A NONPARAMETRIC TEST FOR THE PROBLEM OF SEVERAL SAMPLES

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1. Summary. In this paper, a new nonparametric test for the problem of c samples is offered. It is based upon the numbers of c-plets that can be formed by choosing one observation from each sample such that the observation from the ith sample is the least, $i = 1, 2, \cdots, c$. The asymptotic distribution of the new test statistic is derived by an application of the extension of Hoeffding's theorem [4] on U-statistics to the case of c samples. The asymptotic power and the asymptotic efficiencies of this test relative to the Kruskal-Wallis H-test [7] and the Mood-Brown M-test [10] are computed in standard fashion along the lines of Andrews' paper [1].

2. Introduction. Let $x_{i1}, x_{i2}, \cdots, x_{in_i}$ be independent (real-valued) observations from the ith population with c.d.f. F_i, $i = 1, 2, \cdots, c$, and suppose that these c samples are independent. The F's are assumed to be continuous. We consider a certain nonparametric test for the hypothesis

$$K_0 : F_1 = F_2 = \cdots = F_c .$$

If we assume that the populations are approximately of the same form, in the sense that if they differ it is by a shift or translation, then we may say that we are testing for the equality of location parameters. References to prior work on several-sample tests and some of the recent work may be found in [2], [6], [7], [8], and [10].

Let $v^{(i)}$ be the number of c-plets that can be formed by choosing one observation from each sample such that the observation from the ith sample is the least. Then

$$v^{(i)} = \sum_{j=1}^{n_i} \prod_{r \neq i} \{\text{number of } x_{rs} > x_{ij}, \qquad s = 1, 2, \cdots n_r\}. \tag{2.1}$$

The new test-statistic proposed is

$$V = N(2c - 1)\left[\sum_{i=1}^{c} p_i(u^{(i)} - c^{-1})^2 - \left\{\sum_{i=1}^{c} p_i(u^{(i)} - c^{-1})\right\}^2\right], \tag{2.2}$$

where $N = \sum_i n_i$, $p_i = n_i/N$ and $u^{(i)} = v^{(i)}/(n_1 n_2 \cdots n_c)$. When the hypothesis K_0 is true, it will be seen that the expectation of each $u^{(i)}$ is $1/c$. Thus, V may be considered as a measure of deviation from K_0. The motivation behind the use of the v's is simply to generalize, to the case of several samples, the Wilcoxon

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[12] statistic for two samples (the number of times observations in the first sample are smaller than observations in the second sample). The test consists in rejecting K_0 at a significance level α if V exceeds some predetermined number V_α. In the next section it is shown that, when K_0 is true, V is asymptotically distributed as a χ^2 variable with $c - 1$ degrees of freedom. Thus, a large sample approximation for V_α is provided by the upper α-point of the χ^2 distribution with $c - 1$ degrees of freedom. It is conjectured that this approximation is relatively close even for samples of moderate size.

3. The asymptotic distribution of V under K_0. It will be seen that

$$v \overset{(i)}{=} \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} \cdots \sum_{t_c=1}^{n_c} \phi^{(i)}(x_{1t_1}, x_{2t_2}, \cdots, x_{ct_c}), \tag{3.1}$$

where

$$\phi^{(i)}(x_{1t_1}, x_{2t_2}, \cdots, x_{ct_c}) = \begin{cases} 1 & \text{if } x_{it_i} < x_{kt_k} \text{ for all } k = 1, \cdots, c \text{ except } i \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Thus, $u^{(i)}$ is a generalized U-statistic [11] corresponding to $\phi^{(i)}$. We shall make use of the following generalization of Hoeffding's theorem [4] on U-statistics to the case of c samples:

LEMMA 3.1. *Let X_{ij}, $j = 1, 2, \cdots, n_i$ for a fixed i be independent (real or vector) random variables identically distributed with* c.d.f. F_i. $i = 1, 2, \cdots, c$. *Further, let $\sum_i n_i = N$ and*

$$U_N^{(r)} = \left[\prod_{i=1}^{c} \binom{n_i}{m_i^{(r)}}\right]^{-1} \sum^{*} \phi^{(r)}(X_{1\alpha_1}, \cdots, X_{1\alpha_{m_1^{(r)}}}; X_{2\beta_1}, \cdots, X_{2\beta_{m_2^{(r)}}}; \cdots; X_{c\delta_1}, \cdots, X_{c\delta_{m_c^{(r)}}}), \qquad r = 1, 2, \cdots, g,$$

where each $\phi^{(r)}$ is a function symmetric in each set of its arguments and $\sum^*$ denotes the sum over all combinations $(\alpha_1, \cdots, \alpha_{m_1^{(r)}})$ of $m_1^{(r)}$ integers chosen from $(1, 2, \cdots, n_1)$ and so on for β's, $\cdots$, and δ's. Assume that $\mathcal{E}[\phi^{(r)}] = \eta^{(r)}$ and $\mathcal{E}[\phi^{(r)}]^2 < \infty$. Then

