right hand side of (13) is $\mathbf{E}\{e^{-s\tau_{n+1}} z^{\xi_n + m - \nu_n} \mid \xi_0 = i\}$. To obtain $\mathbf{E}\{e^{-s\tau_{n+1}} z^{\xi_{n+1}} \mid \xi_0 = i\}$ we have to omit from this the terms corresponding to the values $\xi_n + m - \nu_n = -1, -2, \cdots$ and take into consideration that $\xi_{n+1} = 0$ if and only if $\xi_n + m - \nu_n \leqq 0$. Thus we obtain the second term on the right hand side of (13), where

$$C_{ij}^{(n+1)}(s) = \mathbf{P}\{\xi_n + m - \nu_n = -j \mid \xi_0 = i\} \mathbf{E}\{e^{-s\tau_{n+1}} \mid \xi_n + m - \nu_n = -j, \xi_0 = i\}.$$

To obtain (13) we also used the relation

$$\mathbf{E}\{e^{-s\theta_n} z^{-\nu_n}\} = \varphi(s + \mu(1 - (1/z)))$$

if $|z| = 1$.

Now let $\Re(s) \geqq 0$, $|z| \leqq 1$, $|w| < 1$, $|y| < 1$ and define

$$A_i(z, s, w) = \sum_{n=0}^{\infty} \Pi_i^{(n)}(s, z) w^n$$

and

$$A(z, s, w, y) = \sum_{i=0}^{\infty} A_i(z, s, w) y .$$

Clearly

$$A(z, s, w, y) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \pi_{ik}^{(n)}(s) y^i z^k w^n \tag{16}$$

and by definition $A(z, s, w, y)$ is a regular function of z if $|z| \leqq 1$, $\Re(s) \geqq 0$, $|w| < 1$, and $|y| < 1$. If $|z| = 1$ then by (13) we have

$$A_i(z, s, w) = \frac{z^i + \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} C_{ij}^{(n)}(s) w^n (1 - (1/z^j))}{1 - wz^m \varphi(s + \mu(1 - (1/z)))}$$

and hence if $|z| = 1$

$$A(z, s, w, y) = \frac{(1 - zy)^{-1} + \sum_{j=0}^{\infty} C_j(s, w, y)(1 - (1/z^j))}{1 - wz^m \varphi(s + \mu(1 - (1/z)))}, \tag{17}$$

where the coefficients

$$C_j(s, w, y) = \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} C_{ij}^{(n)}(s) w^n y^i, \qquad j = 0, 1, \cdots,$$

satisfy the following condition

$$\sum_{j=0}^{\infty} | C_j(s, w, y) | < |w|/(1 - |w|)(1 - |y|).$$

Now let us define $A(z, s, w, y)$ also for $|z| > 1$ by (17) if $\Re(s) \geqq 0$, $|w| < 1$, and $|y| < 1$. Thus, $A(z, s, w, y)$ has singularities only at $z = 1/y$ and at the zeros of the denominator of (17) outside the unit circle. These zeros evidently agree with the reciprocal values of the roots of (4) inside the unit circle. If we define

$$B(z, s, w, y) = A(z, s, w, y)(1 - zy) \prod_{r=1}^{m} \left(z - \frac{1}{\gamma_r(s, w)}\right) \tag{18}$$

then $B(z, s, w, y)$ will be a regular function of z in the whole complex plane. Since obviously

$$\lim_{|z| \to \infty} [B(z, s, w, y)/|z|^2] = 0$$

therefore $B(z, s, w, y)$ is a linear function of z, that is,

$$B(z, s, w, y) = B_0(s, w, y) + zB_1(s, w, y). \tag{19}$$

$B_0(s, w, y)$ and $B_1(s, w, y)$ can be determined as follows: We have clearly

$$A(1, s, w, y) = \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} [\varphi(s)]^n y^i w^n = \frac{1}{(1 - y)[1 - w\varphi(s)]}$$

and hence by (18)

$$B(1, s, w, y) = \frac{1}{[1 - w\varphi(s)]} \prod_{r=1}^{m} \left(1 - \frac{1}{\gamma_r(s, w)}\right). \tag{20}$$

