SOME MAIN-EFFECT PLANS AND ORTHOGONAL ARRAYS OF STRENGTH TWO[1]

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1. Summary. In this paper we present a method of constructing main-effect plans for symmetrical factorial experiments which can accommodate up to $[2(s^n - 1)/(s - 1) - 1]$ factors, each at $s = p^m$ levels, where p is a prime, with $2s^n$ treatment combinations. As main-effect plans are orthogonal arrays of strength two the method presented permits the construction of the orthogonal arrays $(2s^n, 2[s^n - 1]/[s - 1] - 1, s, 2)$.

2. Introduction. Let there be k factors each of which can assume $s = p^m$ levels, where p is a prime number. An orthogonal array of strength d, of size N, with k constraints and s levels consists of a subset of N treatment combinations from an s^k factorial experiment with the property that all s^d treatment combinations corresponding to any d factors chosen from the k occur an equal number of times in the subset. The array may be denoted by (N, k, s, d).

The concept of orthogonal arrays was first introduced by Rao [1]. He discussed the use of these arrays as fractionally replicated plans for symmetrical factorial experiments which permit the estimation of main-effects and interactions up to order $(d - 2)$ when higher order interactions are negligible.

The plans for fractionally replicated symmetrical factorial experiments which are developed in this paper are orthogonal arrays of strength two. We call these plans main-effect plans because they permit orthogonal estimation of all the main-effects when the interactions are negligible.

The main-effect plans derivable from the system of confounding developed by Fisher [2] can be represented by the orthogonal arrays $(s^n, (s^n - 1)/(s - 1), s, 2)$. These plans fall within the class of optimum multifactorial designs which were considered by Plackett and Burman [3].

It has been shown by Bose [4] that the maximum number of factors that it is possible to accommodate in a symmetrical factorial experiment in which each factor occurs at $s = p^m$ levels and each block is of size s^n, without confounding any d-factor or lower order interaction, is given by the maximum number of points that it is possible to choose in the finite projective geometry PG $(n - 1, p^m)$ so that no d of the chosen points are conjoint. This is equivalent to showing that the maximum number of constraints k in the orthogonal array (s^n, k, s, d) is given by the maximum number of points it is possible to choose in

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PG $(n - 1, p^m)$ so that no d of the chosen points are conjoint. Clearly the maximum number of constraints in the orthogonal array $(s^n, k, s, 2)$ is equal to the number of points of PG $(n - 1, p^m)$. Thus the maximum value of k is $(s^n - 1)/(s - 1)$. These facts are relevant in view of the method of construction to be presented.

3. Preliminary notation and lemmas. The finite projective geometry PG $(n - 1, p^m)$ is a geometrical representation of n factors each at $s = p^m$ levels and their generalized interactions. We shall represent these n factors by $X_1, X_2, \cdots, X_n$ and their generalized interactions by $k_1X_1 + k_2X_2 + \cdots + k_nX_n$ where the k_i can take on any value of the Galois field GF (p^m) and it is understood that the coefficient of the first factor appearing in an interaction is unity.

Let $u_0, u_1, \cdots, u_{s-1}$ represent the elements of GF (p^m) and let $u_0^2, u_1^2, \cdots, u_{s-1}^2$ represent the squares of the elements of GF (p^m). We shall denote the set of squared elements of GF (p^m) by GF^2 (p^m). It is easily verified that apart from the 0 element the set GF^2 (p^m) forms a cyclic Abelian group under multiplication. It follows from the cyclic property that (i) when $p = 2$, GF^2 (p^m) contains each of the elements of GF (p^m) and (ii) when p is an odd prime, the elements of GF^2 (p^m) comprise a subset of $\frac{1}{2}(s + 1)$ distinct elements of GF (p^m), where one element occurs once and $\frac{1}{2}(s - 1)$ elements are duplicated.

