SUFFICIENCY IN THE UNDOMINATED CASE[1]

By D. L. Burkholder

University of Illinois

1. Introduction and summary. In this paper the concept of statistical sufficiency is studied within a general probability setting. It is not assumed that the family of probability measures is dominated. That is, it is not assumed that there is a σ-finite measure μ such that each probability measure in the family is absolutely continuous with respect to μ. In the dominated case, the theory of sufficiency has received a thorough-going and elegant treatment by Halmos and Savage [6], Bahadur [2], and others. Although many families of probability measures of importance for statistical work are dominated, many others are not. Nonparametric statistical work, especially, abounds with undominated families. It seems appropriate, therefore, to see what can be learned about sufficiency in the undominated case.

Let X be a set, $\mathbf{A}$ a σ-field of subsets of X, and P a family of probability measures p on $\mathbf{A}$. The probability structure $(X, \mathbf{A}, P)$ is to be kept in mind throughout the paper and is unrestricted except where specifically stated to the contrary. Any subfield (= sub-σ-field) entering the discussion is implicitly assumed to be a subfield of $\mathbf{A}$. If H is a collection of subfields, let $\vee H$ denote the smallest σ-field containing each member of H. If $\mathbf{A}_1, \mathbf{A}_2, \cdots$ are subfields, write $\mathbf{A}_1 \vee \mathbf{A}_2$ for $\vee\{\mathbf{A}_1, \mathbf{A}_2\}$, $\mathbf{V}_{n=1}^{\infty} \mathbf{A}_n$ for $\vee\{\mathbf{A}_1, \mathbf{A}_2, \cdots\}$, and so forth. A set N is P-null if N is p-null for each p in P, that is, if N is in $\mathbf{A}$ and $p(N) = 0$, $p \varepsilon P$. If f and g are $\mathbf{A}$-measurable functions, write $f = g[p]$ if the set $\{x \mid f(x) \neq g(x)\}$ is p-null and write $f = g[P]$ if this set is P-null. Let $\mathbf{N}$ be the smallest σ-field containing the P-null sets. If $\mathbf{A}_1$ and $\mathbf{A}_2$ are subfields, write $\mathbf{A}_1 \subset \mathbf{A}_2[P]$ if $\mathbf{A}_1 \subset \mathbf{A}_2 \vee \mathbf{N}$, and so forth. A subfield $\mathbf{B}$ is sufficient if, for each bounded $\mathbf{A}$-measurable function f, there is a $\mathbf{B}$-measurable function g such that $\int_B f\,dp = \int_B g\,dp$, $B \varepsilon \mathbf{B}$, $p \varepsilon P$, that is, such that $g = E_p(f \mid \mathbf{B})[p]$, $p \varepsilon P$. Equivalent definitions are obtained if "bounded $\mathbf{A}$-measurable function" is replaced by "$\mathbf{A}$-measurable characteristic function" or by "P-integrable function." Of course, f is P-integrable if f is $\mathbf{A}$-measurable and $\int_X |f|\,dp$ is finite for each p in P. A subfield $\mathbf{B}$ is separable if it contains a countable subcollection such that $\mathbf{B}$ is the smallest σ-field containing the subcollection.

In Section 2, we give an example of a nonsufficient subfield containing a sufficient subfield. This solves a problem posed by Bahadur (Problem 1 on page 441 of [2]). In fact, we show that often the collection of such nonsufficient subfields is much larger than the collection of sufficient subfields. Analogous results hold for statistics. Some of these and later results depend on Theorem 1 which

Received September 9, 1960.

[1] This research was supported by the National Science Foundation under Grant No. G11382.

gives a necessary condition for a subfield to be sufficient in the case that $\mathbf{A}$ is separable.

Let $\mathbf{A}_1, \mathbf{A}_2, \cdots$ be a sequence of sufficient subfields. Are the subfields $\mathbf{A}_1 \cap \mathbf{A}_2$, $\mathbf{A}_1 \vee \mathbf{A}_2$, $\bigcap_{n=1}^{\infty} \mathbf{A}_n$, and $\bigvee_{n=1}^{\infty} \mathbf{A}_n$, necessarily sufficient? This question is investigated in Sections 3 and 4. Using martingale theory, we show that, if the sequence is decreasing (increasing), then $\bigcap_{n=1}^{\infty} \mathbf{A}_n (\bigvee_{n=1}^{\infty} \mathbf{A}_n)$ is sufficient. If the sequence is not necessarily monotone, it is still possible to show that $\mathbf{A}_1 \cap \mathbf{A}_2$ and $\bigcap_{n=1}^{\infty} \mathbf{A}_n$ are sufficient under a small extra assumption involving $\mathbf{N}$. This result rests on a theorem proved in [3] regarding iterates of conditional expectation operators. One consequence of this result is of interest in connection with the theory of minimal sufficient subfields. It is not necessarily true that $\mathbf{A}_1 \vee \mathbf{A}_2$ is sufficient. This is shown in Example 4. Conditions under which $\mathbf{A}_1 \vee \mathbf{A}_2$ is sufficient are examined.

The main result of Section 5 is related to Theorem 1 and indicates that if $\mathbf{A}$ is separable then each sufficient subfield is essentially equal to one of a very special type.

2. On a problem of Bahadur. In [2], Bahadur proves that if the family P of probability measures on $\mathbf{A}$ is dominated, then a subfield of $\mathbf{A}$ containing a sufficient subfield is sufficient, and lists as an unsolved problem the question of whether this is true in general. That this is not true in general we now show by an example.

