SOME FURTHER DESIGNS OF TYPE O:PP

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0. Summary. A new method of deriving designs of type O:PP is described. The method gives rise to some designs previously obtained by other methods, and also to some entirely new designs. These new designs are described in detail and a worked example given.

1. Introduction. Experimental designs with three non-interacting classifications are most likely to be of use in two cases. The first is when the trial area has two physical configurations, such as rows and columns in a field; the second is when a new set of treatments is added to an existing block design and this set is, by its nature, unlikely to interact with the previous treatments. In either case, orthogonal designs, if available, are the best, but frequently the numbers of treatments and other classifications make complete orthogonality impossible. The problems of experimental design are the same in both cases, and suitable experimental designs using total or partial balance were considered by Hoblyn, *et al.* [4]. In their notation, if an experimental design has three classifications, rows, columns and treatments, such that the rows and columns are orthogonal to each other and the treatments are partially balanced with respect to both rows and columns, then the design is said to be of type O:PP.

Designs of type O:PP were discussed in more detail by Freeman [1], who stated that the only practicable designs are those where the designs of type P have two associate-classes only and these two associate-classes are the same for each P. It appears, however, that useful O:PP designs with two associate-classes can sometimes be derived from two designs of type P without these restrictions. One possibility is that the two designs of type P have two associate-classes each but that these are not the same for rows and columns: another is that the two designs of type P each have three associate-classes. Since designs with two associate-classes can be regarded as special cases of those with three associate-classes in which some of the parameters are equal, the first of these possibilities can perhaps be regarded as included within the second; nevertheless, it is probably better to retain the distinction.

In order that the resultant O:PP design shall be analysable by the methods previously given [1], the two designs of type P have to satisfy certain conditions. In the O:PP design let there be n replicates of t treatments on r rows and c columns such that each treatment occurs either f or $f + 1$ times in rows and either g or $g + 1$ times in columns. Let the designs of type P for rows and columns have three associate-classes, the same for each classification, with n_i members in the ith class. Amongst the extra occurrences let ith associates concur λ_i times

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in rows and μ_i times in columns, and write $\nu_i = r\lambda_i + c\mu_i$. Then, for the design of type O:PP to have only two associate-classes, two of ν_1, ν_2 and ν_3 must be equal. If $\nu_i = \nu_j \neq \nu_k$, where i, j, k are 1, 2, 3 in some order, then, in the O:PP design, any one treatment has $n_i + n_j$ associates of one kind and n_k of the other. Which of these are called first associates and which second is largely a matter of convenience, but the same convention as for partially balanced designs should be used when possible. It is also necessary for the parameters of the second kind in the O:PP design to satisfy the usual equations for a partially balanced design, but these parameters cannot in general be derived directly from those of the two separate designs of type P. When the two designs of type P each have only two associate-classes, these not being the same, this is represented by $\lambda_3 = \lambda_1$, $\mu_3 = \mu_2$.

2. Possible designs. Freeman [3] gave a catalogue of useful designs derived in the orthodox manner, "useful" designs being those having more than two replicates or treatments, not more than 30 replicates, treatments, rows or columns, and not more than 150 plots in all. The method of obtaining designs described here does not lead to many useful new designs, while there are some which have the same parameters as those derived by the old method. When these are mentioned in this section they are numbered as in Freeman [3]. Thus, there is a design with 8 replicates of 18 treatments on 12 rows and columns, $n_1 = 1, n_2 = 8, n_3 = 8, \lambda_1 = 8, \lambda_2 = 6, \lambda_3 = 4, \mu_1 = 8, \mu_2 = 4, \mu_3 = 6$. Hence $\nu_1 = 192$, $\nu_2 = \nu_3 = 120$, and so the design has the same parameters as *SS VI* 1.

There are two designs which may be of more use, and one has already been used to design a trial in the field. Both use the principle of having singular group-divisible designs for each of rows and columns, the resultant O:PP design being of Latin square type. All useful Latin square designs so far discovered, whether obtained by this method or the orthodox one, have equal numbers of rows and columns. They are shown in Table I, which thus repeats some of the information given previously [3]. The horizontal line through the middle of Table I separates designs derived by the two methods. Above the line $\nu_1 = r\lambda_1 + c\mu_1$, $\nu_2 = r\lambda_2 + c\mu_2$, as usual, while below the line $\nu_1 = r\lambda_1 + c\mu_1 = r\lambda_2 + c\mu_2$, $\nu_2 = r\lambda_3 + c\mu_3$, that is, $\nu_2 = r\lambda_1 + c\mu_2$.