(i) $\mathcal{E}[U_N^{(r)}] = \eta^{(r)}$,

(ii) $$\operatorname{Cov}[U_N^{(r)}, U_N^{(s)}] = \left[\prod_{i=1}^{c} \binom{n_i}{m_i^{(s)}}\right]^{-1} \sum_{d_1=0}^{m_1^{(rs)}} \cdots \sum_{d_c=0}^{m_c^{(rs)}} \cdot \prod_{i=1}^{c} \binom{m_i^{(r)}}{d_i} \binom{n_i - m_i^{(r)}}{m_i^{(s)} - d_i} \zeta_{d_1, d_2, \cdots, d_c}(r, s),$$

where $m_i^{(rs)} = \min(m_i^{(r)}, m_i^{(s)})$ and

$$(3.3)\quad \begin{aligned}\zeta_{d_1,d_2,\ldots,d_c}(r, s) = \mathcal{E}[\phi^{(r)}(X_{11}, \cdots, X_{1d_1}, X_{1d_1+1}, X_{1m_1^{(r)}}; \cdots ;\\ X_{c1}, \cdots, X_{cd_c}, X_{cd_c+1}, \cdots, X_{cm_c^{(r)}}) \times \phi^{(s)}(X_{11}, \cdots, X_{1d_1},\\ X_{1m_1^{(r)}+1}, \cdots, X_{1m_1^{(r)}+m_1^{(s)}-d_1}; \cdots ; X_{c1}, \cdots, X_{cd_c},\\ X_{cm_c^{(r)}+1}, \cdots, X_{cm_c^{(r)}+m_c^{(s)}-d_c})] - \eta^{(r)}\eta^{(s)},\end{aligned}$$

$r, s = 1, 2, \cdots, g$, *it being understood that* $r = s$ *gives us* $\text{Var}(U_N^{(r)})$, and

(iii) $N^{\frac{1}{2}}[\mathbf{U}_N - \boldsymbol{\eta}]$ *is, in the limit as* $N \to \infty$ *in such a way that* $n_i = Np_i$, *the* p*'s being fixed numbers such that* $\sum_i p_i = 1$, *normally distributed with zero mean and asymptotic covariance matrix* $\boldsymbol{\Sigma} = (\sigma_{rs})$ *given by*

$$(3.4)\quad \sigma_{rs} = \sum_{i=1}^{c} \frac{m_i^{(r)} m_i^{(s)}}{p_i} \zeta_{\underset{(1 \text{ at the } i\text{th place})}{0,\cdots,0,1,0,\cdots,0}}(r, s), \qquad r, s = 1, 2, \cdots, g$$

where $\mathbf{U}_N' = (U_N^{(1)}, \cdots, U_N^{(g)})$ *and* $\boldsymbol{\eta}' = (\eta^{(1)}, \cdots, \eta^{(g)})$.

Proof. The proof of this lemma (concerning generalized **U**-statistics [11]) is a straightforward extension of the proof of Hoeffding's theorem [4] on U-statistics, and the details are omitted.

Now to apply the lemma to our problem, we note from (3.1) that

$$u^{(i)} = v^{(i)}/n_1 n_2 \cdots n_c, \qquad i = 1, 2, \cdots, c,$$

are generalized U-statistics with $g = c$ and $m_1^{(i)} = m_2^{(i)} = \cdots = m_c^{(i)} = 1$. Then if K_0 is true, $\eta^{(i)} = P[X_i < X_k$ for $k = 1, \cdots, c$ except $k = i]$, where the X's are independent and identically distributed random variables, and hence