EXAMPLE 1. Let X be the set of real numbers, $\mathbf{A}$ the collection of Borel subsets of X, and P the set of probability measures p on $\mathbf{A}$ satisfying $p(A) = p(-A)$ for A in $\mathbf{A}$. Here, if $S \subset X$ then $-S$ is the set $\{x \mid -x \,\varepsilon\, S\}$. Let $A_0 = \{A | A \,\varepsilon\, \mathbf{A}, A = -A\}$. Clearly, $\mathbf{A}_0$ is a subfield of $\mathbf{A}$ and if f is a bounded $\mathbf{A}$-measurable function then $2g(x) = f(x) + f(-x)$ defines an $\mathbf{A}_0$-measurable function g satisfying $\int_A f\, dp = \int_A g\, dp$, $A \,\varepsilon\, \mathbf{A}_0$, $p \,\varepsilon\, P$. Hence, $\mathbf{A}_0$ is sufficient.

Suppose that S is a subset of X satisfying $0 \,\varepsilon\, S$ and $S = -S$. Let

$$\mathbf{B} = \{A \cup A_0 \mid A \subset S, A \,\varepsilon\, \mathbf{A}, A_0 \,\varepsilon\, \mathbf{A}_0\}. \tag{1}$$

Clearly, $\mathbf{B}$ satisfies $\mathbf{A}_0 \subset \mathbf{B} \subset \mathbf{A}$. We now show that $\mathbf{B}$ is a σ-field. It is obvious that the union of a countable family of sets in $\mathbf{B}$ is in $\mathbf{B}$. Let $B \,\varepsilon\, \mathbf{B}$. Then there are sets A and A_0 satisfying $B = A \cup A_0$, $A \subset S$, $A \,\varepsilon\, \mathbf{A}$, $A_0 \,\varepsilon\, \mathbf{A}_0$. Let $C_0 = (-A) \cup A$ and $C = C_0 - A$. Since $S = -S$, $C_0 \subset S$ and therefore $C \subset S$. Using primes to denote complements we have that $B' = A' \cap A'_0 = (C \cup C'_0) \cap A'_0 = (C \cap A'_0) \cup (C'_0 \cap A'_0)$ which is the union of a subset of S in $\mathbf{A}$ and a set in $\mathbf{A}_0$. Therefore, $B' \,\varepsilon\, \mathbf{B}$ and $\mathbf{B}$ is a σ-field.

Suppose that $\mathbf{B}$ is a sufficient subfield. Let f be a bounded $\mathbf{A}$-measurable function. Then, since $\mathbf{B}$ is sufficient, there is a $\mathbf{B}$-measurable function g satisfying

$$\int_B f\, dp = \int_B g\, dp, \qquad B \,\varepsilon\, \mathbf{B}, p \,\varepsilon\, P. \tag{2}$$

Let $x \varepsilon S$. Then $\{x\} \varepsilon \mathbf{B}$ and letting $B = \{x\}$ in (2) gives

$$f(x)p(x) = g(x)p(x), \qquad p \varepsilon P,$$

where we write $p(x)$ for $p(\{x\})$. Let $x \varepsilon X - S$. Then $\{x, -x\} \varepsilon \mathbf{B}$ but neither $\{x\}$ nor $\{-x\}$ belongs to $\mathbf{B}$. Accordingly, $g(x) = g(-x)$, since g is $\mathbf{B}$-measurable. Letting $B = \{x, -x\}$ in (2) and using the fact that $p(x) = p(-x)$ gives

$$[f(x) + f(-x)]p(x) = 2g(x)p(x), \qquad p \varepsilon P.$$

If $x \varepsilon X$, then there is a $p \varepsilon P$ such that $p(x) > 0$. Therefore, we have that

$$(3) \qquad \begin{aligned} g(x) &= f(x) && \text{if } x \varepsilon S, \\ &= \tfrac{1}{2}[f(x) + f(-x)] && \text{if } x \varepsilon X - S. \end{aligned}$$

Let $f(x) = -1$ if $x < 0$, $= 1$ if $x \geqq 0$. Then f is $\mathbf{A}$-measurable and the function g of (3) is $\mathbf{B}$-measurable and satisfies $g(x) \neq 0$ if $x \varepsilon S$, $= 0$ if $x \varepsilon X - S$. Thus, $S = X - g^{-1}(\{0\})$ is in $\mathbf{B}$.

Now choose S to be a subset of X satisfying $0 \varepsilon S$, $S = -S$, and $S \not\varepsilon \mathbf{A}$. Such a set exists, of course. Then, if $\mathbf{B}$ is defined by (1), we see that $\mathbf{B}$ cannot be sufficient by the result of the above paragraph, for S does not belong to $\mathbf{A}$ and therefore does not belong to $\mathbf{B}$.

In summary, a subfield can contain a sufficient subfield and yet not be sufficient. We now prove several results which indicate that the probability structure examined in our example is by no means unusual in this respect.

THEOREM 1. *Suppose that* $\mathbf{A}$ *is separable. If* $\mathbf{B}$ *is a sufficient subfield, then there is a separable sufficient subfield* $\mathbf{B}_0$ *satisfying*

$$\mathbf{B}_0 \subset \mathbf{B} \subset \mathbf{B}_0 \vee \mathbf{N}.$$

We recall that $\mathbf{N}$ is the smallest σ-field containing the P-null sets If the only P-null set is the empty set, then $\mathbf{N} = \{\emptyset, X\} \subset \mathbf{B}_0$ and $\mathbf{B}_0 \vee \mathbf{N} = \mathbf{B}_0$. Accordingly, the following result is an immediate consequence of Theorem 1.

