THE MOMENTS OF ELEMENTARY SYMMETRIC FUNCTIONS OF THE ROOTS OF A MATRIX IN MULTIVARIATE ANALYSIS[1]

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1. Introduction and summary. Pillai and Mijares [7] gave the exact expressions for the first four moments of the sum of s non-zero roots of a matrix occurring in multivariate normal analysis as studied independently by R. A. Fisher [3], P. L. Hsu [4] and S. N. Roy [9]. In this paper some properties of completely homogeneous symmetric functions and certain determinantal results (Section 2) are used to give an inverse derivation of those moments (Section 4). The method is further extended to the moments in general of elementary symmetric functions (e.s.f.) of the roots of a matrix in multivariate analysis (Section 6) through the use of certain properties of compound matrices (Section 5).

2. Some results to be used in Sections 4 and 6. Define the completely homogeneous symmetric function (c.h.s.f.) of the pth degree in k arguments by

$$\phi_p(x_1, \cdots, x_k) = \sum_{P_{(p)}} x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k}, \tag{2.1}$$

where $\sum$ extends over all partitions $P_{(p)}$ of a non-negative integer $p = \sum_{i=1}^k p_i$. Define further $\phi_0 = 1$ and $\phi_{p'} = 0$ if $p' < 0$.

Lemma 1.

$$\phi_p(x_1, \cdots, x_k) = \phi_p(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) + x_i \phi_{p-1}(x_1, \cdots, x_k).$$

Proof. Partition $\phi_p(x_1, \cdots, x_k)$ into two groups, one group to contain x_i and the other group not to contain x_i. Factor out x_i from the first group and take the sum of the two groups.

Lemma 2.

$$\begin{aligned}(x_{i+j} - x_i)\phi_{p-1}(x_1, \cdots, x_k) &= \phi_p(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) \\ &\quad - \phi_p(x_1, \cdots, x_i, \cdots, x_{i+j-1}, x_{i+j+1}, \cdots, x_k).\end{aligned}$$

Proof. Use Lemma 1 separately for the x_{i+j} and x_i arguments on

$$\phi_p(x_1, \cdots, x_k),$$

take the difference of the two resulting equations, and transpose the term $(x_{i+j} - x_i)\phi_{p-1}(x_1, \cdots, x_k)$ to the left of the equality sign.

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THEOREM. *Let $r_1 \neq r_2 \neq \cdots \neq r_k$ be non-negative powers of the x's in the successive columns of the k-order determinant given below and let ϕ_j be the* c.h.s.f. *in all k arguments. Then*

$$|x_{k-i+1}^{r_j}| = D\,|\phi_{r_j-k+i}|, \qquad i, j = 1, \cdots, k, \tag{2.2}$$

where

$$D = |x_{k-i+1}^{k-j}| \tag{2.3}$$

*and $x_{k-i+1}^{r_j}$, ϕ_{r_j-k+i}, x_{k-i+1}^{k-j} are the (i, j)*th *elements of the square matrices $(x_{k-i+1}^{r_j})$, (ϕ_{r_j-k+i}), (x_{k-i+1}^{k-j}), respectively.*

PROOF. Perform the following elementary operations on the left determinant of (2.2): ith row $-$ kth row, $i = 1, \cdots, k - 1$. Use Lemma 2 for two arguments to factor out $x_{k-i+1} - x_1$ from the ith row. The result is

$$\prod_{j=2}^{k} (x_j - x_1) \cdot |\phi_{r_{j'}-k+i'}(x_1, x_{k-i'+1})|, \qquad i', j' = 1, \cdots, k, \tag{2.4}$$

where $\phi_{r_{j'}-k+i'}(x_1, x_{k-i'+1})$ is the (i', j')th element of $|\phi_{r_{j'}-k+i'}(x_1, x_{k-i'+1})|$. Thus, the determinantal expression in (2.4) has c.h.s.f. in two arguments for its elements, except for the elements in the last row which have x_1 only as argument.

Next, repeat the operation on the determinant of (2.4) with ith row $-$ $(k - 1)$th row, $i = 1, \cdots, k - 2$ and use Lemma 2 for three arguments this time in factoring out $x_{k-i+1} - x_2$ from the ith row as the resulting determinant. Repeat the operation until finally we have 1st row $-$ 2nd row and the same lemma is used but for k arguments. After factoring out $x_k - x_{k-1}$ from the last determinant, the expression (2.4) reduces finally to

$$\prod_{i>j} (x_i - x_j) \cdot |\phi_{r_{j'}-k+i'}(x_1, x_2, \cdots, x_{k-i'+1})|, \qquad i', j' = 1, \cdots, k, \tag{2.5}$$

with the i'th row of elements $\phi_{r_{j'}-k+i'}$ containing arguments $x_1, x_2, \cdots, x_{k-i'+1}$.