$$\eta^{(i)} = c^{-1};$$

$$(3.5)\quad \begin{aligned}&\zeta_{\underset{(1 \text{ at the } i\text{th place})}{0,\cdots,0,1,0,\cdots,0}}(i, i)\\ &= \mathcal{E}[\phi^{(i)}(X_1, \cdots, X_i, \cdots X_c)\phi^{(i)}(X_1', \cdots, X_i, \cdots, X_c')] - c^{-2},\end{aligned}$$

where again the X's and X''s are independent and identically distributed random variables, so that

$$(3.6)\quad \begin{aligned}&\zeta_{\underset{(1 \text{ at the } i\text{th place})}{0,\cdots,0,1,0,\cdots,0}}(i, i)\\ &= P[X_i < X_k, X_i < X_k' \text{ all } k = 1, \cdots, c \text{ except } k = i] - c^{-2}\\ &= \frac{(c-1)^2}{c^2(2c-1)};\end{aligned}$$

$$(3.7)\quad \begin{aligned}&\zeta_{\underset{(1 \text{ at the } j\text{th place})}{0,\cdots,0,1,0,\cdots,0}}(i, i)\\ &= P[X_i < X_k, X_i' < X_j, X_i' < X_l \text{ for all } k = 1, \cdots, c\\ &\qquad \text{except } i \text{ and all } l = 1, \cdots, c \text{ except } i \text{ and } j] - c^{-2}\\ &= [c^2(2c-1)]^{-1};\end{aligned}$$

and similarly,

$$\zeta_{\underset{(i\neq j)}{0,\cdots,0,1,0,\cdots,0}}(i, j)$$

$$(3.8)\qquad = \begin{cases} -\dfrac{c-1}{c^2(2c-1)} & \text{if 1 is at the } i\text{th or the } j\text{th place in the row of 0's} \\ [c^2(2c-1)]^{-1} & \text{otherwise.} \end{cases}$$

Thus, if K_0 is true, from (3.4) we have

$$(3.9)\qquad \sigma_{ii} = [c^2(2c-1)]^{-1}\left[\frac{(c-1)^2}{p_i} + \sum_{k\neq i}\frac{1}{p_k}\right],$$

and

$$(3.10)\qquad \sigma_{ij} = [c^2(2c-1)]^{-1}\left[\left(\sum_k \frac{1}{p_k}\right) - \frac{c}{p_i} - \frac{c}{p_j}\right], \qquad i \neq j.$$

The above two relations give us

$$(3.11)\quad c^2(2c-1)\mathbf{\Sigma} = (\textstyle\sum_k 1/p_k)\mathbf{J}_{c,c} + c^2\mathbf{D} - c\mathbf{q}\mathbf{J}_{1,c} - c\mathbf{J}_{c,1}\mathbf{q}',$$

where $\mathbf{D}$ = diagonal $(1/p_k, k = 1, 2, \cdots, c)$, $\mathbf{q}' = (1/p_1, \cdots, 1/p_c)$ and $\mathbf{J}_{r,s} = (1)_{r,s}$. Hence from (iii) in Lemma 3.1 it follows that $N^{\frac{1}{2}}[\mathbf{U} - \mathbf{J}_{c,1}/c]$, where $\mathbf{U}' = (u^{(1)}, \cdots, u^{(c)})$, has a limiting normal distribution with zero means and asymptotic covariance matrix $\mathbf{\Sigma}$ given by (3.11). But $\sum_i v^{(i)} = n_1n_2 \cdots n_c$, and hence u's are subject to one linear constraint, *viz.*, $\sum_i u^{(i)} = 1$. Thus the distribution of u's is singular and hence the asymptotic distribution is also singular. Then $\mathbf{\Sigma}$ is singular; in fact it can be easily verified from (3.11) that $\mathbf{J}_{1,c}\mathbf{\Sigma} = \mathbf{0}$. Let

$$N^{\frac{1}{2}}[\mathbf{U}' - \mathbf{J}_{1,c}/c] = \mathbf{b}' = (b_1, \cdots, b_{c-1}, b_c) = (\mathbf{b}_0', b_c).$$

Then it follows that $\mathbf{b}_0'\mathbf{\Sigma}_0^{-1}\mathbf{b}_0$ has a limiting χ^2 distribution with $c - 1$ degrees of freedom, where $\mathbf{\Sigma}_0$ denotes the asymptotic covariance matrix of $\mathbf{b}_0$. From (3.11) we have

$$(3.12)\qquad c^2(2c-1)\mathbf{\Sigma}_0 = a\mathbf{J}_{c-1,c-1} + c^2\mathbf{D}_0 - c\mathbf{q}_0\mathbf{J}_{1,c-1} - c\mathbf{J}_{c-1,1}\mathbf{q}_0',$$

where $\mathbf{D}_0$ = diagonal $(1/p_k, k = 1, 2, \cdots, c-1)$, $\mathbf{q}_0' = (1/p_1, \cdots, 1/p_{c-1})$ and $a = \sum_{k=1}^{c} 1/p_k$.