Further by (17)

$$\lim_{z \to 1/y} (1 - zy)A(z, s, w, y) = y^m/[y^m - w\varphi(s + \mu(1 - y))]$$

and hence by (18)

$$B\left(\frac{1}{y}, s, w, y\right) = \frac{1}{[y^m - w\varphi(s + \mu(1 - y))]} \prod_{r=1}^{m} \left(1 - \frac{y}{\gamma_r(s, w)}\right). \tag{21}$$

Thus (19) is determined by (20) and (21). Finally $A(z, s, w, y)$ can be obtained by (18). So we get (11) which was to be proved. It is to be remarked that in the above proof we did not exploit the fact that the roots $\gamma_r(s, w)$, $r = 1, 2, \cdots, m$, are distinct.

REMARK. If we restrict ourselves to the case $y = 0$ in proving (11) then we have

$$\lim_{|z| \to \infty} [B(z, s, w, 0)/|z|] = 0,$$

i.e., $B(z, s, w, 0)$ is independent of z and thus it is determined by (20). In this case we obtain by (18) that

$$[1 - w\varphi(s)] \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \pi_{0k}^{(n)}(s) z^k w^n = \prod_{r=1}^{m} \left(\frac{1 - \gamma_r(s, w)}{1 - z\gamma_r(s, w)}\right) \tag{22}$$

where $\gamma_r(s, w)$, $r = 1, 2, \cdots, m$ are defined in Theorem 2.

To prove Theorem 1 let us note that $p_{ik}^{(n)} = \pi_{ik}^{(n)}(0)$ and thus if $s = 0$ in (11) then we get (9). In particular if $s = 0$ in (22) then we get

$$(1 - w) \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p_{0k}^{(n)} z^k w^n = \prod_{r=1}^{m} \left(\frac{1 - g_r(w)}{1 - zg_r(w)}\right) \tag{23}$$

where $g_r(w)$, $r = 1, 2, \cdots, m$ are defined in Theorem 1.

4. The limiting distribution of $\{\xi_n\}$. Using Theorem 1 we shall prove

THEOREM 3. *If $\mu\alpha > m$ then the limiting probability distribution*

$$\lim_{n \to \infty} \mathbf{P}\{\xi_n = k\} = P_k, \qquad k = 0, 1, \cdots,$$

exists irrespective of the initial distribution. We have

$$\sum_{k=0}^{\infty} P_k z^k = \prod_{r=1}^{m} \left(\frac{1 - \omega_r}{1 - z\omega_r}\right) \tag{24}$$

where ω_r, $r = 1, 2, \cdots, m$, are the m roots in z of the equation

$$z^m = \varphi(\mu(1 - z)) \tag{25}$$

in the unit circle $|z| < 1$.

PROOF. Since $\{\xi_n\}$ is an irreducible and aperiodic Markov chain, the limit $\lim_{n\to\infty} p_{ik}^{(n)} = P_k$ always exists irrespective of i and either every $P_k > 0$ and $\{P_k\}$ is a probability distribution or every $P_k = 0$. Let $i = 0$. Using (23) by Abel's theorem we get

$$\sum_{k=0}^{\infty} P_k z^k = \lim_{w\to 1} (1 - w) \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p_{0k}^{(n)} z^k w^n = \prod_{r=1}^{m} \left(\frac{1 - \omega_r}{1 - z\omega_r}\right).$$

If $\mu\alpha > m$ then $\{P_k\}$ is a proper probability distribution, because $|\omega_r| < 1$, $r = 1, 2, \cdots, m$. If $\mu\alpha \leqq m$ then $\omega_m = 1$ and therefore $P_k = 0$ for every k.

Another consequence of Theorem 1 is

THEOREM 4. *Denote by $f_{00}^{(n)}$ the probability that in the Markov chain $\{\xi_n\}$ starting from state E_0 the first return occurs at the nth step. If $|w| < 1$ then*

$$\sum_{n=1}^{\infty} f_{00}^{(n)} w^n = 1 - \frac{1 - w}{\prod_{r=1}^{m} [1 - g_r(w)]} \tag{26}$$

where $g_r(w)$, $r = 1, 2, \cdots, m$, are the m roots in z of the equation

$$z^m = w\varphi(s + \mu(1 - z)) \tag{27}$$

in the unit circle $|z| < 1$.