Consider one of the factors X_i in a main-effect plan in which each X_i has s levels, each occurring s^{n-1} times in a total of s^n treatment combinations. Let X_i^2 be a pseudo-factor obtained by squaring the levels of X_i. We now present the following lemmas:

LEMMA 1. *When p is an odd prime, $X_i^2 + kX_i$ (k an element of* GF (p^m)) *contains $\frac{1}{2}(s + 1)$ distinct levels, one level occurring s^{n-1} times and $\frac{1}{2}(s - 1)$ levels occurring $2s^{n-1}$ times in s^n treatment combinations.*

LEMMA 2. *When $p = 2$, X_i^2 contains each of the s levels s^{n-1} times.*

LEMMA 3. *When $p = 2$, $X_i^2 + kX_i$ (k any element of* GF (p^m) *except* 0) *contains $\frac{1}{2}s$ distinct levels each occurring $2s^{n-1}$ times.*

Lemma 3 can be proved as follows. Let x_i range over the elements of GF (p^m) which represent the s levels of X_i. As x_i ranges over the elements of the field so does $x_i + k$ where k is an element of GF (p^m). Also if $x_i + k = x_j \pmod 2$ then $x_j + k = x_i \pmod 2$. Hence $x_i(x_i + k) = x_ix_j$ and $x_j(x_j + k) = x_ix_j$. Thus whatever values of $x_i(x_i + k)$ are achieved they are achieved for at least two values of x_i.

It will now be shown that the values of $x_i(x_i + k)$ are achieved for exactly two values of x_i. Let y be the generator of the field and let $x_i = y^\alpha$ and $k = y^\beta$. Thus $x_i(x_i + k) = y^\alpha(y^\alpha + y^\beta)$. Suppose that

$$y^\alpha(y^\alpha + y^\beta) = y^\gamma(y^\gamma + y^\beta)$$

where

$$y^\alpha \neq y^\gamma \quad \text{and} \quad y^\alpha + y^\beta \neq y^\gamma.$$

Hence

$$(y^\alpha)^2 + y^\alpha y^\beta = (y^\gamma)^2 + y^\gamma y^\beta$$
$$(y^\alpha + y^\gamma)^2 + (y^\alpha + y^\gamma)y^\beta = 0$$
$$(y^\alpha + y^\gamma)(y^\alpha + y^\gamma + y^\beta) = 0.$$

This implies that either $y^\alpha + y^\gamma = 0$ and therefore $y^\alpha = y^\gamma$ which is a contradiction or that $y^\alpha + y^\gamma + y^\beta = 0$ and therefore $y^\alpha + y^\beta = y^\gamma$ which is a contradiction. Hence the values of $x_i(x_i + k)$ are achieved for exactly two values of x_i and Lemma 3 is proved.

Lemma 4. *The factor represented by* $X_i^2 + k_iX_i + \sum_{j\neq i} k_jX_j$, ($k_i$ *and* k_j *elements of* GF (p^m)) *where at least one* $k_i \neq 0$, *contains each of the s levels* s^{n-1} *times.*

Lemma 5. *The levels of* $X_i^2 + k_1X_i + k_2X_j$ *which occur in a plan with the* u_t *level of* $a_1X_i + a_2X_j$, *where* k_1, k_2, a_1 *and* a_2 *are elements of* GF (p^m) *and* $a_2 \neq 0$ *are given by the values of* $x_i^2 + k_1x_i + k_2x_j + c(a_1x_i + a_2x_j) - cu_t$ *where* $k_2 + ca_2 = 0$ *and* x_i *ranges over the elements of* GF (p^m).

Proof. When $a_1X_i + a_2X_j$ takes on the u_t level then $a_1x_i + a_2x_j = u_t$ and thus

$$x_j = (u_t - a_1x_i)/a_2 .$$

Hence the levels of the factor $X_i^2 + k_1X_i + k_2X_j$ which occur with the level u_t of $a_1X_i + a_2X_j$ can be represented by

$$x_i^2 + k_1x_i + k_2x_j = x_i^2 + k_1x_i + k_2(u_t - a_1x_i)/a_2$$
$$= x_i^2 + (k_1 - k_2a_1/a_2)x_i + (k_2/a_2)u_t .$$

Since $k_2 + ca_2 = 0$, then $c = -k_2/a_2$. Thus

$$x_i^2 + (k_1 - k_2a_1/a_2)x_i + (k_2/a_2)u_t = x_i^2 + k_1x_i + k_2x_j + c(a_1x_i + a_2x_j) - cu_t ,$$

and the lemma is proved.