COROLLARY 1. *Suppose that* $\mathbf{A}$ *is separable and the only* P*-null set is the empty set. If* $\mathbf{B}$ *is sufficient, then* $\mathbf{B}$ *is separable.*

PROOF OF THEOREM 1. Since $\mathbf{A}$ is separable, there is a countable field $\mathbf{A}_0$ such that $\mathbf{A}$ is the smallest σ-field containing $\mathbf{A}_0$. Let $\mathbf{B}$ be a sufficient subfield. Then, if $A \varepsilon \mathbf{A}_0$, there is a $\mathbf{B}$-measurable function g_A such that $p(A \cap B) = \int_B g_A \, dp$, $B \varepsilon \mathbf{B}$, $p \varepsilon P$. Let $\mathbf{B}_0$ be the smallest σ-field with respect to which each of the functions g_A, $A \varepsilon \mathbf{A}_0$, is measurable. Since $\mathbf{A}_0$ is countable, it is clear that $\mathbf{B}_0$ is separable. Also, $\mathbf{B}_0 \subset \mathbf{B}$.

Let $\mathbf{A}_1$ be the collection such that $A \varepsilon \mathbf{A}_1$ if and only if $A \varepsilon \mathbf{A}$ and there is a $\mathbf{B}_0$-measurable function g satisfying

$$(4) \qquad p(A \cap B) = \int_B g \, dp, \qquad B \varepsilon \mathbf{B}, p \varepsilon P.$$

Then $\mathbf{A}_0 \subset \mathbf{A}_1 \subset \mathbf{A}$. Clearly, $\mathbf{A}_1$ is a monotone class. Accordingly, $\mathbf{A}_1 = \mathbf{A}$ since

$\mathbf{A}$ is the smallest monotone class containing $\mathbf{A}_0$. From the definition of $\mathbf{A}_1$ and the relation $\mathbf{B}_0 \subset \mathbf{B}$, we conclude that $\mathbf{B}_0$ is sufficient.

We now show that $\mathbf{B} \subset \mathbf{B}_0 \vee \mathbf{N}$. Suppose that $A \varepsilon \mathbf{B}$. Then $A \varepsilon \mathbf{A} = \mathbf{A}_1$ and there is a $\mathbf{B}_0$-measurable function g satisfying (4). In particular,

$$0 = p(A \cap (X - A)) = \int_{X-A} g \, dp, \qquad p \varepsilon P,$$

$$p(A) = p(A \cap A) = \int_A g \, dp, \qquad p \varepsilon P.$$

Therefore, if h is the characteristic function of A, we have that $g = h[P]$, using the fact that $0 \leqq g \leqq 1[P]$, an immediate consequence of (4). Thus, $h - g$ is $\mathbf{N}$-measurable, and $h = g + (h - g)$, being the sum of two $\mathbf{B}_0 \vee \mathbf{N}$-measurable functions, is $\mathbf{B}_0 \vee \mathbf{N}$-measurable. Consequently, $A \varepsilon \mathbf{B}_0 \vee \mathbf{N}$. This completes the proof.

In the following theorem let

$$a_x = \bigcap\{A \mid x \varepsilon A \varepsilon \mathbf{A}\}, \qquad a_{0x} = \bigcap\{A \mid x \varepsilon A \varepsilon \mathbf{A}_0\},$$

where $\mathbf{A}_0$ is a sufficient subfield. Let c be the cardinal number of the set of real numbers, c_0 the cardinal number of the collection of sufficient subfields, and c_1 the cardinal number of the collection of subfields containing $\mathbf{A}_0$ that are not sufficient. Since we now know that $0 < c_1$ is possible, it will not be too surprising to find out that sometimes $c_0 < c_1$.

THEOREM 2. *Suppose that* $\mathbf{A}$ *is separable,* $\mathbf{A}_0$ *is a sufficient subfield, the only* P*-null set is the empty set, and*

$$\text{card}\,\{a_{0x} \mid x \varepsilon X,\, a_x \neq a_{0x}\} \geqq c. \tag{5}$$

Then,

$$c_0 \leqq c < 2^c \leqq c_1.$$

PROOF. By Corollary 1, each sufficient subfield must be separable. Therefore,

$$c_0 \leqq \text{card}\,\{\mathbf{B} \mid \mathbf{B} \text{ is a separable subfield}\}. \tag{6}$$

Since $\mathbf{A}$ is separable, card $\mathbf{A} \leqq c$ (see Problem 9 on page 26 of [5]). There is a one-to-one function from the set of separable subfields of $\mathbf{A}$ to the set of countable subcollections of $\mathbf{A}$. If $\mathbf{B}$ is a separable subfield, the value at $\mathbf{B}$ of this function may be, for example, any particular countable subcollection of $\mathbf{A}$ such that $\mathbf{B}$ is the smallest σ-field containing the subcollection. Since card $\mathbf{A} \leqq c$, the set of countable subcollections of $\mathbf{A}$ has cardinal number less than or equal to c. Thus, the right hand side of (6) is less than or equal to c, implying that $c_0 \leqq c$.

We now show that $2^c \leqq c_2$ where c_2 is the cardinal number of the collection of subfields containing $\mathbf{A}_0$. Consequently, $c_0 < c_2$, $c_2 = c_2 - c_0$, $c_1 = c_2$, and $2^c \leqq c_1$.