CASE (i): $n_1 = n_2 = \cdots = n_c$. Then $p_i = 1/c$ and (3.12) gives $(2c-1)\mathbf{\Sigma}_0 = c\mathbf{I} - \mathbf{J}_{c-1,c-1}$, so that

$$\mathbf{\Sigma}_0^{-1} = (2c - 1/c)[\mathbf{I} + \mathbf{J}_{c-1,c-1}],$$

and hence,

$$(3.13)\qquad \mathbf{b}_0'\,\mathbf{\Sigma}_0^{-1}\,\mathbf{b}_0 = \frac{N(2c-1)}{c}\sum_{i=1}^{c}\left(u^{(i)} - \frac{1}{c}\right)^2.$$

CASE (ii): Not all n's are equal. Then $\mathbf{q}_0$ and $\mathbf{J}_{c-1,1}$ are linearly independent and from (3.12) we have

$$c^2(2c-1)\mathbf{\Sigma}_0 = c^2\mathbf{D}_0 - EF',$$

where

$$\mathbf{\Sigma} = [c\mathbf{q}_0, \mathbf{J}_{c-1,1}], \qquad \mathbf{F} = [\mathbf{J}_{c-1,1}, c\mathbf{q}_0 - a\mathbf{J}_{c-1,1}]$$

are both of full rank *viz.*, two. Then

$$[c^2(2c-1)]^{-1}\mathbf{\Sigma}_0^{-1} = c^{-2}\mathbf{D}_0^{-1} - \mathbf{D}_0^{-1}\mathbf{E\Lambda F'D}_0^{-1},$$

where $\mathbf{\Lambda}$ is given by

$$c^2[\mathbf{F'D}_0^{-1}\mathbf{E} - c^2\mathbf{I}]\mathbf{\Lambda} = \mathbf{I}.$$

After simplification we finally have

$$\text{(3.14)} \qquad \mathbf{b}_0'\mathbf{\Sigma}_0^{-1}\mathbf{b}_0 = N(2c-1)\left[\sum_{i=1}^{c} p_i\left(u^{(i)} - \frac{1}{c}\right)^2 - \left\{\sum_{i=1}^{c} p_i\left(u^{(i)} - \frac{1}{c}\right)\right\}^2\right].$$

It may be seen that the above expression reduces to (3.13) when $n_1 = n_2 = \cdots = n_c$. It may be noted that the above expression is invariant under any choice of $(c-1)$ linearly independent u's. We have thus proved Theorem 3.1.

THEOREM 3.1. *If $F_1 = F_2 = \cdots = F_c$ and $n_i = Np_i$, where the p's are fixed numbers such that $\sum_i p_i = 1$, then the statistic V, defined by (2.2), has a limiting χ^2 distribution with $c-1$ degrees of freedom as $N \to \infty$.*

4. Consistency of the V-test. As mentioned earlier, if we assume that the populations are approximately of the same form, then we may say that we are testing for the equality of location parameters. Thus, we are primarily interested in translation-type alternatives $F_i(x) = F(x - \theta_i)$, $i = 1, 2, \cdots, c$, where the θ's are not all equal. We shall show that the V-test is consistent against this class of alternatives.

We first state, without proof, the following straightforward extensions of a lemma of Lehmann ([9], p. 169).

LEMMA 4.1. *Let $\eta = f(F_1, F_2, \cdots, F_c)$ be a real-valued function such that $f(F, F, \cdots, F) = \eta_0$ for all $(F, F, \cdots, F)$ in a class $\mathcal{C}_0$. Let*

$$T_{n_1,\ldots,n_c} = t(X_{11}, \cdots, X_{1n_1}; \cdots; X_{c1}, \cdots, X_{cn_c})$$

be a sequence of real-valued statistics such that $T_{n_1,\ldots,n_c}$ tends to η in probability as $\min(n_1, \cdots, n_c) \to \infty$. Suppose that $f(F_1, F_2, \cdots, F_c) \neq \eta_0 (> \eta_0)$ for all $(F_1, F_2, \cdots, F_c)$ in a class $\mathcal{C}_1$. Then the sequence of tests which reject when $|T_{n_1,\ldots,n_c} - \eta_0| > c_{n_1,\ldots,n_c}$ (when $T_{n_1,\ldots,n_c} - \eta_0 > c'_{n_1,\ldots,n_c}$) is consistent for testing $H: \mathcal{C}_0$ at every fixed level of significance against the alternatives $\mathcal{C}_1$.