Two factors X_i and X_j are said to be orthogonal to each other if each level of X_j occurs the same number of times with every level of X_i. Two factors X_i and X_j are said to be semi-orthogonal to each other if (i) for p an odd prime, one level of X_j occurs s^{n-2} times and $\frac{1}{2}(s - 1)$ levels of X_j each occur $2s^{n-2}$ times with each level of X_i and (ii) for $p = 2$, $\frac{1}{2}s$ levels of X_j each occur $2s^{n-2}$ times with each level of X_i.

It follows from Lemmas 1, 3, and 5 that when p is an odd prime or when $k_1 - k_2a_1/a_2 \neq 0$, then $a_1X_i + a_2X_j$ is semi-orthogonal to $X_i^2 + k_1X_i + k_2X_j$. It follows from Lemmas 2 and 5 that when $p = 2$ and $k_1 - k_2a_1/a_2 = 0$ then $a_1X_i + a_2X_j$ is orthogonal to $X_i^2 + k_1X_i + k_2X_j$. Employing an argument similar to that used in Lemma 5 it can be deduced that $kX_i^2 + k_1X_i + X_j$ and $kX_i^2 + k_2X_i + X_j$ are orthogonal to each other when $k_1 \neq k_2$.

Lemma 5 can be generalized to include more than two factors as stated in Lemma 5a.

LEMMA 5a. *The levels of* $X_i^2 + k_iX_i + \sum_{j\neq i} k_jX_j$ *which occur in a plan with the* u_t *level of* $a_iX_i + \sum_{j\neq i} a_jX_j$ *are given by the values of*

$$x_i^2 + k_ix_i + \sum_{j\neq i} k_jx_j + c(a_ix_i + \sum_{j\neq i} a_jx_j) - cu_t$$

where $k_j + ca_j = 0$ *for all* $j \neq i$. *If the* a_j *and the* k_j *are not of such a form that* $k_j + ca_j = 0$ *for all* $j \neq i$ *and some* c *contained in* GF (p^m) *then the two factors are orthogonal.*

LEMMA 6. *When* p *is a prime the complements in* GF (p^m) *to the elements in* $\mathrm{GF}^2(p^m)$ *are the set of elements in* $\mathrm{GF}^2(p^m)$ *each multiplied by an element of* $\mathrm{GF}(p^m)$ *which is not an element of* $\mathrm{GF}^2(p^m)$. *If the set of elements in* $\mathrm{GF}^2(p^m)$ *and their set of complements are taken together in one set, the elements of* $\mathrm{GF}(p^m)$ *are obtained.*

PROOF. From abstract group theory (see Birkhoff and Mac Lane [5]) we employ a lemma which states that two right cosets of a subgroup are either identical or without common elements. Now the elements of $\mathrm{GF}^2(p^m)$ form an Abelian subgroup of the elements of $\mathrm{GF}(p^m)$. Hence multiplying each element of $\mathrm{GF}^2(p^m)$ by an element of $\mathrm{GF}(p^m)$ which is not an element of $\mathrm{GF}^2(p^m)$ yields the complementary set to $\mathrm{GF}^2(p^m)$.

It is clear from Lemma 2 that when $p = 2$ the set complementary to $\mathrm{GF}^2(p^m)$ is the null set.

4. Construction of main-effect plans.

THEOREM 1. *There exists a main-effect plan for* $[2(s^n - 1)/(s - 1) - 1]$ *factors, each at* $s = p^m$ *levels, with* $2s^n$ *treatment combinations.*

PROOF. In order to facilitate the presentation of the proof of Theorem 1, let $n = 2$. First construct an orthogonal main effect plan for $(s^2 - 1)/(s - 1)$ factors each at s levels in s^2 trials, represented by the two factors X_1 and X_2 and their generalized interactions $X_1 + X_2$, $X_1 + 2X_2$, $\cdots$, $X_1 + (s - 1)X_2$, where the coefficients 1, 2, $\cdots$, $(s - 1)$ are elements of $\mathrm{GF}(p^m)$, addition and multiplication being performed within this field. To these add

$$[(s^2 - 1)/(s - 1) - 1]$$

factors represented by

$$X_1^2 + X_2\,, X_1^2 + X_1 + X_2\,, X_1^2 + 2X_1 + X_2\,, \cdots, X_1^2 + (s - 1)X_1 + X_2\,.$$

These $[2(s^n - 1)/(s - 1) - 1]$ factors in s^n observations represent the first half of the main-effect plan.