Let S be a subset of X such that $X - S$ is the union of some subcollection of $\{a_{0x} \mid x \varepsilon X,\, a_x \neq a_{0x}\}$. Clearly, the collection of such sets S has cardinal number greater than or equal to 2^c, using (5). Let

$$\mathbf{B} = \{A \cup A_0 \mid A \subset S, A \,\varepsilon\, \mathbf{A}, A_0 \,\varepsilon\, \mathbf{A}_0\}. \tag{7}$$

Since $\mathbf{A}$ is separable, if $x \,\varepsilon\, X$ then a_x is the intersection of a countable number of sets in $\mathbf{A}$ and hence is in $\mathbf{A}$. (Note that the partition $\{a_x\}$ of X, induced by $\mathbf{A}$, is also the partition induced by any field with the property that $\mathbf{A}$ is the smallest σ-field containing it. Here, since $\mathbf{A}$ is separable, a countable field with this property exists.) Since $\mathbf{A}_0$ is sufficient, $\mathbf{A}_0$ is also separable. Accordingly, $a_{0x} \,\varepsilon\, \mathbf{A}_0$, $x \,\varepsilon\, X$. From these facts, it follows that

$$S = \{x \mid a_x \,\varepsilon\, \mathbf{B}\}. \tag{8}$$

For if $x \,\varepsilon\, S$ then $a_x \subset a_{0x} \subset S$ and a_x is the union of a subset of S in $\mathbf{A}$, itself, with a set in $\mathbf{A}_0$, the empty set, and hence is in $\mathbf{B}$. If x is not in S then $a_x \neq a_{0x} \subset X - S$, a_x is not in $\mathbf{A}_0$ (for otherwise $a_{0x} = a_x$), and a_x does not have the right form to be a set in $\mathbf{B}$. From (8) it follows that the mapping $S \rightarrow \mathbf{B}$ described in (7) is one-to-one. Thus the cardinal number of the collection of $\mathbf{B}$'s is greater than or equal to 2^c.

Let $\mathbf{B}$ be as in (7). Then $\mathbf{A}_0 \subset \mathbf{B} \subset \mathbf{A}$. It remains to show that $\mathbf{B}$ is a σ-field. Let $B \,\varepsilon\, \mathbf{B}$. Then there are sets A and A_0 satisfying $B = A \cup A_0$, $A \subset S$, $A \,\varepsilon\, \mathbf{A}$, $A_0 \,\varepsilon\, \mathbf{A}_0$. Let $C_0 = \cup\{a_{0x} \mid x \,\varepsilon\, A\}$ and $C = C_0 - A$. If C_0 is in $\mathbf{A}_0$ then $X - B$ is in $\mathbf{B}$ and $\mathbf{B}$ is a σ-field by the same reasoning as in Example 1. We now prove that C_0 is in $\mathbf{A}_0$. Since $\mathbf{A}_0$ is sufficient there is an $\mathbf{A}_0$-measurable function g satisfying

$$p(A \cap a_{0x}) = \textstyle\int_{a_{0x}} g\, dp, \quad x \,\varepsilon\, X, \; p \,\varepsilon\, P. \tag{9}$$

If $x \,\varepsilon\, X$ then g is constant on a_{0x} since g is $\mathbf{A}_0$-measurable and from (9) we have that

$$p(A \cap a_{0x}) = g(x)p(a_{0x}), \qquad p \,\varepsilon\, P.$$

Since the only P-null set is the empty set, if $A \cap a_{0x}$ is empty then $g(x) = 0$ and if $A \cap a_{0x}$ is nonempty then $g(x) > 0$. It is clear from the definition of C_0 that if $x \,\varepsilon\, C_0$ then $A \cap a_{0x}$ is nonempty and if x is not in C_0 then $A \cap a_{0x}$ is empty. Thus, $X - g^{-1}(\{0\}) = C_0$, implying that C_0 is in $\mathbf{A}_0$. This completes the proof.

REMARK 1. In Example 1, $a_x = \{x\}$ and $a_{0x} = \{x, -x\}$, and it is clear that the conditions of Theorem 2 are satisfied. Many probability structures relevant for nonparametric statistical work satisfy the conditions, hence the conclusion, of Theorem 2. Among these, in addition to the one described in Example 1, the following is typical:

EXAMPLE 2. Let n be an integer > 1, X Euclidean n-space, $\mathbf{A}$ the collection of Borel subsets of X, and P the set of all probability measures p on $\mathbf{A}$ of the form $p = q \times \cdots \times q$, where q is a probability measure on the σ-field of Borel subsets of the real line. If $x = (x_1, \cdots, x_n) \,\varepsilon\, X$, let $t_0(x)$ be the set of all points $(x_{i_1}, \cdots, x_{i_n})$, where $(i_1, \cdots, i_n)$ is a permutation of $(1, \cdots, n)$. Let $\mathbf{A}_0$ be the subfield of $\mathbf{A}$ induced by the statistic t_0. That is, $\mathbf{A}_0$ is the collection such that $A \varepsilon \mathbf{A}_0$ if and only if $A \varepsilon \mathbf{A}$ and there is a subset D of the range of t_0 such that $t_0^{-1}(D) = A$. Here, $a_{0x} = t_0(x)$ and $a_x = \{x\}$, and the assumptions of Theorem 2 are satisfied.

REMARK 2. With reference to Example 1, let t_0 and t be functions on X satisfying $t_0(x) = |x|$ if $x \varepsilon X$, $t(x) = x$ if $x \varepsilon S$, $= |x|$ if $x \varepsilon X - S$. The statistics t_0 and t induce the subfields $\mathbf{A}_0$ and $\mathbf{B}$, respectively, of Example 1. Since a statistic is sufficient if and only if its induced subfield is sufficient, we have that t_0 is sufficient but t need not be sufficient. This is in spite of the fact that $t_0 = F(t)$ for some function F.

Or with reference to Theorem 2, let $t_0(x) = a_{0x}$ if $x \varepsilon X$, $t(x) = a_x$ if $x \varepsilon S$, $= a_{0x}$ if $x \varepsilon X - S$, where S is as described in the proof of Theorem 2. One can proceed as in the above paragraph and obtain a similar conclusion.

REMARK 3. Example 1, Theorem 2, and the above remarks indicate that sometimes a nonsufficient subfield or statistic can be as "informative" as a sufficient subfield or statistic. Accordingly, the definition of sufficiency in terms of conditional expectations, like most definitions, does not seem to capture all of the intuitive content commonly associated with the concept being defined. Needless to say, this, in itself, is not necessarily regrettable.

