TABLE I

$\lim_{n\to\infty} P[H/n^{\frac{1}{2}} < t]$*

t	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0	.00000	.00000	.00000	.00000	.00000	.00000	.00008	.00090	.00452
1.	.01438	.03387	.06497	.10785	.16120	.22280	.29008	.36046	.43156	.50132
2.	.56807	.63054	.68782	.73939	.78501	.82472	.85874	.88745	.91135	.93095
3.	.94682	.95949	.96948	.97726	.98324	.98778	.99119	.99371	.99556	.99690
4.	.99786	.99854	.99901	.99934	.99956	.99971	.99981	.99988	.99992	.99995

* computed on the McGill I.B.M. 650

REFERENCES

[1] Bahadur, R. R. (1960). Stochastic comparison of tests. *Ann. Math. Statist.* **31** 276–295.
[2] Daniels, H. E. (1954). A distribution-free test for regression parameters. *Ann. Math. Statist.* **25** 499–513.
[3] Feller, William (1957). *An Introduction to Probability Theory and its Applications.* **1** 2nd ed. Wiley, New York.
[4] Hill, B. M. (1960). A relationship between Hodges' bivariate sign test and a nonparametric test of Daniels. *Ann. Math. Statist.* **31** 1190–1192.
[5] Hodges, J. L., Jr. (1955). A bivariate sign test. *Ann. Math. Statist.* **26** 523–527.
[6] Kemperman, J. H. B. (1959). Asymptotic expansions for the Smirnov test and for the range of cumulative sums. *Ann. Math. Statist.* **30** 448–462.
[7] Klotz, Jerome (1959). Null distribution of the Hodges bivariate sign test. *Ann. Math. Statist.* **30** 1029–1033.
[8] Lévy, Paul (1948). *Processus Stochastiques et Mouvement Brownien.* Gauthier-Villars, Paris.
[9] Parzen, E. (1954). On uniform convergence of families of sequences of random variables. *Univ. of California Pubs. in Statist.* **2** 23–54, Univ. of California Press.

NOTE ON MULTIVARIATE GOODNESS-OF-FIT TESTS[1]

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1. Introduction and summary. Let $X_1, X_2, \cdots, X_n$ be m-dimensional statistically independent random vectors with common distribution function F. It is frequently desirable to test the hypothesis that F is a member of some class of distribution functions $\mathcal{H}_o$. For the scalar case, $(m = 1)$, much research has been done; see for example [1], [2], [3]. For $m > 1$ comparatively little has been accomplished, and a useful extension of the techniques used for $m = 1$ awaits

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the solution of certain problems in stochastic processes with a vector parameter; see, for example, [4].

In this paper consistent tests are developed for any given class $\mathcal{H}_o$. These tests can be constructed to have size α and prescribed power $1 - \beta$ against alternatives whose probability assignment to at least one of a certain given class of sets $\{B(v)\}$ differs from that of each member of $\mathcal{H}_o$ by at least a prescribed value K. The range of such alternatives is seen in Section 3 to be rather wide, so that at least in theory, the suggested tests would seem to be rather useful. The tests are constructed by mapping the set of all m-dimensional distribution functions in a one to one measurable manner into a subset of the set of one-dimensional distribution functions. Such mappings are, of course, reasonably well known; see Halmos [7], p. 153. The purpose of this note is to show how such mappings offer sufficient flexibility for the construction of a class of tests which are useful for most ordinary purposes. Simpson [6] suggested tests based on mapping bivariate distributions into univariate distributions. However no mention was made there of consistency, or of power in terms of the type of alternative here mentioned.