PROOF. By the theory of Markov chains it follows that

$$\sum_{n=1}^{\infty} f_{00}^{(n)} w^n = \frac{\sum_{n=1}^{\infty} p_{00}^{(n)} w^n}{\sum_{n=0}^{\infty} p_{00}^{(n)} w^n}$$

and (26) can be obtained from (23) with $z = 0$.

5. The determination of $F_{ik}^{(n)}(x)$.

Let

$$F_{ik}^{(n)}(x) = \mathbf{P}\{\tau_n \leqq x, \xi_n = k, \xi_{n-1} > 0, \cdots, \xi_1 > 0 \mid \xi_0 = i\}.$$

The Laplace-Stieltjes transform

$$\Phi_{ik}^{(n)}(s) = \int_0^{\infty} e^{-sx}\, dF_{ik}^{(n)}(x).$$

is given by

THEOREM 5. *If $\Re(s) \geqq 0$ and $|w| < 1$ then we have*

$$\sum_{n=1}^{\infty} \Phi_{ik}^{(n)}(s) w^n = \sum_{n=1}^{\infty} \pi_{ik}^{(n)}(s) w^n - \frac{\left(\sum_{n=1}^{\infty} \pi_{i0}^{(n)}(s) w^n\right)\left(\sum_{n=1}^{\infty} \pi_{0k}^{(n)}(s) w^n\right)}{1 + \sum_{n=1}^{\infty} \pi_{00}^{(n)}(s) w^n} \tag{28}$$

where the expressions on the right hand side are defined by (11).

PROOF. By the theorem of total probability we get

$$P_{ik}^{(n)}(x) = F_{ik}^{(n)}(x) + \sum_{j=1}^{n-1} \int_0^x P_{0k}^{(n-j)}(x-y)\, dF_{i0}^{(j)}(y)$$

and forming Laplace-Stieltjes transforms we have

$$\pi_{ik}^{(n)}(s) = \Phi_{ik}^{(n)}(s) + \sum_{j=1}^{n-1} \pi_{0k}^{(n-j)}(s)\Phi_{i0}^{(j)}(s).$$

Hence

$$(29)\quad \sum_{n=1}^{\infty} \pi_{ik}^{(n)}(s)w^n = \sum_{n=1}^{\infty} \Phi_{ik}^{(n)}(s)w^n + \left(\sum_{n=1}^{\infty} \pi_{0k}^{(n)}(s)w^n\right)\left(\sum_{n=1}^{\infty} \Phi_{i0}^{(n)}(s)w^n\right).$$

If $k = 0$ in (29) then we get (28) for $k = 0$, whence (28) follows for every k by (29).

By (28) and (11) we conclude

THEOREM 6. *If* $\Re(s) \geqq 0$ *and* $|w| < 1$ *then we have*

$$(30)\quad \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \Phi_{0k}^{(n)}(s)z^k w^n = \prod_{r=1}^{m}\left(\frac{1}{1 - z\gamma_r(s, w)}\right) - \frac{1 - w\varphi(s)}{\prod_{r=1}^{m} [1 - \gamma_r(s, w)]}$$

where $\gamma_r(s, w)$, $r = 1, 2, \cdots, m$, *are defined in Lemma* 1.

In particular if $z = 0$ in (30) then we have

$$(31)\quad \sum_{n=1}^{\infty} \Phi_{00}^{(n)}(s)w^n = 1 - \frac{1 - w\varphi(s)}{\prod_{r=1}^{m} [1 - \gamma_r(s, w)]}.$$

REMARK. If $F_{00}(x)$ denotes the probability that the distance between two consecutive transitions $E_0 \to E_m$ is $\leqq x$, then we have

$$F_{00}(x) = \sum_{n=1}^{\infty} F_{00}^{(n)}(x),$$

and if $\Phi_{00}(s)$ denotes its Laplace-Stieltjes transform then by (31) we get

$$(32)\quad \Phi_{00}(s) = 1 - \frac{1 - \varphi(s)}{\prod_{r=1}^{m} [1 - \gamma_r(s)]},$$

where $\gamma_r(s) = \gamma_r(s, 1)$, $r = 1, 2, \cdots, m$.