Note from the preceeding lemmas that when p is a prime number, $X_1 + a_iX_2$ and $X_1^2 + k_iX_1 + X_2$ are semi-orthogonal and also that X_2 and $X_1^2 + k_iX_1 + X_2$ are semi-orthogonal for all a_i and k_i in GF (p^m) except $a_i = 0$. All other pairs of factors are clearly orthogonal. If $p = 2$ and $(k_i - a_1/a_i) = 0$, then $a_1X_1 + a_iX_2$ and $X_1^2 + k_iX_1 + X_2$ are orthogonal.

The second half of the plan is chosen so that the pairs of factors which are orthogonal in the first half are also orthogonal in the second half and pairs of factors which are semi-orthogonal in the first half are semi-orthogonal in a complementary manner in the second half. The factors in the second half which correspond to the factors of the first half can be denoted by

$$X_1, X_2, X_1 + X_2 + b_1, X_1 + 2X_2 + b_2, \cdots, X_1 + (s-1)X_2 + b_{s-1},$$
$$kX_1^2 + X_2, kX_1^2 + k_1X_1 + X_2 + c_1,$$
$$kX_1^2 + k_2X_1 + X_2 + c_2, \cdots, kX_1^2 + k_{(s-1)}X_1 + X_2 + c_{s-1}$$

where the coefficients $b_1, b_2, \cdots, b_{s-1}, k, k_1, k_2, \cdots, k_{s-1}, c_1, c_2, \cdots, c_{s-1}$, which are to be determined, are elements of $\text{GF}(p^m)$.

From Lemma 5, it is seen that the levels of $X_1^2 + X_2$ which occur with the u_t level of X_2 are given by the values of $x_1^2 + u_t$ where x_1 takes on the values of the elements of GF (p^m). Without loss of generality we may let $u_t = u_0 = 0$. When p is an odd prime, the values of $kX_1^2 + X_2$, where k is an element of GF (p^m) but not an element of GF^2 (p^m), which occur with the $u_t = 0$ level of X_2 are given by the values of kx_1^2. As shown in Lemma 6, kx_1^2 complements x_1^2.

Thus, when p is an odd prime k can take on the value of any element in $\text{GF}(p^m)$ which is not an element of $\text{GF}^2(p^m)$. If $p = 2$ it is clear from Lemma 2 that $k = 1$.

A method for determining the constants $b_1, b_2, \cdots, b_{s-1}, k_1, k_2, \cdots, k_{s-1}, c_1, c_2, \cdots, c_{s-1}$, when $s = p^m$ and p is an odd prime is now presented. In order that the levels of $kX_1^2 + X_2$ which occur with the 0 level of $X_1 + a_iX_2 + b_i$ be the complements of the levels of $X_1^2 + X_2$ which occur with the 0 levels of $X_1 + a_iX_2$, b_i must be such that the values which $kx_1^2 - (1/a_i)x_1 - b_i/a_i$ takes when x_1 ranges over the field GF (p^m) complements the values which $x_1^2 - (1/a_i)x_1$ takes. Now $x_1^2 - (1/a_i)x_1$ consists of one element of GF (p^m) occurring once and $\frac{1}{2}(s-1)$ elements occurring twice. Let the unique element of GF (p^m) be u_1. Then $x_1^2 - (1/a_i)x_1 = u_1$ must have only one solution as x_1 ranges over the elements of GF (p^m). Thus $1/a_i^2 + 4u_1 = 0$ and hence $4u_1 = -1/a_i^2$. Since $kx_1^2 - (1/a_i)x_1 - b_i/a_i$ must complement $x_1^2 - (1/a_i)x_1$, the equation

$$kx_1^2 - (1/a_i)x_1 - b_i/a_i = u_1$$

must also have only one solution. Therefore

$$1/a_i^2 + 4k(b_i/a_i + u_1) = 0.$$

Substituting $4u_1 = -1/a_i^2$ in this equation and solving for b_i we get

$$b_i = (k-1)/4ka_i. \tag{1}$$

To find the levels of $X_1^2 + d_iX_1 + X_2$ which occur with the 0 levels of X_2 note that there exists an element of GF (p^m), u_2 say, such that $x_1^2 + d_ix_1 = u_2$ has only one solution.