2. Preliminary theory. Let $\{B(v): -\infty < v < \infty\}$ be a collection of Borel measurable sets in R^m such that for $h > 0$,

$$B(-h) = \emptyset, \quad B(v) \subset B(v + h), \quad \text{and} \quad \lim_{v\to\infty} B(v) = R^m.$$

THEOREM 2.1. *For any class $\mathcal{H}_o$ of distribution functions on R^m and each K, α, β in* $(0, 1)$, *there is a test of $\mathcal{H}_o$ of size α satisfying*

$$P_F\,\{\text{rej } \mathcal{H}_o\} \geqq 1 - \beta$$

for all F satisfying

$$\inf_{F_o \varepsilon \mathcal{H}_o} \sup_v |P_{F_o}\{B(v)\} - P_F\{B(v)\}| > K. \tag{2.1}$$

PROOF. For each m-dimensional distribution function F^*, define the function G_{F^*} by

$$G_{F^*}(v) = P_{F^*}\{B(v)\}. \tag{2.2}$$

Clearly G_{F^*} is a univariate distribution function. Let $G_n(v) = P_n\{B(v)\} \equiv$ proportion of observed values of $X_1, \cdots, X_n$ in $B(v)$. It is seen that when the X_i have distribution function F, defining the random variables Y_i by

$$Y_i(\omega) = \inf\{v : X_i(\omega)\, \varepsilon B(v)\},$$

that the Y_i are independent with common distribution function G_F, and that G_n is the empirical (one dimensional) distribution function formed from the Y_i. The condition $\inf_{F_o\varepsilon\mathcal{H}_o} \sup_v |P_{F_o}\{B(v)\} - P_F\{B(v)\}| > K$ is equivalent to the condition $\inf_{F_o\varepsilon\mathcal{H}_o} \sup_v |G_{F_o}(v) - G_F(v)| \equiv \inf_{F_o\varepsilon\mathcal{H}_o} d_1(G_{F_o}, G_F) > K$, ($d_1$ being the uniform metric). Then we can use the modified Kolmogorov-Smirnov tests [5], "reject $\mathcal{H}_o^*: G_F \varepsilon \{G_{F_o} : F_o \varepsilon \mathcal{H}_o\}$ if and only if $\inf_{F_o\varepsilon\mathcal{H}_o} n^{\frac{1}{2}}\, d_1(G_{F_o}, G_n) > h_{1,\alpha,n}$, where $h_{1,\alpha,n}$ is chosen so that for all $F \varepsilon \mathcal{H}_o$, $P_F\{\inf_{F_o\varepsilon\mathcal{H}_o} n^{\frac{1}{2}}\, d_1(G_{F_o}, G_n) > h_{1,\alpha,n}\}$

$\leqq \alpha$, and n is chosen so that if $\inf_{F_o \varepsilon \mathcal{H}_o} d_1(G_{F_o}, G_F) > K$, then

$$P_F\{\inf_{F_o \varepsilon \mathcal{H}_o} n^{\frac{1}{2}} d_1(G_{F_o}, G_n) > h_{1,\alpha,n}\} \geqq 1 - \beta,\text{''}$$

as a test of $\mathcal{H}_o$ of size α satisfying the assertion of this theorem.

THEOREM 2.2. *If the sets $B(v)$ generate (by means of the operations of complementation and countable union) the Borel sets in R^m, then the mapping $M_B:F^* \rightarrow G_{F^*}$ defined by (2.2), is one to one. In such a case the tests of the previous theorem are consistent.*

PROOF. Any probability distributions which agree on the sets $B(v)$ must agree on the sets of the form $B(v + h) - B(v)$, since the $B(v)$ are non-decreasing. Hence they must agree on the algebra of finite disjoint unions and complements of finite disjoint unions of such sets. By the well known extension theorem they must therefore agree on the σ-algebra generated by this algebra, which by hypothesis is the collection of Borel sets in R^m.

It follows that if $F_1 \neq F_2$, there exists v such that

$$G_{F_1}(v) = P_{F_1}\{B(v)\} \neq P_{F_2}\{B(v)\} = G_{F_2}(v).$$

Hence $F_1 \neq F_2$ implies $G_{F_1} \neq G_{F_2}$, and the first assertion follows. The second assertion follows from the consistency of the modified Kolmogorov-Smirnov tests.

3. Construction of the $B(v)$. In this section we give a flexible method of constructing non-decreasing collections of sets $B(v)$ which generate the Borel sets in R^m.