6. The probability law of the busy period. Denote by $G_n(x)$ the probability that a busy period consists in serving n batches and its length is $\leqq x$. Write

$$\Gamma_n(s) = \int_0^{\infty} e^{-sx}\, dG_n(x)$$

if $\Re(s) \geqq 0$. We shall prove

THEOREM 7. *If $\Re(s) \geqq 0$ and $|w| < 1$ then we have*

$$\sum_{n=1}^{\infty} \Gamma_n(s) w^n = 1 - \frac{1 - w\left(\frac{\mu}{\mu + s}\right)^m}{\prod_{r=1}^{m}\left(1 - \frac{\mu}{\mu + s}\gamma_r(s, w)\right)} \tag{33}$$

where $\gamma_r(s, w)$, $r = 1, 2, \cdots, m$, are the m roots in z of (4) in the unit circle $|z| < 1$.

PROOF. If $w = 0$ then (33) is evidently true. Thus we suppose that $w \neq 0$. By the theorem of total probability we can write that

$$G_1(x) = \mu \int_0^x [1 - F(y)] e^{-\mu y} \frac{(\mu y)^{m-1}}{(m-1)!} dy,$$

and if $n = 2, 3, \cdots$, then

$$G_n(x) = \mu \sum_{k=1}^{\infty} \int_0^x F_{0k}^{(n-1)}(x - y)[1 - F(y)] e^{-\mu y} \frac{(\mu y)^{k+m-1}}{(k+m-1)!} dy.$$

Hence

$$\Gamma_1(s) = \mu \int_0^{\infty} e^{-(\mu+s)x} \frac{(\mu x)^{m-1}}{(m-1)!} [1 - F(x)]\, dx$$

and

$$\Gamma_n(s) = \mu \sum_{k=1}^{\infty} \Phi_{0k}^{(n-1)}(s) \int_0^{\infty} e^{-(\mu+s)x} \frac{(\mu x)^{k+m-1}}{(k+m-1)!} [1 - F(x)]\, dx$$

if $n = 2, 3, \cdots$. Forming the generating function of $\{\Gamma_n(s)\}$ we get

$$\sum_{n=1}^{\infty} \Gamma_n(s) w^n = \mu w \sum_{k=0}^{\infty} C_k(s, w) \int_0^{\infty} e^{-(\mu+s)x} \frac{(\mu x)^k}{k!} [1 - F(x)]\, dx, \tag{34}$$

where $C_k(s, w) \equiv 0$ if $k = 0, 1, \cdots, m-2$; $C_{m-1}(s, w) \equiv 1$ and

$$C_{k+m-1}(s, w) = \sum_{n=0}^{\infty} \Phi_{0k}^{(n)}(s) w^n, \qquad k = 1, 2, \cdots.$$

Thus by (30) we have

$$\sum_{k=0}^{\infty} C_k(s, w) z^k = \frac{1}{z} \prod_{r=1}^{m} \left(\frac{z}{1 - z\gamma_r(s, w)}\right). \tag{35}$$

For fixed s and w write

$$f(z) = \prod_{r=1}^{m} \left(\frac{z}{1 - z\gamma_r(s, w)}\right). \tag{36}$$

Then

$$C_k(s, w) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^{k+2}} dz, \qquad k = 0, 1, \cdots,$$

and by (34) we get

$$\sum_{n=1}^{\infty} \Gamma_n(s)w^n = \frac{\mu w}{2\pi i} \oint_{|z|=1} f(z) \frac{\left[1 - \varphi\left(s + \mu\left(1 - \frac{1}{z}\right)\right)\right]}{z[(\mu + s)z - \mu]} dz. \tag{37}$$

We can integrate term by term because the series is uniformly convergent on the circle $|z| = 1$. Now the integral on the right hand side of (37) can be evaluated as $-2\pi i$ times the sum of the residues of the integrand at the poles $z = 1/\gamma_r(s, w)$, $r = 1, 2, \cdots, m$, outside the unit circle. The residue at $z = 1/\gamma_r(s, w)$ depends on the value $\varphi(s + \mu(1 - (1/z)))$ at $z = 1/\gamma_r(s, w)$, but if $z = 1/\gamma_r(s, w)$ then

$$\varphi(s + \mu(1 - (1/z))) = 1/wz^m.$$

Accordingly (37) remains unchanged by the substitution

$$\varphi(s + \mu(1 - (1/z))) = 1/wz^m.$$

Hence

$$\sum_{n=1}^{\infty} \Gamma_n(s)w^n = \frac{\mu}{2\pi i} \oint_{|z|=1} f(z) \frac{(w - (1/z^m))}{z[(\mu + s)z - \mu]} dz. \tag{38}$$