The product $\prod_{i>j} (x_i - x_j)$ is equal to a determinant of Vandermonde and given directly by (2.3). The determinantal part of (2.5) can be reduced into a determinant in the ϕ's, with complete arguments $x_1, \cdots, x_k$, by the repeated application of Lemma 1. Take any element at the intersection of the ith row, $i = 2, \cdots, k$, and a given jth column of the determinant in (2.5), and note that

$$\begin{aligned} &\phi_{r_j-k+i}(x_1, \cdots, x_{k-i+1}) + x_{k-i+2}\phi_{r_j-k+i-1}(x_1, \cdots, x_{k-i+2}) \\ &\qquad\qquad = \phi_{r_j-k+i}(x_1, \cdots, x_{k-i+2}) \end{aligned} \tag{2.6}$$

after using Lemma 1 for $k - i + 2$ arguments. Hence, perform ith row $+$ $x_{k-i+2} \cdot (i - 1)$th row successively for $i = k, k - 1, \cdots, 2$ in the order given and use Lemma 1 on $k - i + 2$ arguments. This increases by one the number of arguments of the ϕ's in each row. Now, perform ith row $+$ $x_{k-i+3} \cdot (i - 1)$th row for $i = k, k - 1, \cdots, 3$ and use Lemma 1 on $k - i + 3$ arguments, etc., until finally we have $(k-1)$th row $+ x_k \cdot k$th row with Lemma 1 used for k arguments. This completes all the arguments in every ϕ of the determinant in (2.5).

3. The mathematical expectation of the sum of the roots. The well-known distribution of the s non-zero roots obtained independently by R. A. Fisher [3], P. L. Hsu [4] and S. N. Roy [9] can be written [7] in the form

$$f(\theta_1, \cdots, \theta_s) = c(s, m, n) \prod_{i=1}^{s} \theta_i^m (1-\theta_i)^n \prod_{i>j} (\theta_i - \theta_j), \qquad 0 < \theta_1 \leqq \cdots \leqq \theta_s < 1, \tag{3.1}$$

where m and n have various interpretations which depend on the null-hypothesis [6] and

$$c(s, m, n) = \pi^{\frac{1}{2}s} \prod_{i=1}^{s} \frac{\Gamma\{\frac{1}{2}(2m + 2n + s + i + 2)\}}{\Gamma\{\frac{1}{2}(2m + i + 1)\}\Gamma\{\frac{1}{2}(2n + i + 1)\}\Gamma(\frac{1}{2}i)}. \tag{3.2}$$

Denote the sum of the roots $\sum_i^s \theta_i$ by $V_1^{(s)}$, the subscript 1 indicating the first e.s.f. (Pillai uses the $V^{(s)}$ notation.) Then the mathematical expectation of $\exp(tV^{(s)})$ can easily be shown to take the determinantal form

$$\begin{aligned} E(e^{tV_1^{(s)}}) &= \int \cdots \int f(\theta_1, \cdots, \theta_s) e^{tV_1^{(s)}} \prod_i d\theta_i \\ &= c(s, m, n) \int_0^1 d\theta_s \int_0^{\theta_s} d\theta_{s-1} \cdots \int_0^{\theta_2} d\theta_1 \\ &\quad \cdot |\theta_{s-i+1}^{m+s-j} (1 - \theta_{s-i+1})^n e^{t\theta_{s-i+1}}|, \qquad i, j = 1, \cdots, s, \end{aligned} \tag{3.3}$$

where the (i, j)th element of the determinantal expression is given by $\theta_{s-i+1}^{m+s-j}(1 - \theta_{s-i+1})^n e^{t\theta_{s-i+1}}$, by first noting that the product $\prod_{i>j}(\theta_i - \theta_j)$ of $f(\theta_1, \cdots, \theta_s)$ is a Vandermonde determinant like (2.3) and then multiplying the ith row of this determinant by $\theta_{s-i+1}^m(1 - \theta_{s-i+1})^n e^{t\theta_{s-i+1}}$, $i = 1, \cdots, s$.