Divide R^m into a disjoint union of Borel measurable sets $A_{0,}, A_{1,}, \cdots$, the integer to the left of the comma written to the base $s + 1$. Recursively define $A_{i_1,i_2,\cdots,i_k i_{k+1}}$ ($i_1 = 0, 1, 2, \cdots, i_j = 0, 1, \cdots, s$ for $j > 1$, written to base $s + 1$), by subdividing $A_{i_1,i_2\cdots i_k}$ into a disjoint union of Borel sets, making sure that the collection of all sets $A_{i_1,}, A_{i_1,i_2}, \cdots$ generates the m-dimensional Borel sets. (For example this can be done by letting A_{i_1}, be unit m-cubes and $A_{i_1,i_2,\cdots,i_k}$ be m-cubes of volume $2^{-m(k-1)}$.)

Let the subscript $i_1, i_2 \cdots i_k$ be identified with the interval

$$[i_1.i_2 \cdots i_k, i_1.i_2 \cdots i_k + (s + 1)^{-k}],$$

whose end points are rational numbers written to base $s + 1$.

Let $R(\mu)$ be the right end point of the interval identified with the subscript μ, and it is clear that the collection of sets $B(v) = \bigcup_{R(\mu)<v} A_\mu$ generates the Borel sets, since any one of the sets A_μ' is the difference of two sets from

$$\{B(v):v \text{ rational, terminating when its decimal part is written to base } s + 1\}.$$

It is seen, for example, that if A_j, is the spherical shell with inner radius kj, outer radius $k(j + 1)$, center at some point (x, y), then the sets $B(v)$ include all spheres about (x, y) of radius a multiple of k. It is clear, therefore, that from various choices of the A_j, one can generate a wide variety of tests.

REFERENCES

[1] Anderson, T. W. and Darling, D. A. (1952). Asymptotic theory of certain 'goodness-of-fit' criteria based on stochastic processes. *Ann. Math. Statist.* **23** 193–212.

[2] Birnbaum, Z. W. (1953). On the power of a one sided test of fit for continuous probability functions. *Ann. Math. Statist.* **24** 484–489.

[3] Rosenblatt, Judah (1959). Goodness-of-fit tests for approximate hypotheses, microfilmed Ph.D. thesis, Columbia Univ.

[4] Kiefer, J. and Wolfowitz, J. (1958). On the deviations of the empiric distribution function of vector chance variables. *Trans. Amer. Math. Soc.*, **87** 173–186.

[5] Rosenblatt, Judah (1962). Some modified Kolmogorov-Smirnov tests of approximate hypotheses and their properties. *Ann. Math. Statist.* **33** 513–524.

[6] Simpson, Paul B. (1951). Note on the estimation of a bivariate distribution function. *Ann. Math. Statist.* **23** 476–478.

[7] Halmos, Paul R. (1958). *Measure Theory.* Van Nostrand, New York.

THE UNIQUENESS OF THE SPACING OF OBSERVATIONS IN POLYNOMIAL REGRESSION FOR MINIMAX VARIANCE OF THE FITTED VALUES

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1. Introduction. A spacing of $n(p + 1)$ observations in the interval $[-1, 1]$ in order to minimize the maximum variance of $\hat{u}(x)$, a pth degree polynomial fitted by least squares, in the interval $[-1, 1]$ has been given by P. G. Guest [1]. The spacing places n observations at -1, 1, and each of the $p - 1$ zeros of $P'_p(x)$, where $P_p(x)$ is the Legendre polynomial of degree p. The purpose of this note is to establish the uniqueness of P. G. Guest's solution when the observations are made at $p + 1$ distinct points.

2. Statement of problem. Guest defines

$$F(x) = \prod_{j=0}^{p} (x - x_j) \tag{1}$$

where the x_j are the distinct points in the interval $[-1, 1]$ at which the observations are to be made. The minimax variance condition requires

$$x_0 = -1, \qquad x_p = 1 \tag{2}$$

and

$$F''(x_j) = 0, \qquad \text{for } j = 1, 2, \cdots, p - 1. \tag{3}$$

After defining $\phi(x)$ by the equation

$$F(x) = \alpha(x^2 - 1)\phi(x), \tag{4}$$

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