TABLE I

Useful designs in family LL

Design	m	w	λ_1	λ_2	λ_3	μ_1	μ_2	μ_3	ν_1	ν_2	Rep	Tr	Row	Col
LL 1	2	3	3	2	—	3	2	—	36	24	4	9	6	6
LL 2	2	3	3	2	—	2	3	—	30	30	4	9	6	6
LL 3	3	4	6	7	—	6	7	—	144	168	9	16	12	12
LL 4	3	4	6	9	6	9	6	6	180	120	9	16	12	12
LL 5	2	5	1	4	1	4	1	1	50	20	4	25	10	10

The most general Latin square design with equal numbers of rows and columns has m^2 replicates of w^2 treatments on mw rows and columns. When the designs of type P are singular group-divisible a given treatment in them has $w - 1$ first associates and $w(w - 1)$ second associates. Further, if $fm < w < (f + 1)m$, $\lambda_1 = m^2 - fmw$, and so

$$\lambda_2 = \frac{(m^2 - fmw)(m - fw - 1)}{w - 1},$$

in order to satisfy the usual constraint on parameters of the first kind. λ_2 has to be integral, and this condition imposes some limitation on the possible values of m and w. All singular group-divisible designs are derived from balanced incomplete block designs by replacing one treatment of the balanced design by a group of treatments, λ_2 in the partially balanced design equalling λ in the balanced design; it is easy to construct corresponding O:PP designs, which are very numerous. Thus, any balanced incomplete block design with m replicates and plots per block and w treatments and blocks gives rise to a design with the required parameters. The smallest resultant O:PP design has 4 replicates of 9 treatments on 6 rows and columns, but this has the same parameters as *LL* 1. The next in this series has 9 replicates of 16 treatments on 12 rows and columns, and is shown as *LL* 4 in Table I. There is a group-divisible O:PP design, *SS I* 15, with 9 replicates of 16 treatments, in 4 groups of 4, on 12 rows and columns, but this has all other parameters different. Also, the design shown as *LL* 3, with the same values of m and w, which is derived by orthodox methods, is new, although it should have been included in the 1958 paper [3].

Another set of designs arises from the balanced incomplete block designs with m^2 replicates of w treatments on mw blocks of m plots each. The smallest design in this series has $m = 2$, $w = 5$, and gives rise to *LL* 5. This is particularly noteworthy in that no other O:PP design is possible with these numbers of rows, columns, treatments and replicates. It seems at first sight that there should be a singular group-divisible O:PP design using the same designs of type P for rows and columns, but this design is excluded by Theorem 1 [2]. This same Theorem excludes singular group-divisible designs with the same parameters as other Latin square designs in this series, for example, those with $m = 3$, $w = 10$, and $m = 3$, $w = 19$.

When a $w \times w$ factorial experiment is to be laid out in rows and columns, a Latin square O:PP design derived from two singular designs of type P may be particularly suitable. The main effects of the two factors can be associated with the rows and the columns, and the corresponding sums of squares in the analysis take a fairly simple form. Competing designs for the same numbers of treatments and replicates will include lattice squares, but the two designs may well require different numbers of rows and columns, which are often pre-determined. Further, two error variances have to be calculated in a lattice square, and only one in an O:PP design. The example which follows illustrates the method of analysis for a general O:PP design, with the modification required for a $w \times w$ factorial experiment.