Now, denote (3.3) by $U(m + s - 1, \cdots, m; n; t)$ and more conveniently by $U(s - 1, s - 2, \cdots, 0)$ with the m's and n omitted when $t = 0$. To obtain in general the moments of the sum of the roots in determinantal form, $U(m + s - 1, \cdots, m; n; t)$ is differentiated [5, 8] successively with respect to t and t is set equal to zero after each differentiation. The lower-order moments through the fourth moment are given by equations (3.2) through (3.5) of [8]. For the purpose of this paper, the third moment is reproduced below with $V^{(s)}$ changed to $V_1^{(s)}$,

$$\begin{aligned} E[(V_1^{(s)})^3] = c(s, m, n)[&U(s + 2, s - 2, s - 3, \cdots, 1, 0) \\ &+ 2U(s + 1, s - 1, s - 3, \cdots, 1, 0) \\ &+ U(s, s - 1, s - 2, s - 4, \cdots, 1, 0)]. \end{aligned} \tag{3.4}$$

4. An alternative derivation of the moments of $V_1^{(s)}$. We indicate in this section an alternative way to derive the moments of $V_1^{(s)}$, and we extend this method later in Section 6 to obtain the moments of $V_j^{(s)}$, the jth e. s. f. of s roots, which are not yet available in the current literature except for $j = 1$ and s.

Consider the classes of functions of $\theta_1, \cdots, \theta_s$ of form

$$U'(q_s, q_{s-1}, \cdots, q_1; t) = |\theta_{s-i+1}^{q_{s-j+1}} e^{t\theta_{s-i+1}}|, \quad q_s > \cdots > q_1 \geqq m, \tag{4.1}$$

where $i, j = 1, \cdots, s$, and are respectively the indices of the rows and columns of the determinantal expression. Denote those of form (4.1) for $t = 0$ and $q_i' = q_i - m$ simply by $U'(q_s', q_{s-1}', \cdots, q_1')$. Then $U(m + s - 1, m + s - 2, \cdots, m; n; t)$ of Section 3, i.e., (3.3), may be rewritten as

$$\begin{aligned} &c(s, m, n) \int_0^1 d\theta_s \int_0^{\theta_s} d\theta_{s-1} \int_0^{\theta_{s-1}} \cdots \int_0^{\theta_2} d\theta_1 \\ &\qquad\qquad \cdot U'(m + s - 1, \cdots, m; t) \cdot \prod_{i=1}^{s} (1 - \theta_i)^n. \end{aligned} \tag{4.2}$$

The class $U(q_s, q_{s-1}, \cdots, q_1; n; t)$ of functions of $\theta_1, \cdots, \theta_s$ generated by the successive differentiations of $U(m + s - 1, m + s - 2, \cdots, m; n; t)$ with respect to t can be represented by the class $U'(q_s, q_{s-1}, \cdots, q_1; t)$ of functions generated by successive differentiations of $U'(m + s - 1, \cdots, m; t)$.

Since $U'(m + s - 1, \cdots, m; t)$ may be verified to be equal to

$$\prod_{i=1}^{s} \theta_i^m e^{tV_1^{(s)}} \prod_{i>j} (\theta_i - \theta_j)$$

by comparing (3.3) and (4.2),

$$\begin{aligned} E[(V_1^{(s)})]^r &= d^r/dt^r\{E(e^{tV_1^{(s)}})\}|_{t=0} \\ &= c(s, m, n) \int \cdots \int \left\{\prod_{i=1}^{s} \theta_i^m (V_1^{(s)})^r \prod_{i>j} (\theta_i - \theta_j)\right\} \{\prod_i (1 - \theta_i)^n d\theta_i\}, \end{aligned} \tag{4.3}$$

where the right side of the first equality indicates evaluation of the rth derivative at $t = 0$. Obviously, the factor $(V_1^{(s)})^r \prod_{i>j} (\theta_i - \theta_j)$ is a linear combination of functions in the class $U'(q_s', q_{s-1}', \cdots, q_1')$ and so moments of $V_1^{(s)}$ may be derived alternatively by finding the necessary linear combination in this class with the aid of the theorem in Section 2 applied in the reverse manner.