On the other hand this integral can be evaluated as $2\pi i$ times the sum of the residues of the integrand at the poles $z = 0$ and $z = \mu/(\mu + s)$ inside the unit circle. Proceeding in this way, we get

$$\sum_{n=1}^{\infty} \Gamma_n(s)w^n = 1 - \left[\left(\frac{\mu + s}{\mu}\right)^m - w\right] f\left(\frac{\mu}{\mu + s}\right)$$

where $f(z)$ is defined by (36). This completes the proof of the theorem.

We remark that the above proof would be also valid in the case of multiple roots $\gamma_r(s, w)$, $r = 1, 2, \cdots, m$. For, if $z = \gamma_r(s, w)$ is a root of order ν then the residue of the integrand in (37) at $z = 1/\gamma_r(s, w)$ depends on the values

$$\frac{d^i\varphi(s + \mu(1 - (1/z)))}{dz^i} = \frac{d^i[1/wz^m]}{dz^i}, \qquad i = 0, 1, \cdots, \nu - 1,$$

at $z = 1/\gamma_r(s, w)$.

REMARK. Denote by $G(x)$ the distribution function of the length of the busy period and let $\Gamma(s)$ be its Laplace-Stieltjes transform. Evidently $\Gamma(s) = \sum_{n=1}^{\infty} \Gamma_n(s)$ and therefore if $w \to 1$ in (33) we get

$$\Gamma(s) = 1 - \frac{1 - \left(\frac{\mu}{\mu + s}\right)^m}{\prod_{r=1}^{m}\left(1 - \frac{\mu}{\mu + s}\gamma_r(s)\right)}, \tag{39}$$

where $\gamma_r(s) = \gamma_r(s, 1)$, $r = 1, 2, \cdots, m$.

The probability that a busy period consists in serving n batches is $f_{00}^{(n)} = \Gamma_n(0)$ and therefore by (33)

$$\sum_{n=1}^{\infty} f_{00}^{(n)} w^n = 1 - \frac{(1-w)}{\prod_{r=1}^{m} [1 - g_r(w)]} \tag{40}$$

in agreement with (26).

THEOREM 8. *Denote by $P_{00}(t)$ the probability that the server is idle at the instant t given that he was idle at $t = 0$. If $\Re(s) > 0$ then*

$$\int_0^\infty e^{-st} P_{00}(t)\, dt = \frac{1}{s} - \frac{\left[1 - \left(\frac{\mu}{\mu + s}\right)^m\right]}{s[1 - \varphi(s)]} \prod_{r=1}^{m} \left(\frac{1 - \gamma_r(s)}{1 - \frac{\mu}{\mu + s}\gamma_r(s)} \right), \tag{41}$$

where $\gamma_r(s)$, $r = 1, 2, \cdots, m$, are the m roots in z of the equation (4) *in the unit circle.*

PROOF. Clearly $P_{00}(t) = \mathbf{P}\{\xi(t) = 0 \mid \xi(0) = 0\}$. Denote by $M_{00}(t)$ the expectation of the number of transitions $E_0 \to E_m$ occurring in the time interval $[0, t]$, given that $\xi(0) = 0$. Then we can write that

$$P_{00}(t) = 1 - \int_0^t [1 - G(t-x)]\, dM_{00}(x), \tag{42}$$

where

$$M_{00}(t) = I(t) + F_{00}(t) + F_{00}(t) * F_{00}(t) + \cdots$$

and $I(t) = 1$ if $t \geqq 0$, $I(t) = 0$ if $t < 0$. Since

$$\int_0^\infty e^{-st}\, dM_{00}(t) = \frac{1}{1 - \Phi_{00}(s)},$$

we get by (42) that

$$\int_0^\infty e^{-st} P_{00}(t)\, dt = \frac{1}{s}\left[1 - \frac{1 - \Gamma(s)}{1 - \Phi_{00}(s)}\right]$$

where $\Phi_{00}(s)$ is defined by (32) and $\Gamma(s)$ by (39).

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