Thus $d_i^2 + 4u_2 = 0$ and hence $4u_2 = -d_i^2$. In order that the levels of $kX_1^2 + k_iX_1 + X_2 + c_i$ which occur with the 0 levels of X_2 complement those given by

$x_1^2 + d_i x_1$, then $kx_1^2 + k_i x_1 + c_i = u_2$ must have only one solution. Substituting $4u_2 = -d_i^2$ in this equation and solving for c_i we get

$$c_i = k_i^2/4k - d_i^2/4. \tag{2}$$

To find the levels of $X_1^2 + d_i X_1 + X_2$ which occur with the 0 levels of $X_1 + a_i X_2$ note that there exists an element of GF (p^m), u_3 say, such that $x_1^2 + (d_i - 1/a_i)x_1 = u_3$ has only one solution. Thus

$$(d_i - 1/a_i)^2 + 4u_3 = 0 \text{ and } 4u_3 = -(d_i - 1/a_i)^2.$$

Since $kx_1^2 + (k_1 - 1/a_i)x_1 + (c_i - b_i/a_i)$ must complement $x_1^2 + (d_i - 1/a_i)x_1$, the equation

$$kx_1^2 + (k_1 - 1/a_i)x_1 + (c_i - b_i/a_i) = u_3$$

must also have only one solution as x_1 ranges over the elements of GF (p^m). Therefore

$$(k_i - 1/a_i)^2 - 4k[(c_i - b_i/a_i) - u_3] = 0.$$

Substituting $4u_3 = -(d_i - 1/a_i)^2$ and equations (1) and (2) into this equation we get

$$k_i = kd_i. \tag{3}$$

Hence equation (2) can be rewritten as

$$c_i = d_i^2 \tfrac{1}{4}(k - 1). \tag{4}$$

Thus k is determined by choosing an element of GF (p^m) which is not an element of GF^2 (p^m). By letting $a_i = 1, 2, \cdots, s - 1$ we can determine $b_1, b_2, \cdots, b_{s-1}$ from equation (1). Then setting $d_i = 1, 2, \cdots, s - 1$ we determine $k_1, k_2, \cdots, k_{s-1}$ from equation (3) and $c_1, c_2, \cdots, c_{s-1}$ from equation (4).

The procedure employed above cannot be applied when $p = 2$ since $x_1^2 + cx_1$ consists of $\frac{1}{2}s$ elements of GF (2^m), each occurring twice. Thus there exists no element u such that $x_1^2 + cx_1 = u$ must have only one solution.

We deduce from Lemma 2 that when $p = 2$, then $k = 1$. In order that the levels of $X_1^2 + X_2$ which occur with the 0 levels of $X_1 + a_i X_2 + b_i (a_i = 1, 2, 3, \cdots, s - 1)$ complement the levels of $X_1^2 + X_2$ which occur with the 0 levels of $X_1 + a_i X_2$ then the levels given by $x_1^2 - (1/a_i)x_1 - b_i/a_i$ must complement the levels given by $x_1^2 - (1/a_i)x_1$ when x_1 ranges over GF (2^m). It is easily verified that b_i can be any one of the 2^{m-1} elements of GF (2^m) which are not given by $x_1^2 - (1/a_i)x_1$.

In order that the levels of $X_1^2 + k_i X_1 + X_2 + c_i$ which occur with the 0 levels of X_2 complement the levels of $X_1^2 + d_i X_1 + X_2$ which occur with the 0 levels of X_2, then the values given by $x_1^2 + k_i x_1 + c_i$ must complement the values given by $x_1^2 + d_i x_1$. It can be shown that $k_i = d_i$ and c_i can be any one of the 2^{m-1} elements of GF (2^m) which are not given by the values of $x_1^2 + d_i x_1$.

By finding the values of $X_1^2 + k_iX_1 + X_2 + c_i$ which occur with the 0 levels of $X_1 + a_iX_2 + b_i$ and which complement the values of $X_1^2 + d_iX_1 + X_2$ that occur with the 0 levels of $X_1 + a_iX_2$ a set of b_i and c_i which satisfy all the requirements to have the second half of the plan complement the first half of the plan can be determined.