3. Sufficiency in the general case. Throughout this section, except in Example 3, $(X, \mathbf{A}, P)$ is any probability structure. Making no further assumptions, we now prove several results about the sufficient subfields of $\mathbf{A}$. These results are easily shown to be true if P is assumed to be dominated. Without this assumption, these results and their proofs become somewhat more interesting.

THEOREM 3. *Suppose that* $\mathbf{A}_1, \mathbf{A}_2, \cdots$ *are sufficient subfields.*

(i) *If* $\mathbf{A}_1 \supset \mathbf{A}_2 \supset \cdots$, *then* $\bigcap_{n=1}^{\infty} \mathbf{A}_n$ *is sufficient.*

(ii) *If* $\mathbf{A}_1 \subset \mathbf{A}_2 \subset \cdots$, *then* $\bigvee_{n=1}^{\infty} \mathbf{A}_n$ *is sufficient.*

PROOF. Let f be a bounded $\mathbf{A}$-measurable function. There is, for each n, an $\mathbf{A}_n$-measurable function g_n such that $g_n = E_p(f \mid \mathbf{A}_n)[p]$, $p \varepsilon P$. Let $g(x) = \lim_{n\to\infty} g_n(x)$ for all x at which the limit exists, $= 0$ otherwise.

Suppose that $\mathbf{A}_1 \supset \mathbf{A}_2 \supset \cdots$. Then g is $\bigcap_{n=1}^{\infty} \mathbf{A}_n$-measurable. By the continuity theorem for conditional expectations [4, p. 331], $\lim_{n\to\infty} g_n = E_p(f \mid \bigcap_{n=1}^{\infty} \mathbf{A}_n)[p]$, $p \varepsilon P$. Therefore, $g = E_p(f \mid \bigcap_{n=1}^{\infty} \mathbf{A}_n)[p]$, $p \varepsilon P$. Hence, $\bigcap_{n=1}^{\infty} \mathbf{A}_n$ is sufficient. The proof of (ii) is similar.

THEOREM 4. *If* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are sufficient subfields and* $\mathbf{N}$ *is contained in at least one of these subfields, then the subfield* $\mathbf{A}_1 \cap \mathbf{A}_2$ *is sufficient.*

PROOF. Suppose that $\mathbf{A}_1$ and $\mathbf{A}_2$ are sufficient subfields and, without loss of generality, that $\mathbf{N} \subset \mathbf{A}_2$. If n is a positive integer let $\mathbf{A}_{2n-1} = \mathbf{A}_1$ and $\mathbf{A}_{2n} = \mathbf{A}_2$. Let f be a bounded $\mathbf{A}$-measurable function. Define $g_1, g_2, \cdots$ inductively as follows: Let g_1 be an $\mathbf{A}_1$-measurable function satisfying $g_1 = E_p(f \mid \mathbf{A}_1)[p]$, $p \varepsilon P$. If g_{n-1} has been defined, let g_n be an $\mathbf{A}_n$-measurable function satisfying $g_n = E_p(g_{n-1} \mid \mathbf{A}_n)[p]$, $p \varepsilon P$. Such a sequence $g_1, g_2, \cdots$ exists because $\mathbf{A}_1, \mathbf{A}_2, \cdots$ are sufficient subfields. Let $g(x) = \lim_{n\to\infty} g_{2n-1}(x)$ for all x at which the limit exists, $= 0$ otherwise. Let $h(x) = \lim_{n\to\infty} g_{2n}(x)$ for all x at which the limit exists, $= 0$ otherwise. Then g is $\mathbf{A}_1$-measurable and h is $\mathbf{A}_2$-measurable. If p is in P let $\mathbf{A}_{np}$ be the smallest σ-field containing $\mathbf{A}_n$ and the collection of p-null sets. Then, by a theorem proved in [3], $\lim_{n\to\infty} g_n = E_p(f \mid \mathbf{A}_{1p} \cap \mathbf{A}_{2p})[p]$, $p \varepsilon P$, implying that $g = E_p(f \mid \mathbf{A}_{1p} \cap \mathbf{A}_{2p})[p]$, $p \varepsilon P$, and that $\{x \mid g(x) \neq h(x)\}$ is in $\mathbf{N}$. Thus, since $\mathbf{N} \subset \mathbf{A}_2$, we have that $g - h$ is $\mathbf{A}_2$-measurable. Therefore, $g = h + (g - h)$ is the sum of two $\mathbf{A}_2$-measurable functions, hence is $\mathbf{A}_2$-measurable. Since g is measurable with respect to both $\mathbf{A}_1$ and $\mathbf{A}_2$, it is $\mathbf{A}_1 \cap \mathbf{A}_2$-measurable. Moreover, for $p \varepsilon P$,

$$\begin{aligned} g &= E_p(g \mid \mathbf{A}_1 \cap \mathbf{A}_2)[p] \\ &= E_p(E_p(f \mid \mathbf{A}_{1p} \cap \mathbf{A}_{2p}) \mid \mathbf{A}_1 \cap \mathbf{A}_2)[p] \\ &= E_p(f \mid \mathbf{A}_1 \cap \mathbf{A}_2)[p], \end{aligned}$$

since $\mathbf{A}_1 \cap \mathbf{A}_2$ is contained in $\mathbf{A}_{1p} \cap \mathbf{A}_{2p}$. Thus, $\mathbf{A}_1 \cap \mathbf{A}_2$ is sufficient.