LEMMA 4.2. *Let $\eta^{(i)} = f^{(i)}(F_1, F_2, \cdots, F_c)$, $i = 1, 2, \cdots, g$, be real-valued functions such that $f^{(i)}(F, F, \cdots, F) = \eta_0^{(i)}$ for all $(F, F, \cdots, F)$ in a class $\mathcal{C}_0$. Let $T^{(i)}_{n_1,\ldots,n_c} = t^{(i)}(X_{11},\ldots,X_{1n_1}; \cdots; X_{c1}, \cdots, X_{cn_c})$, $i = 1, 2, \cdots, g$, be* *sequences of real-valued statistics such that* $T^{(i)}_{n_1}, \cdots, n_c$ *tends to* $\eta^{(i)}$ *in probability as* $\min(n_1, \cdots, n_c) \to \infty$. *Suppose that at least one* $f^{(i)}(F_1, F_2, \cdots, F_c) \neq \eta_0^{(i)}$ *for all* $(F_1, F_2, \cdots, F_c)$ *in a class* $\mathcal{C}_1$. *Further, let*

$$W_{n_1,\ldots,n_c} = \omega(T^{(1)}_{n_1\ldots,n_c}; \cdots; T^{(g)}_{n_1\ldots,n_c})$$

be a nonnegative function which is zero if, and only if, $T^{(i)}_{n_1\ldots,n_c} = \eta_0^{(i)}$ *for all* $i = 1, \cdots, g$. *Then the sequence of tests which reject when*

$$W_{n_1,\ldots,n_c} > d_{n_1,\ldots,n_c}$$

is consistent for testing $H:\mathcal{C}_0$ *at every fixed level of significance against the alternatives* $\mathcal{C}_1$.

If we take $\eta^{(i)} = P[X_i < X_j \text{ for all } j = 1, \cdots, c \text{ except } i]$, where the X's are independent random variables with continuous c.d.f. $F_1, F_2, \cdots, F_c$, respectively, and $T^{(i)}_{n_1\ldots,n_c} = u^{(i)}$, $i = 1, \cdots, c$, then the convergence in probability of $u^{(i)}$ to $\eta^{(i)}$ follows from (iii) in Lemma 3.1. For the class $\mathcal{C}_1$ of translation-type alternatives $F_i(x) = F(x - \theta_i)$, where the θ's are not all equal, it may be easily seen that $\eta^{(r)} > 1/c$, where θ_r is the (or one of the) least among $\theta_1, \cdots, \theta_c$. The V-test, thus, is seen to be consistent against the class of translation-type alternatives.

More generally, the V-test is consistent against the wider class of alternatives for which $P[X_i < X_j \text{ for all } j = 1, \cdots, c \text{ except } i] \neq 1/c$ for at least one i among $(1, \cdots, c)$, where the X's are independent random variables with continuous c.d.f. $F_1, F_2, \cdots, F_c$, respectively.

5. The asymptotic distribution of V under translation-type alternatives. Andrews [1] has investigated the asymptotic efficiencies of Kruskal's H-test and Mood's M-test and has concluded that the asymptotic efficiency of one relative to the other is $\geqq$ or $\leqq$ 1, for the translation-type alternatives, depending on the distribution function. It will be interesting (as suggested by Hoeffding and the referee) to carry out similar studies on this test with respect to the two previous tests. It is expected that the same type of conclusion will be reached.

Let us study the distribution of V, assuming a sequence of translation-type alternative hypotheses K_n for $n = 1, 2, \cdots$. The hypothesis K_n specifies that $F_i(x) = F(x - n^{-\frac{1}{2}}\theta_i)$, $i = 1, 2, \cdots, c$, where not all θ's are equal. The letter n will be used to index a sequence of situations in which K_n is the true hypothesis. The limiting probability distribution will then be found as $n \to \infty$.