When $n > 2$ the same procedures will yield the desired plans if Lemma 5a is utilized in place of Lemma 5. Thus the theorem is proved.

5. Examples. Some of the more useful orthogonal arrays which can be constructed by the above procedures are: (18, 7, 3, 2), (54, 25, 3, 2), (32, 9, 4, 2), (128, 41, 4, 2), (50, 11, 5, 2), (250, 61, 5, 2), (98, 15, 7, 2), (128, 17, 8, 2) and (162, 19, 9, 2).

Bose and Bush [6] have constructed the arrays (18, 7, 3, 2) and (32, 9, 4, 2) by other procedures and have shown that $[2(s^n - 1)/(s - 1) - 1]$ is the maximum number of constraints that arrays of size $2s^n$ can accommodate.

We now present two examples of the construction of main effect plans for $[2(s^n - 1)/(s - 1) - 1]$ factors each at $s = p^m$ levels with $2s^n$ treatment combinations. The first example illustrates the construction of a plan for eleven factors, each at five levels with fifty treatment combinations. This plan is the orthogonal array (50, 11, 5, 2).

The eleven factors which represent the first twenty-five treatment combinations are denoted by X_1, X_2, $X_1 + X_2$, $X_1 + 2X_2$, $X_1 + 3X_2$, $X_1 + 4X_2$, $X_1^2 + X_2$, $X_1^2 + X_1 + X_2$, $X_1^2 + 2X_1 + X_2$, $X_1^2 + 3X_1 + X_2$ and $X_1^2 + 4X_1 + X_2$. The corresponding eleven factors representing the second half of the plan are denoted by X_1, X_2, $X_1 + X_2 + b_1$, $X_1 + 2X_2 + b_2$, $X_1 + 3X_2 + b_3$, $X_1 + 4X_2 + b_4$, $kX_1^2 + X_2$, $kX_1^2 + k_1X_1 + X_2 + c_1$, $kX_1^2 + k_2X_1 + X_2 + c_2$, $kX_1^2 + k_3X_1 + X_2 + c_3$ and $kX_1^2 + k_4X_1 + X_2 + c_4$.

The elements of GF (5) are 0, 1, 2, 3 and 4. Hence the elements of $\text{GF}^2(5)$ are 0, 1, 4, 4, 1. From Lemma 6, therefore, $k = 2$ or $k = 3$. Let us choose $k = 3$. Hence, from equation (1)

$$b_i = 1/a_i.$$

Thus, when

$$a_i = 1 \quad \text{then} \quad b_1 = 1$$

$$a_i = 2 \quad \text{then} \quad b_2 = 3$$

$$a_i = 3 \quad \text{then} \quad b_3 = 2$$

$$a_i = 4 \quad \text{then} \quad b_4 = 4.$$

Now, from equations (3) and (4)

$$k_i = 3d_i \text{ and } c_i = 3d_i^2.$$

Thus, when

$$d_i = 1, \quad \text{then} \quad k_1 = 3, c_1 = 3$$

$$d_i = 2, \quad \text{then} \quad k_2 = 1, c_2 = 2$$

$$d_i = 3, \quad \text{then} \quad k_3 = 4, c_3 = 2$$

$$d_i = 4, \quad \text{then} \quad k_4 = 2, c_4 = 3.$$

The eleven factor representations for the second half of the plan are therefore given by: X_1, X_2, $X_1 + X_2 + 1$, $X_1 + 2X_2 + 3$, $X_1 + 3X_2 + 2$, $X_1 + 4X_2 + 4$, $3X_1^2 + X_2$, $3X_1^2 + 3X_1 + X_2 + 3$, $3X_1^2 + X_1 + X_2 + 2$, $3X_1^2 + 4X_1 + X_2 + 2$ and $3X_1^2 + 2X_1 + X_2 + 3$.

The second example will illustrate the construction of the plan for nine factors each at four levels with thirty-two treatment combinations. This plan is the orthogonal array (32, 9, 4, 2).