COROLLARY 2. *If* $\mathbf{A}_1, \mathbf{A}_2, \cdots$ *are sufficient subfields such that* $\mathbf{N} \subset \mathbf{A}_n$, $n = 1, 2, \cdots$, *then the subfield* $\bigcap_{n=1}^{\infty} \mathbf{A}_n$ *is sufficient.*

PROOF. Let $\mathbf{B}_n = \bigcap_{k=1}^{n} \mathbf{A}_k$. Then, by induction and Theorem 4, each subfield $\mathbf{B}_n$ is sufficient. Applying part (i) of Theorem 3 now gives the desired result.

Consider the following two properties which a sufficient subfield $\mathbf{A}_0$ may or may not have:

I. If $\mathbf{B}$ is a sufficient subfield satisfying $\mathbf{B} \subset \mathbf{A}_0$ then $\mathbf{A}_0 \subset \mathbf{B}[P]$.

II. If $\mathbf{B}$ is a sufficient subfield then $\mathbf{A}_0 \subset \mathbf{B}[P]$.

A sufficient subfield $\mathbf{A}_0$ satisfying (II) is sometimes termed a minimal sufficient subfield. It might be at least as appropriate, however, especially if the discussion is restricted to subfields containing $\mathbf{N}$, to use "least" or "smallest" in place of "minimal" and, instead, apply the adjective "minimal" to any sufficient subfield $\mathbf{A}_0$ satisfying (I). Whether this is true or not hardly matters in the light of the following result:

COROLLARY 3. *If* $\mathbf{A}_0$ *is a sufficient subfield satisfying* (I), *then* $\mathbf{A}_0$ *satisfies* (II).

PROOF. Suppose that $\mathbf{A}_0$ is sufficient and satisfies (I). Let $\mathbf{B}$ be sufficient. Let $\mathbf{A}_1 = \mathbf{B} \vee \mathbf{N}$. It is easy to see that $\mathbf{A}_1$ is sufficient. By Theorem 4, $\mathbf{A}_0 \cap \mathbf{A}_1$ is sufficient, and, therefore, using (I), $\mathbf{A}_0 \subset (\mathbf{A}_0 \cap \mathbf{A}_1) \vee \mathbf{N} \subset \mathbf{A}_1 \vee \mathbf{N} = \mathbf{B} \vee \mathbf{N}$, the desired result.

REMARK 4. The condition involving $\mathbf{N}$ in Theorem 4 cannot be eliminated entirely as the following example shows:

EXAMPLE 3. Let X be Euclidean 2-space, $\mathbf{A}$ the collection of Borel subsets of X, and P the family of all probability measures p on $\mathbf{A}$ satisfying $p(D) = 1$ where

$$D = \{x \mid x = (x_1, x_2) \varepsilon X, x_1 = x_2\}.$$

For $i = 1, 2$, let $\mathbf{A}_i$ be the subfield of $\mathbf{A}$ induced by t_i where $t_i(x) = x_i$, $x \varepsilon X$. It is easy to check that $\mathbf{A}_1$ and $\mathbf{A}_2$ are sufficient but that $\mathbf{A}_1 \cap \mathbf{A}_2 = \{\emptyset, X\}$ is not sufficient.

REMARK 5. The uncountable analogue of Corollary 2 is not true. That is, there can exist a family H of sufficient subfields, each containing $\mathbf{N}$, such that $\bigcap\{\mathbf{B} \mid \mathbf{B} \in H\}$ is not sufficient. Such an example is given by Pitcher [7]. If no such example existed, there would always exist a minimal sufficient subfield, contrary to fact [7].

LEMMA 1. *If* $\mathbf{B}$ *is a sufficient subfield and* A *belongs to* $\mathbf{A}$, *then the smallest* σ*-field containing* $\mathbf{B} \cup \{A\}$ *is sufficient.*

PROOF. Let $\mathbf{C}$ be the smallest σ-field containing $\mathbf{B} \cup \{A\}$. Then

$$\mathbf{C} = \{(B_1 \cap A) \cup (B_2 \cap A') \mid B_i \in \mathbf{B}, i = 1, 2\}$$

where $A' = X - A$. For the purposes of this proof, if h is P-integrable let h' denote any $\mathbf{B}$-measurable function satisfying $h' = E_p(h \mid \mathbf{B})[p]$, $p \in P$. Let $r = 1 - s$ be the characteristic function of A.

We now show that $\mathbf{C}$ is sufficient. Let f be an $\mathbf{A}$-measurable function into [0, 1]. Let

$$g_1(x) = (rf)'(x)/r'(x) \quad \text{if} \quad r'(x) \neq 0,$$
$$= 0 \quad \text{if} \quad r'(x) = 0,$$

and let g_2 be defined similarly using s in place of r. Then g_1 and g_2 are $\mathbf{B}$-measurable and

$$g = rg_1 + sg_2$$

is a $\mathbf{C}$-measurable function. Since $0 \leqq rf \leqq r$, we have that $0 \leqq (rf)' \leqq r'$ $[P]$, $r'g_1 = (rf)'$ $[P]$, and $0 \leqq g_1 \leqq 1$ $[P]$. Similar results hold for s and g_2. Let $C = (B_1 \cap A) \cup (B_2 \cap A')$ where $B_i \in \mathbf{B}$, $i = 1, 2$. Then, for $p \in P$,

$$\int_{B_1 \cap A} g\, dp = \int_{B_1} rg_1\, dp = \int_{B_1} (rg_1)'\, dp = \int_{B_1} r' g_1\, dp$$
$$= \int_{B_1} (rf)'\, dp = \int_{B_1} rf\, dp = \int_{B_1 \cap A} f\, dp.$$

Similarly, $\int_{B_2 \cap A'} g\, dp = \int_{B_2 \cap A'} f\, dp$, $p \in P$. Hence $\int_C f\, dp = \int_C g\, dp$, $p \in P$. The sufficiency of $\mathbf{C}$ follows.