To illustrate now the alternative method of obtaining the moments of $V_1^{(s)}$, take the case of the third moment given by (3.4). Let Φ_p be the equivalent c. h. s. f. of ϕ_p in Section 2 when the arguments in x's are replaced by arguments in θ's. The initial choice of the s-order determinant in the class $U'(q_s', q_{s-1}', \cdots, q_1')$ is suggested by Φ_1^3 which is equivalent to $(V_1^{(s)})^3$. Choose the U'-determinant such that we have the elements $\Phi_1, \Phi_1, \Phi_1, \Phi_0, \cdots, \Phi_0$ along the principal diagonal. The product of these diagonal elements is Φ_1^3 since $\Phi_0 = 1$ by definition. We have

$$
(4.4) \qquad \begin{vmatrix} \Phi_1 & \Phi_0 & 0 & 0 & \cdots & 0 \\ \Phi_2 & \Phi_1 & \Phi_0 & 0 & \cdots & 0 \\ \Phi_3 & \Phi_2 & \Phi_1 & 0 & \cdots & 0 \\ \Phi_4 & \Phi_3 & \Phi_2 & \Phi_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Phi_s & \Phi_{s-1} & \Phi_{s-2} & \Phi_{s-4} & \cdots & \Phi_0 \end{vmatrix} = \Phi_1^3 + \Phi_3 - 2\Phi_1 \Phi_2 .
$$

It may be remarked that if the determinant of (4.4) is multiplied by $\prod_{i>j}(\theta_i - \theta_j)$ and the theorem of Section 2 is applied, the determinant reduces to $U'(s, s-1, s-2, s-4, \cdots, 1, 0)$.

We next wish to eliminate the product $\Phi_1\Phi_2$ in the right-hand side of (4.4). This suggests taking the s-order determinant in the U'-class with elements in the principal diagonal given by $\Phi_2, \Phi_1, \Phi_0, \cdots, \Phi_0$. Thus

$$
(4.5) \qquad \begin{vmatrix} \Phi_2 & \Phi_0 & 0 & \cdots & 0 \\ \Phi_3 & \Phi_1 & 0 & \cdots & 0 \\ \Phi_4 & \Phi_2 & \Phi_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Phi_{s+1} & \Phi_{s-1} & \Phi_{s-3} & \cdots & \Phi_0 \end{vmatrix} = \Phi_1 \Phi_2 - \Phi_3 .
$$

By the same principle as (4.4) above, the left determinant of (4.5) can be reduced to $U'(s+1, s-1, s-3, \cdots, 1, 0)$. Finally, after inspecting right sides of (4.4) and (4.5), we need

$$
(4.6) \qquad \begin{vmatrix} \Phi_3 & 0 & 0 & \cdots & 0 \\ \Phi_4 & \Phi_0 & 0 & \cdots & 0 \\ \Phi_5 & \Phi_1 & \Phi_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Phi_{s+2} & \Phi_{s-2} & \Phi_{s-3} & \cdots & \Phi_0 \end{vmatrix} = \Phi_3 ,
$$

which gives $U'(s+2, s-2, s-3, \cdots, 1, 0)$. Combining properly the right-hand sides of (4.4) through (4.6), we see the equivalence

$$
(4.7) \qquad \Phi_1^3 = (\Phi_1^3 + \Phi_3 - 2\Phi_1\Phi_2) + 2(\Phi_1\Phi_2 - \Phi_3) + \Phi_3 .
$$

Now multiply (4.4) through (4.6) by $\prod_{i=1}^{s} \theta_i^m (1-\theta_i)^n \prod_{i>j} (\theta_i - \theta_j)$ and integrate over proper limits. We have

$$
(4.8) \qquad \begin{aligned} E(\Phi_1)^3 = c(s, m, n) \int \cdots \int \{ & U'(s, s-1, s-2, s-4, \cdots, 1, 0) \\ & + 2U'(s+1, s-1, s-3, \cdots, 1, 0) \\ & + U'(s+2, s-2, s-3, \cdots, 1, 0)\} \{\prod_i \theta_i^m (1-\theta_i)^n \, d\theta_i\}, \end{aligned}
$$

which reduces to (3.4) after noting how the U-class there and the U'-class here have been defined. It may be remarked that the linear combination in the U'-class is really $(V_1^{(s)})^3 \prod_{i>j} (\theta_i - \theta_j)$ by comparison with (4.3).

5. The kth compound of a matrix. In order to extend our results to the moments of $V_j^{(s)}$, $j = 2, \cdots, s$, we need an important property of compound matrices.