The nine factors which represent the first sixteen treatment combinations are denoted by X_1, X_2, $X_1 + X_2$, $X_1 + 2X_2$, $X_1 + 3X_2$, $X_1^2 + X_2$, $X_1^2 + X_1 + X_2$, $X_1^2 + 2X_1 + X_2$ and $X_1^2 + 3X_1 + X_2$. The corresponding nine factors representing the second half of the plan are denoted by X_1, X_2, $X_1 + X_2 + b_1$, $X_1 + 2X_2 + b_2$, $X_1 + 3X_2 + b_3$, $X_1^2 + X_2$, $X_1^2 + k_1X_1 + X_2 + c_1$, $X_1^2 + k_2X_1 + X_2 + c_2$ and $X_1^2 + k_3X_1 + X_2 + c_3$. The coefficients are elements of GF (2^2) and all additions and multiplications are performed within this field.

Solving for c_1 so that the levels of $X_1^2 + X_2$ which occur with the 0 level of $X_1 + X_2 + b_1$ complements the levels of $X_1^2 + X_2$ which occur with the 0 level of $X_1 + X_2$ we find that $b_1 = 2$ or $b_1 = 3$. Similarly we find that $b_2 = 1$ or $b_2 = 2$ and that $b_3 = 1$ or $b_3 = 3$.

As we wish the levels of $X_1^2 + k_iX_1 + X_2 + c_i$ which occur with the 0 level of $X_1 + a_iX_2$ to be complements to the levels of $X_1^2 + k_iX_1 + X_2$ which occur with the 0 level of $X_1 + a_iX_2$ we find that

$$k_1 = 1, \qquad k_2 = 2, \qquad k_3 = 3,$$

$$b_1 + c_2 = 1 \text{ or } 3, \qquad b_1 + c_3 = 1 \text{ or } 2, \qquad 3b_2 + c_1 = 1 \text{ or } 2,$$

$$3b_2 + c_2 = 2 \text{ or } 3, \qquad 2b_3 + c_1 = 1 \text{ or } 3 \quad \text{and} \quad 2b_3 + c_3 = 2 \text{ or } 3.$$

Values of b_1, b_2, b_3, c_1, c_2 and c_3 which are consistent with all the above equations are $b_1 = c_1 = 2$, $b_2 = c_2 = 1$ and $b_3 = c_3 = 3$. A second set of solutions is $b_1 = c_1 = 3$, $b_2 = c_2 = 2$ and $b_3 = c_3 = 1$. These are the only two possible sets of solutions for this plan.

Since the coefficients satisfy all the properties required to make the second half of the plan complement the first half every pair of factors is orthogonal.

6. Some useful orthogonal arrays. In this final section we present the factors which represent the first and second halves of the arrays (18, 7, 3, 2) and (54, 25, 3, 2) and the treatment combinations which constitute the array (50, 11, 5, 2).

The factors representing the first half of the orthogonal array (18, 7, 3, 2) are:

$$X_1, X_2, X_1 + X_2, X_1 + 2X_2, X_1^2 + X_2, X_1^2 + X_1 + X_2, X_1^2 + 2X_1 + X_2.$$

The factors representing the second half of this array are:

$$X_1, X_2, X_1 + X_2 + 2, X_1 + 2X_2 + 1, 2X_1^2 + X_2, 2X_1^2 + 2X_1 + X_2 + 1, 2X_1^2 + X_1 + X_2 + 1.$$

The factors representing the first half of the orthogonal array (54, 25, 3, 2) are:

$$X_1, X_2, X_1 + X_2, X_1 + 2X_2, X_3, X_1 + X_3, X_1 + 2X_3, X_2 + X_3, X_2 + 2X_3, X_1 + X_2 + X_3, X_1 + X_2 + 2X_3, X_1 + 2X_2 + X_3, X_1 + 2X_2 + 2X_3, X_1^2 + X_2, X_1^2 + X_1 + X_2, X_1^2 + 2X_1 + X_2, X_1^2 + X_3, X_1^2 + X_1 + X_3, X_1^2 + 2X_1 + X_3, X_1^2 + X_2 + X_3, X_1^2 + X_1 + X_2 + X_3, X_1^2 + 2X_1 + X_2 + X_3, X_1^2 + X_2 + 2X_3, X_1^2 + X_1 + X_2 + 2X_3; X_1^2 + 2X_1 + X_2 + 2X_3.$$