THEOREM 5. *If* $\mathbf{A}_1$ *is a sufficient subfield and* $\mathbf{A}_2$ *is a separable subfield, then* $\mathbf{A}_1 \vee \mathbf{A}_2$ *is sufficient. In particular, if* $\mathbf{B}$ *is a separable subfield containing a sufficient subfield, then* $\mathbf{B}$ *is sufficient.*

PROOF. Let $A_1, A_2, \cdots$ be sets in $\mathbf{A}$ such that $\mathbf{A}_2$ is the smallest σ-field containing $\{A_1, A_2, \cdots\}$. Let $\mathbf{B}_0 = \mathbf{A}_1$ and define $\mathbf{B}_1, \mathbf{B}_2, \cdots$ inductively as follows: If n is a positive integer and $\mathbf{B}_{n-1}$ has been defined, let $\mathbf{B}_n$ be the smallest σ-field containing $\mathbf{B}_{n-1} \cup \{A_n\}$. Using Lemma 1, it follows that each of $\mathbf{B}_1, \mathbf{B}_2, \cdots$ is sufficient. Clearly, $\mathbf{B}_1 \subset \mathbf{B}_2 \subset \cdots$ and $\bigvee_{n=1}^{\infty} \mathbf{B}_n = \mathbf{A}_1 \vee \mathbf{A}_2$. Thus, by Theorem 3, $\mathbf{A}_1 \vee \mathbf{A}_2$ is sufficient. The second assertion of the theorem is an immediate consequence of the first.

4. On the smallest subfield containing two sufficient subfields.

Let $\mathbf{A}_1$ and

$\mathbf{A}_2$ be sufficient subfields containing $\mathbf{N}$. Then $\mathbf{A}_1 \cap \mathbf{A}_2$ is sufficient by Theorem 4. Is $\mathbf{A}_1 \vee \mathbf{A}_2$ also sufficient? It is perhaps somewhat surprising to discover that $\mathbf{A}_1 \vee \mathbf{A}_2$ need not be.

EXAMPLE 4. Let X be the set of all points $x = (x_1, x_2)$ of Euclidean 2-space satisfying $|x_1| = |x_2|$ and $x_1 \neq 0$. Let $r_1(x) = (x_1, -x_2)$, $r_2(x) = (-x_1, x_2)$, $a_{ix} = \{x, r_i(x)\}$, $x \varepsilon X$, and $\mathbf{A}_i$ be the smallest σ-field containing $\{a_{ix} \mid x \varepsilon X\}$, $i = 1, 2$. Let $\mathbf{B} = \mathbf{A}_1 \vee \mathbf{A}_2$, $D = \{x \mid x \varepsilon X, x_1 = x_2\}$, and $\mathbf{A}$ be the smallest σ-field containing $\mathbf{B} \cup \{D\}$. If $x \varepsilon X$, let p_x be the probability measure on $\mathbf{A}$ putting probability $\frac{1}{4}$ on each of the points x, $(x_1, -x_2)$, $(-x_1, x_2)$, $(-x_1, -x_2)$. Finally, let $P = \{p_x \mid x \varepsilon X\}$. The set A is in $\mathbf{A}_i$ if and only if there is a countable set $S \subset X$ such that $\bigcup\{a_{ix} \mid x \varepsilon S\}$ is either A or A', primes being used to denote complements. A subset B of X is in $\mathbf{B}$ if and only if B or B' is countable. Thus, if $x \varepsilon X$ then $\{x\} \varepsilon \mathbf{B}$ but D is not in $\mathbf{B}$. Clearly,

$$\mathbf{A} = \{(B_1 \cap D) \cup (B_2 \cap D') \mid B_i \varepsilon \mathbf{B}, i = 1, 2\}.$$

Here $\mathbf{N} = \{\emptyset, X\}$, hence is contained in any subfield.

Let $i = 1$ or 2. Then $\mathbf{A}_i$ is sufficient, as we now show. Let $f = f_1 + f_2$ where f_1 is the characteristic function of $B_1 \cap D$, f_2 is the characteristic function of $B_2 \cap D'$, and B_1 and B_2 belong to $\mathbf{B}$. Let $g_1 = f_1 + f_1(r_i)$, $g_2 = f_2 + f_2(r_i)$, and $g = (g_1 + g_2)/2$. If B_1 is countable, then $\{x \mid g_1(x) \neq 0\}$ is countable. If B_1' is countable, then $\{x \mid g_1(x) \neq 1\}$ is countable. Therefore, in either case, since $g_1 = g_1(r_i)$, g_1 is $\mathbf{A}_i$-measurable. Similarly, g_2 is $\mathbf{A}_i$-measurable implying that g is $\mathbf{A}_i$-measurable. If $A_i \varepsilon \mathbf{A}_i$ and $p \varepsilon P$, it is clear that $\int_{A_i} f(r_i)\, dp = \int_{A_i} f\, dp$, implying that

$$\int_{A_i} g\, dp = \int_{A_i} \tfrac{1}{2}(f + f(r_i))\, dp = \int_{A_i} f\, dp.$$

Therefore, $\mathbf{A}_i$ is a sufficient subfield.

However, $\mathbf{B} = \mathbf{A}_1 \vee \mathbf{A}_2$ is not sufficient. Otherwise, there would exist a $\mathbf{B}$-measurable function g satisfying $p(D \cap B) = \int_B g\, dp$, $B \varepsilon \mathbf{B}$, $p \varepsilon P$. In particular, $p_x(D \cap \{x\}) = \int_{\{x\}} g\, dp_x$, $x \varepsilon X$, implying that g is the characteristic function of D. This is a contradiction since D is not in $\mathbf{B}$.