THEOREM 5.1. *For each index n assume that* $n_i = ns_i$, *with* s_i *a positive integer and the truth of* K_n.

If F possesses a continuous derivative f and there exists a function g such that

$$|[f(y + h) - f(y)]/h| \leqq g(y)$$

and

$$\int_{-\infty}^{\infty} g(y)f(y)\, dy < \infty,$$

then, for $n \to \infty$, the statistic V has a limiting noncentral χ^2 distribution with $c - 1$ degrees of freedom and the noncentrality parameter[1]

$$(5.1) \qquad (2c-1)c^2 \sum_{i=1}^{c} s_i(\theta_i - \bar{\theta})^2 \left[\int_{-\infty}^{\infty} [1 - F(y)]^{c-2} f^2(y)\, dy \right]^2,$$

where $\bar{\theta} = \sum_i s_i\theta_i / \sum_i s_i$.

PROOF: Let $\eta_n^{(i)} = \mathcal{E}[\phi^{(i)}(X_1, X_2, \cdots, X_c) \mid K_n]$; then it can be easily shown that

$$\eta_n^{(i)} = \frac{1}{c} - \frac{\delta_i}{n^{\frac{1}{2}}}\lambda + O(n^{-1}),$$

where

$$\delta_i = c\theta_i - \sum_{k=1}^{c} \theta_k,$$

and

$$(5.2) \qquad \lambda = \int_{-\infty}^{\infty} [1 - F(y)]^{c-2} f^2(y)\, dy.$$

Similarly, it may be shown that

$$(5.3) \qquad \mathbf{\Sigma}_n = \mathbf{\Sigma} + \mathbf{O}(n^{-\frac{1}{2}}),$$

where $\mathbf{\Sigma}$ is given by (3.11) and $\mathbf{O}(n^{-\frac{1}{2}})$ denotes a matrix whose elements are $O(n^{-\frac{1}{2}})$.

Then, in view of Lemma 3.1, $N^{\frac{1}{2}}(\mathbf{U} - \mathbf{n}_n)$ is, in the limit as $n \to \infty$, distributed with zero means and covariance matrix $\mathbf{\Sigma}_n$, or, in view of (5.3), with asymptotic covariance matrix $\mathbf{\Sigma}$. Hence $N^{\frac{1}{2}}(\mathbf{U} - c^{-1}\mathbf{J}_{c,1})$ has a limiting normal distribution with mean-vector $-(\sum_k s_k)^{\frac{1}{2}}\lambda\boldsymbol{\delta}$, where $\boldsymbol{\delta}' = (\delta_1, \cdots, \delta_c)$, and covariance matrix $\mathbf{\Sigma}$. Thus V, in the limit as $n \to \infty$, is distributed as a noncentral χ^2 with $c - 1$ degrees of freedom and the noncentrality parameter

$$\lambda_V = (\textstyle\sum_k s_k)\lambda^2 \boldsymbol{\delta}_0' \mathbf{\Sigma}_0^{-1} \boldsymbol{\delta}_0,$$

in the notation of Section 3. Since $\sum_k \delta_k = 0$, arguing exactly as from (3.11) to (3.14), we see that λ_V reduces to (5.1).

6. Asymptotic relative efficiency. Andrews [1] has shown that the H-statistic, the M-statistic and the F-statistic are asymptotically distributed as noncentral χ^2 with $c - 1$ degrees of freedom and noncentrality parameters λ_H, λ_M and λ_F, respectively, where

$$\lambda_H = 12\left\{ \int_{-\infty}^{\infty} F'(x)\, dF(x) \right\}^2 \sum_{i=1}^{c} s_i(\theta_i - \bar{\theta})^2,$$

$$\lambda_M = 4[F'(a)]^2 \sum_{i=1}^{c} s_i(\theta_i - \bar{\theta})^2,$$

[1] This was also obtained independently by Y. S. Sathe.

and

$$\lambda_F = \sum_i s_i[(\theta_i - \bar{\theta})/\sigma_F]^2,$$

where a is the median of F.