Consider the expansion of $[\prod_{i=1}^{k} (1 - x_i t)]^{-1}$ into a power series

$$[\prod_i (1 - x_i t)]^{-1} = 1 + \phi_1 t + \phi_2 t^2 + \cdots + \phi_r t^r + \cdots \tag{5.1}$$

where ϕ_i is a c.h.s.f. in k arguments. Let $a_j = \sum x_1 \cdots x_j$ be the jth e.s.f. with k arguments in x's. Multiplying both sides of (5.1) by $\prod_i (1 - x_i t) = \sum_{j=0}^{s} (-1)^j a_j t^j$ and equating coefficients, we have

$$\begin{aligned}
&\phi_1 - a_1 &&= 0\\
&\phi_2 - \phi_1 a_1 + a_2 &&= 0\\
&\phi_3 - \phi_2 a_1 + \phi_1 a_2 - a_3 &&= 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &&\\
&\phi_k - \phi_{k-1} a_1 + \cdots + (-1)^k a_k &&= 0.
\end{aligned} \tag{5.2}$$

If we define two $(k + 1)$th order triangular matrices by

$$(a) = \begin{pmatrix} a_0 & 0 & \cdots & 0\\ -a_1 & a_0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ (-1)^k a_k & (-1)^{k-1} a_{k-1} & \cdots & a_0 \end{pmatrix}$$

$$(\phi) = \begin{pmatrix} \phi_0 & 0 & \cdots & 0\\ \phi_1 & \phi_0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ \phi_k & \phi_{k-1} & \cdots & \phi_0 \end{pmatrix}$$

where $a_0 = \phi_0 = 1$, then it may be checked that an alternative form of (5.2) is

$$(a)(\phi) = (\phi)(a) = \mathbf{I} \tag{5.3}$$

where $\mathbf{I}$ is a unit matrix of order $(k + 1)$.

Now, consider the kth *compound of a matrix* (b), denoted by $(b)^{(k)}$, defined by a matrix whose elements are k-order minors of $\det(b)$ arranged in Aitken's lexical sense (see [1], p. 90), i.e., minors which come from the same group of k rows from (b) are placed in the same rows in $(b)^{(k)}$, the order being decided by the columns of (b) that the minors contain in the same manner that words are arranged in a dictionary. For instance, the minor containing columns 1, 3, 4 of (b) is preceded in the row of $(b)^{(3)}$ by those minors containing columns 1, 2, q for $q \geqq 3$. The same definition holds if the words *row*, *rows* are replaced by words *column*, *columns*, respectively, and *vice versa*.

Further, define the kth *adjugate compound of* (b), denoted by $\operatorname{adj}^{(k)}(b)$, as the transpose of the matrix formed from $(b)^{(k)}$ after replacing every element in $(b)^{(k)}$ by its cofactor in $|b|$.

It may be noted that from the way $(b)^{(k)}$ and $\operatorname{adj}^{(k)}(b)$ are defined above,

$(b)^{(1)} = (b)$ and $\mathrm{adj}^{(1)}(b) = \mathrm{adj}(b)$. Furthermore, every element in the product $(b)^{(k)}\,\mathrm{adj}^{(k)}(b)$ is a Laplace Development according to k-ordered minors and their cofactors in $\det(b)$. It may be checked easily that only the diagonal elements in the product $(b)^{(k)}\,\mathrm{adj}^{(k)}(b)$ are expansions in minors by their *algebraic complements* (for definition, see [2], p. 23) and each is equal to $\det(b)$. The off-diagonal elements are sums of products of minors by the algebraic complements of some other minors and each sum is therefore equal to zero. Hence, if (b) is of order n,

$$(b)^{(k)}\mathrm{adj}^{(k)}(b) = |b|\,\mathbf{I}, \tag{5.4}$$

where $\mathbf{I}$ is a unit matrix of order $n!/\{k!(n-k)!\}$.

Consider now the product $(a)(\phi)$ in (5.3). On applying the Binet-Cauchy theorem (see [1], p. 93) and multiplying both sides of the equation by $(\mathrm{adj}(a))^{(k)}$, we have

$$(\phi)^{(k)}(a)^{(k)}(\mathrm{adj}(a))^{(k)} = (\mathrm{adj}(a))^{(k)}\mathbf{I}. \tag{5.5}$$

It may be recalled that $(a)\,\mathrm{adj}(a) = |a|\mathbf{I}$ and by the Binet-Cauchy theorem, the kth compound is

$$(a)^{(k)}(\mathrm{adj}(a))^{(k)} = |a|^{k}\mathbf{I}. \tag{5.6}$$

Furthermore, comparing (*op. cit.*, p. 98) the equality in (5.6) with (5.4) after replacing (b) by (a), we have

$$(\mathrm{adj}(a))^{(k)} = |a|^{k-1}\mathrm{adj}^{(k)}(a). \tag{5.7}$$

Using (5.6) and (5.7) and noting that $|a| = 1$, (5.5) reduces finally to

$$\begin{aligned}(\phi)^{(k)} &= (\mathrm{adj}(a))^{(k)}\\ &= \mathrm{adj}^{(k)}(a).\end{aligned} \tag{5.8}$$

From the nature of the construction of $(\phi)^{(k)}$ and $\mathrm{adj}^{(k)}(a)$, the last equality of (5.8) reveals an inner relationship of minors of $|\phi|$ and $|a|$ which plays a key role in the next section.