REMARK 6. The proof of Theorem 4 was based on a theorem proved in [3] which gives a simple way of obtaining the operator $E_p(\cdot \mid \mathbf{A}_1 \cap \mathbf{A}_2)$ from the operators $E_p(\cdot \mid \mathbf{A}_1)$ and $E_p(\cdot \mid \mathbf{A}_2)$. That there can be no closely analogous result for obtaining $E_p(\cdot \mid \mathbf{A}_1 \vee \mathbf{A}_2)$ from $E_p(\cdot \mid \mathbf{A}_1)$ and $E_p(\cdot \mid \mathbf{A}_2)$ is implied by the above example.

Of course, certain extra assumptions, in addition to the assumption that $\mathbf{A}_1$ and $\mathbf{A}_2$ are sufficient, imply that $\mathbf{A}_1 \vee \mathbf{A}_2$ is sufficient. One such extra assumption is that P be a dominated family of measures. Another is that either $\mathbf{A}_1$ or $\mathbf{A}_2$ be separable (see Theorem 5). Still another is given in the following theorem.

THEOREM 6. *Suppose that* $\mathbf{A}$ *is separable. If* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are sufficient subfields, then* $\mathbf{A}_1 \vee \mathbf{A}_2$ *is sufficient.*

PROOF. By Theorem 1, there are separable sufficient subfields $\mathbf{B}_1$ and $\mathbf{B}_2$ such that

$$\mathbf{B}_i \subset \mathbf{A}_i \subset \mathbf{B}_i \vee \mathbf{N}, \qquad i = 1, 2.$$

Therefore, $\mathbf{B}_1 \vee \mathbf{B}_2$ is sufficient, by Theorem 5, and

$$\mathbf{B}_1 \vee \mathbf{B}_2 \subset \mathbf{A}_1 \vee \mathbf{A}_2 \subset \mathbf{B}_1 \vee \mathbf{B}_2 \vee \mathbf{N},$$

implying that $\mathbf{A}_1 \vee \mathbf{A}_2$ is sufficient.

COROLLARY 4. *Suppose that* $\mathbf{A}$ *is separable. If* $\mathbf{A}_1, \mathbf{A}_2, \cdots$ *are sufficient subfields, then* $\bigvee_{n=1}^{\infty} \mathbf{A}_n$ *is sufficient.*

PROOF. It follows from Theorem 6 that $\bigvee_{k=1}^{n} \mathbf{A}_k$ is sufficient for each positive integer n. By Theorem 3, the desired result follows.

5. Separability and sufficiency. Separability of $\mathbf{A}$ or of one of its subfields plays an important role in Theorems 1, 5, 6, and elsewhere in the above sections. Even so, probably less can be said about sufficiency in the separable case than about sufficiency in the dominated case. Whether or not this is true, it should be kept in mind that nearly all, if not all, of the probability structures of importance in statistical work satisfy the condition that $\mathbf{A}$ is separable, but many do not satisfy the condition that P is dominated.

As usual, let $(X, \mathbf{A}, P)$ be any probability structure. Let $\mathbf{D}$ be the collection of Borel subsets of the real line. If $\mathbf{B}_0$ is a separable subfield of $\mathbf{A}$ then there is an $\mathbf{A}$-measurable function f such that $f^{-1}(\mathbf{D}) = \{f^{-1}(D) \mid D \, \varepsilon \, \mathbf{D}\} = \mathbf{B}_0$. (See Lemma 4 of [1], for example. Bahadur's blanket assumption that X is Euclidean, and so forth, is, of course, not needed and not used in his proof of Lemma 4.) Therefore, as an immediate consequence of Theorem 1, we have the following:

THEOREM 7. *Suppose that* $\mathbf{A}$ *is separable. If* $\mathbf{B}$ *is a sufficient subfield, then there is an* $\mathbf{A}$*-measurable function* f *such that*

$$f^{-1}(\mathbf{D}) = \mathbf{B}[P].$$

This should be compared to a result of Bahadur: If $\mathbf{A}$ is separable, P is dominated, and $\mathbf{B}$ is a subfield, then there is an $\mathbf{A}$-measurable function f such that $f^{-1}(\mathbf{D}) = \mathbf{B}[P]$. (This follows from Lemmas 3 and 4 of [1].) Theorem 7 indicates that if one adds the assumption that $\mathbf{B}$ is sufficient, then one can drop the assumption that P is dominated.

Of course, in Theorem 7 and the above, f could equally well be a measurable transformation into any Euclidean space with $\mathbf{D}$ again denoting the collection of Borel subsets of the space.

REFERENCES

[1] R. R. BAHADUR, "Statistics and subfields," *Ann. Math. Stat.*, Vol. 26 (1955), pp. 490–497.
[2] R. R. BAHADUR, "Sufficiency and statistical decision functions," *Ann. Math. Stat.*, Vol. 25 (1954), pp. 423–462.
[3] D. L. BURKHOLDER AND Y. S. CHOW, "Iterates of conditional expectation operators," *Proc. Amer. Math. Soc.*, Vol. 12 (1961), pp. 490–495.
[4] J. L. DOOB, *Stochastic Processes*, John Wiley and Sons, New York, 1953.
[5] PAUL R. HALMOS, *Measure Theory*, D. Van Nostrand, New York, 1950.
[6] PAUL R. HALMOS AND L. J. SAVAGE, "Application of the Radon-Nikodym theorem to the theory of sufficient statistics," *Ann. Math. Stat.*, Vol. 20 (1949), pp. 225–241.
[7] T. S. PITCHER, "Sets of measures not admitting necessary and sufficient statistics or subfields," *Ann. Math. Stat.*, Vol. 28 (1957), pp. 267–268.