It is now well known ([1], [3]) that in such cases the asymptotic efficiency of one statistic relative to the other is equal to the ratio of their noncentrality parameters. Hence, we have the asymptotic efficiencies of the V-statistic relative to the H, M and F statistics as follows:

$$\epsilon_{V,H} = (2c - 1)\, c^2\lambda^2/12\left\{\int_{-\infty}^{\infty} F'(x)\, dF(x)\right\}^2,$$

$$\epsilon_{V,M} = (2c - 1)c^2\lambda^2/4[F'(a)]^2,$$

and

$$\epsilon_{V,F} = (2c - 1)c^2\lambda^2\sigma_F^2,$$

respectively, where λ is given by (5.2). These expressions are seen to be independent of the scale parameter. For the uniform distribution the efficiencies are given by

$$\epsilon_{V,H} = \epsilon_{V,F} = \epsilon_{V,M}/3 = (2c - 1)c^2/12(c - 1)^2,$$

so that we have

c	2	3	4	5	6	10	∞
$\epsilon_{V,H}$	1.00	0.94	1.04	1.17	1.32	1.95	∞.

For the exponential distribution, $f(y) = e^{-y}$, $0 \leqq y < \infty$, $\epsilon_{V,H} = \epsilon_{V,M}/3 = \epsilon_{V,F}/3 = (2c - 1)/3$, so that

c	2	3	4	5	10	∞
$\epsilon_{V,H}$	1.00	1.66	2.33	3.00	6.33	∞.

For the normal distribution λ can be computed from the Table I given by Hojo [5] for $c \leqq 13$. We have

c	2	3	4	5	6	7	8	10	12	13
$\epsilon_{V,H}$	1.00	0.94	0.86	0.80	0.74	0.69	0.65	0.58	0.53	0.51 ,

while $\epsilon_{V,M} = 3\epsilon_{V,H}/2$ and $\epsilon_{V,F} = 3\epsilon_{V,H}/\pi$.

For the normal distribution, the asymptotic efficiency of the V-statistic relative to the Kruskal-Wallis H-statistic tends to zero as the number of populations tends to infinity. I am thankful to the referee for supplying the following indication of the proof.

Outline of the Proof. We must show that

$$n^{\frac{1}{2}} \int_{-\infty}^{\infty} [\Phi(x)]^n \varphi^2(x)\, dx \to 0 \qquad \text{as } n \to \infty.$$

On integrating by parts, it is seen that

$$\int_{-\infty}^{\infty} [\Phi(x)]^n \varphi^2(x)\, dx = \frac{1}{n+1} \int_{-\infty}^{\infty} x[\Phi(x)]^{n+1} \varphi(x)\, dx$$

$$= \frac{1}{n+1} \int_0^1 y^{n+1} \Phi^{-1}(y)\, dy.$$

It is therefore enough to prove that

$$n^{\frac{1}{2}} \int_0^1 x^n \Phi^{-1}(x)\, dx \to 0 \qquad \text{as} \quad n \to \infty.$$

We shall prove this using the fact that

$$\int_0^1 x^n [\log (x^{-1})]^{-\frac{1}{2}}\, dx = \left(\frac{\pi}{n+1}\right)^{\frac{1}{2}}.$$

It is easily seen by de l'Hospital's rule that

$$\frac{\Phi^{-1}(x)}{[\log (x^{-1})]^{-\frac{1}{2}}} \to 0 \qquad \text{as} \quad x \to 1.$$

Given any $\epsilon > 0$ there exists therefore a constant $a(\frac{1}{2} < a < 1)$ such that

$$\Phi^{-1}(x) \leqq \epsilon[\log (x^{-1})]^{-\frac{1}{2}} \qquad \text{for} \quad a < x < 1,$$

and hence

$$\int_a^1 x^n \Phi^{-1}(x)\, dx \leqq \epsilon \int_a^1 x^n [\log (x^{-1})]^{-\frac{1}{2}}\, dx \leqq \epsilon \left(\frac{2\pi}{n+1}\right)^{\frac{1}{2}}.$$

Finally it is easily seen that for fixed a $\int_0^a x^n\ \Phi^{-1}(x)dx$ tends to 0 at a faster rate than $\int_a^1 x^n\ \Phi^{-1}(x)dx$ as $n \to \infty$. Given ϵ and hence a, there therefore exist n_0 so that $n \geqq n_0$ implies

$$\int_0^1 x^n \Phi^{-1}(x)\, dx \leqq 2 \int_a^1 x^n \Phi^{-1}(x)\, dx \leqq 2\epsilon \left(\frac{2\pi}{n+1}\right)^{\frac{1}{2}}$$

and this completes the proof.

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