If the columns of elements of $(\phi)^{(k')}$, which are k'-order minors of $(k+1)$-order determinant $|\phi|$, are labelled by their highest suffixes occurring in the columns and if the same method of labelling is used for the elements of $\mathrm{adj}^{(k')}(a)$ which are $(k+1-k')$-order minors of $|a|$, then the two sets of suffixes form a bicomplementary set with respect to the highest index. Specifically, we restrict our use of (5.8) only to those minors with consecutive suffixes in the columns.

For example, if an element of $(\phi)^{(3)}$ has column indices labelled 4, 2, 1 then the indices missing in the sequence of numbers 4, 3, 2, 1, 0 are 3, 0. Thus the complementary indices with respect to the highest index 4 in the corresponding $\mathrm{adj}^{(3)}(a)$ have labels 1, 4 in reverse order. Hence,

$$\begin{vmatrix}\phi_2 & \phi_0 & 0\\ \phi_3 & \phi_1 & \phi_0\\ \phi_4 & \phi_2 & \phi_1\end{vmatrix} = \begin{vmatrix}-a_3 & a_0\\ a_4 & -a_1\end{vmatrix}.$$

6. The mathematical expectations of the e.s.f.'s. From the property (5.8) and the theorem in Section 2, the inverse derivation of the moments of the first e.s.f. of s roots may now be extended to any e.s.f. As an illustration, take the second moment of the second e.s.f. for the case of $s = 3$.

If the arguments in x's of the ϕ's and a's of Section 5 are now replaced by arguments in θ's, then $\phi_i \to \Phi_i$ and $a_i \to V_i^{(3)} = V_i$, say, with superscripts omitted if the meaning is clear from the context. The second moment suggests taking the V-determinant with V_2, V_2 in the principal diagonal so that the corresponding Φ-determinant is of order 3 (since $s = 3$). Thus

$$\begin{vmatrix} V_2 & -V_1 \\ -V_3 & V_2 \end{vmatrix} = V_2^2 - V_1 V_3 \tag{6.1}$$

which suggests to add to (6.1) a V-determinant with V_1, V_3 in the principal diagonal. Thus, we have only

$$\begin{vmatrix} V_2 & -V_1 \\ -V_3 & V_2 \end{vmatrix} + \begin{vmatrix} -V_3 & V_2 \\ V_4 & -V_1 \end{vmatrix} = V_2^2 \tag{6.2}$$

since $V_4 = 0$ in the case of $s = 3$. Using (5.8) the equivalent Φ-determinants of the V-determinants are

$$\begin{vmatrix} \Phi_2 & \Phi_1 & 0 \\ \Phi_3 & \Phi_2 & 0 \\ \Phi_4 & \Phi_3 & \Phi_0 \end{vmatrix} + \begin{vmatrix} \Phi_1 & \Phi_0 & 0 \\ \Phi_2 & \Phi_1 & \Phi_0 \\ \Phi_4 & \Phi_2 & \Phi_1 \end{vmatrix} \tag{6.3}$$

on noting that the subscripts (in reverse order) of the last rows of the V-determinants form a bicomplementary set with respect to the highest index 4 with the *missing* subscripts in the last rows of the Φ-determinants.

The next obvious step to find $\mathrm{E}[(V_2^{(3)})^2]$ is to employ directly the method of Section 4 on (6.3). This gives

$$\begin{aligned} E[(V_2^{(3)})^2] &= c(3, m, n) \int \cdots \int \{U'(4, 3, 0) + U'(4, 2, 1)\{\prod_i \theta_i^m (1 - \theta_i)^n \, d\theta_i\} \\ &= c(3, m, n)[U(4, 3, 0) + U(4, 2, 1)]. \end{aligned}$$

The author is presently tabulating the lower-order moments of $V_i^{(s)}$, $i = 2, 3, 4$ and $s = 2, 3, 4$ for values of $2m = -1(1)10(10)60(20)120$ and $2n = 10(10)200$. The results will be reported at some future time.

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