3. Example. The design *LL* 5 has been used for a trial on yams (*Dioscorea* sp.) conducted by Mr. E. F. I. Baker, Research Division, Ministry of Agriculture and Natural Resources, Western Region of Nigeria. Yams are grown from setts, pieces of root tuber with adventitious buds, and the purpose of this trial was to find the effect of planting setts of different weights at varying populations per acre. A 5 × 5 factorial system was used, the levels of one factor being sett weights per acre and of the other factor populations per acre. Representing the 25 treatment combinations by

$$\begin{matrix} A & F & K & P & U \\ B & G & L & Q & V \\ C & H & M & R & W, \\ D & I & N & S & X \\ E & J & O & T & Y \end{matrix}$$

the rows were lettered *a*, *b*, *c*, *d*, *e* and represented sett weights per acre, and the columns, numbered 1, 2, 3, 4, 5, were populations per acre. Furthermore, the treatments down the leading diagonal, *A*, *G*, *M*, *S*, *Y* all had the same individual sett weight, though varying in population and weight per acre. The lay-out of the O:PP design in the field, after randomisation of rows and columns, was that shown in Table II.

TABLE II

E	*T*	*J*	*D*	*S*	*Y*	*X*	*N*	*O*	*I*	*d e*
X	*N*	*C*	*R*	*I*	*S*	*W*	*M*	*D*	*H*	*c d*
D	*S*	*I*	*A*	*F*	*P*	*U*	*X*	*N*	*K*	*a d*
V	*Q*	*D*	*S*	*G*	*X*	*I*	*L*	*B*	*N*	*b d*
U	*R*	*F*	*C*	*P*	*W*	*H*	*K*	*A*	*M*	*a c*
C	*L*	*B*	*Q*	*H*	*R*	*V*	*W*	*M*	*G*	*b c*
B	*K*	*A*	*P*	*Q*	*V*	*G*	*U*	*L*	*F*	*a b*
A	*P*	*E*	*T*	*J*	*U*	*F*	*Y*	*K*	*O*	*a e*
Y	*O*	*G*	*B*	*T*	*Q*	*J*	*V*	*E*	*L*	*b e*
W	*M*	*H*	*E*	*R*	*T*	*Y*	*O*	*C*	*J*	*c e*
1 5	3 4	1 2	1 4	2 4	4 5	2 5	3 5	1 3	2 3	

It is seen that all sett weights per acre in two populations were arranged in each column, these being shown as 1 5, 3 4, etc., while all populations with two sett weights per acre occurred in each row, these being *d e*, *c d*, etc. First associates of any treatment were thus those treatments with either the same population or the same sett weight per acre.

The analysis follows the lines previously given [1], the notation used being the same. Thus, in the O:PP design,

$$p_{ij}^{1} = \begin{pmatrix} 3 & 4 \\ 4 & 12 \end{pmatrix}, \qquad p_{ij}^{2} = \begin{pmatrix} 2 & 6 \\ 6 & 9 \end{pmatrix}, \qquad N = n(cr - r - c) = 320.$$

If treatment, row and column totals are represented by *D*, *B* and *C* respectively, G being the grand total, the treatment total for A adjusted for rows and columns is P_A, where

$$P_A = 100T_A - 10(B_3 + B_5 + B_7 + B_8) - 10(C_1 + C_3 + C_4 + C_9) + 4G, \text{ etc.}$$

Further, $A_{12} = 340$, $B_{12} = -30$, $A_{22} = -180$, $B_{22} = 310$, $\Delta = 100000$. Then, the treatment parameter for A is $\delta_A = (31P_A + 3\sum P_{A1})/10000$, where $\sum P_{A1}$ is the sum of the P's for the first associates of A.

The treatment sum of squares is $\sum \delta P/100$, and row and column sums of squares are unadjusted.

The variance of the difference between the means of two treatments that are ith associates is obtained by multiplying the error variance by ψ_i^2, where $\psi_1^2 = \frac{14}{25}$, $\psi_2^2 = \frac{31}{50}$.

In this trial the main comparison of importance was between first associates, these being particular levels of one factor for a given level of the other. The most important comparison for second associates was that among the treatments A, G, M, S, Y with the same individual sett weight. It was also necessary to consider the main effects of populations and sett weights per acre. The sum of squares for the main effect of populations has the simple form

$$\sum (P_A + P_B + P_C + P_D + P_E)^2/125000$$

minus the correction factor, and similarly for sett weights. The multiplying factor for the error variance when comparing the means of any two levels of one factor, taken over all levels of the other, is $\frac{4}{25}$.

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