The factors representing the second half of this array are:

$$X_1, X_2, X_1 + X_2 + 2, X_1 + 2X_2 + 1, X_3, X_1 + X_3 + 2, X_1 + 2X_3 + 1, X_2 + X_3, X_2 + 2X_3, X_1 + X_2 + X_3 + 2, X_1 + X_2 + 2X_3 + 2, X_1 + 2X_2 + X_3 + 1, X_1 + 2X_2 + 2X_3 + 1, 2X_1^2 + X_2, 2X_1^2 + 2X_1 + X_2 + 1, 2X_1^2 + X_1 + X_2 + 1, 2X_1^2 + X_3, 2X_1^2 + 2X_1 + X_3 + 1, 2X_1^2 + X_1 + X_3 + 1, 2X_1^2 + X_2 + X_3, 2X_1^2 + 2X_1 + X_2 + X_3 + 1, 2X_1^2 + X_1 + X_2 + X_3 + 1, 2X_1^2 + X_2 + 2X_3, 2X_1^2 + 2X_1 + X_2 + 2X_3 + 1, 2X_1^2 + X_1 + X_2 + 2X_3 + 1.$$

The factors representing the orthogonal array (50, 11, 5, 2) were deduced in Section 5. The following fifty treatment combinations constitute a main-effect plan for eleven factors each at five levels and the array (50, 11, 5, 2). The treatment combinations are divided into two sets of twenty-five, the first set being the first half of the plan and the second set being the second half of the plan (see Table 1 following references).

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TABLE 1

0	0	0	0	0	0	0	0	0	0	0
0	1	1	2	3	4	1	1	1	1	1
0	2	2	4	1	3	2	2	2	2	2
0	3	3	1	4	2	3	3	3	3	3
0	4	4	3	2	1	4	4	4	4	4
1	0	1	1	1	1	1	2	3	4	0
1	1	2	3	4	0	2	3	4	0	1
1	2	3	0	2	4	3	4	0	1	2
1	3	4	2	0	3	4	0	1	2	3
1	4	0	4	3	2	0	1	2	3	4
2	0	2	2	2	2	4	1	3	0	2
2	1	3	4	0	1	0	2	4	1	3
2	2	4	1	3	0	1	3	0	2	4
2	3	0	3	1	4	2	4	1	3	0
2	4	1	0	4	3	3	0	2	4	1
3	0	3	3	3	3	4	2	0	3	1
3	1	4	0	1	2	0	3	1	4	2
3	2	0	2	4	1	1	4	2	0	3
3	3	1	4	2	0	2	0	3	1	4
3	4	2	1	0	4	3	1	4	2	0
4	0	4	4	4	4	1	0	4	3	2
4	1	0	1	2	3	2	1	0	4	3
4	2	1	3	0	2	3	2	1	0	4
4	3	2	0	3	1	4	3	2	1	0
4	4	3	2	1	0	0	4	3	2	1
---	---	---	---	---	---	---	---	---	---	---
0	0	1	3	2	4	0	3	2	2	3
0	1	2	0	0	3	1	4	3	3	4
0	2	3	2	3	2	2	0	4	4	0
0	3	4	4	1	1	3	1	0	0	1
0	4	0	1	4	0	4	2	1	1	2
1	0	2	4	3	0	3	4	1	4	3
1	1	3	1	1	4	4	0	2	0	4
1	2	4	3	4	3	0	1	3	1	0
1	3	0	0	2	2	1	2	4	2	1
1	4	1	2	0	1	2	3	0	3	2
2	0	3	0	4	1	2	1	1	2	4
2	1	4	2	2	0	3	2	2	3	0
2	2	0	4	0	4	4	3	3	4	1
2	3	1	1	3	3	0	4	4	0	2
2	4	2	3	1	2	1	0	0	1	3
3	0	4	1	0	2	2	4	2	1	1
3	1	0	3	3	1	3	0	3	2	2
3	2	1	0	1	0	4	1	4	3	3
3	3	2	2	4	4	0	2	0	4	4
3	4	3	4	2	3	1	3	1	0	0
4	0	0	2	1	3	3	3	4	1	4
4	1	1	4	4	2	4	4	0	2	0
4	2	2	1	2	1	0	0	1	3	1
4	3	3	3	0	0	1	1	2	4	2
4	4	4	0	3	4	2	2	3	0	3