That is, we have

$$\phi(t) = \exp [\pm\{1 - (1 - 2iqm^2t)^{\frac{1}{2}}\}/(qm)] \tag{12}$$

to be true for all real values of t for which $\phi(t) \neq 0$ and $\phi'(t) \neq 0$. It was already demonstrated that (12) is valid in an interval around the origin in which $\phi(t) \neq 0$. Since $\phi(t)$ is continuous, one has $\lim_{h \downarrow 0} \phi(t_0 - h) = \phi(t_0) = 0$, however, it follows from (12) that this limit is not zero and this contradiction proves that $\phi(t) \neq 0$ for all t, so that (12) is valid for all t. On account of the uniqueness theorem on the characteristic function ([1], (10.3.1) p. 93), we have the distribution of x as Inverse Gaussian, with the parameters q^{-1} and $(\pm m)$.

Lastly, the converse result, namely if $x_1, x_2, \cdots, x_n$ are independently and identically distributed as Inverse Gaussian then y and z are independently distributed, is proved by M. C. K. Tweedie [2].

Thus, the theorem is completely proved.

3. Acknowledgment. The author is indebted to the referee for his guidance in the preparation of the paper.

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NULL DISTRIBUTION AND BAHADUR EFFICIENCY OF THE HODGES BIVARIATE SIGN TEST

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1. Summary. This note presents a simplified expression for the exact null distribution of the Hodges bivariate sign test. The form is suitable for computing both for small and large sample size and gives the limiting distribution easily. A small table is given for the limiting distribution. The Bahadur limiting efficiency of the test is computed relative to the bivariate Hotelling T^2 test for normal alternatives. The value $2/\pi$ is obtained, which is the same as for the one dimensional sign test relative to the t-test.

2. Introduction. Let $V_i = (X_i - X_i', Y_i - Y_i'), i = 1, 2, \cdots, n$, be a sample of n bivariate observations from a continuous distribution. The bivariate sign test was proposed by Hodges [5] in 1955 to test the hypothesis that the median is zero for the joint distribution. The test is based upon the statistic M, which

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is the maximum number of vectors that can have positive projections on some directed line through the origin. Equivalently the statistics $K = n - M$ or $H_n = 2M - n = n - 2K$ may be used. As shown by Hodges [5], the problem of calculating the null distribution can be translated into a random walk problem. The vectors V_i and $-V_i$ are plotted. To the extremity of V_i and $-V_i$ we assign a plus and minus sign respectively and from the resulting configuration construct a walk in the (X, Y)-plane. Starting at V_1 and moving counterclockwise we take a step in the Y-direction for each plus sign and a step in the X-direction for each minus sign. The walk consists of $2n$ steps but is completely determined after n steps since opposite signs have been assigned to V_i and $-V_i$. Let $(X(j), Y(j))$ represent the coordinates of the walk after j steps. Then $X(0) = Y(0) = 0$ and

$$X(j) + Y(j) = j, \tag{2.1}$$

$$\begin{aligned} X(n + j) &= X(n) + Y(j), \\ Y(n + j) &= Y(n) + X(j), \qquad 0 \leqq j \leqq n. \end{aligned} \tag{2.2}$$

From the geometry of the walk it can be seen that

$$H_n = \max_{0 \leqq j \leqq 2n} (X(j) - Y(j)) + \max_{0 \leqq j \leqq 2n} (Y(j) - X(j))$$

is the distance between the two 45° lines which enclose the walk (measured in the X or Y direction). Letting $Z_j = 2X(j) - j$ and using (2.1) and (2.2) we can write

$$H_n = \max(-Z_n + 2 \max_{0 \leqq j \leqq n} Z_j, Z_n - 2 \min_{0 \leqq j \leqq n} Z_j), \tag{2.3}$$

where Z_j is a sum of j independent and identically distributed random variables taking values ± 1 with probability $\frac{1}{2}$ under the null hypothesis.

3. The null distribution of H_n. It is well known that for the random walk

$$\begin{aligned} P_n(i, j, c) &= P[Z_n = j, 0 < Z_j < c, j = 1, \cdots, n \mid Z_0 = i] \\ &= P[Z_n = j - i, \quad -i < Z_l < c - i, \quad l = 1, \quad \cdots, n \mid Z_0 = 0] \\ &= \frac{2}{c} \sum_{k=1}^{c-1} \sin \frac{k\pi i}{c} \sin \frac{k\pi j}{c} \left(\cos \frac{k\pi}{c}\right)^n \end{aligned} \tag{3.1}$$

(See for example Kemperman [6] or Feller [3], prob. 5, p. 335.)

From (3.1) the distribution of H_n is easily derived by summing over the proper range of Z_j. Since H_n and n are of the same parity, upon interchanging the order of summation and using elementary trigonometric identities, we obtain the expressions

$$\begin{aligned} P[H_{2m} \leqq 2r] &= 2 \sum_{k=1}^{r} (-1)^{k+1} \left(\cos \frac{k\pi}{2(r + 1)}\right)^{2m}, \qquad r = 1, \cdots, m, \\ P[H_{2m+1} \leqq 2r + 1] &= 2 \sum_{k=1}^{r+1} (-1)^{k+1} \left(\cos \frac{k\pi}{2r + 3}\right)^{2m+1}, \quad r = 0, 1, 2, \cdots, m. \end{aligned} \tag{3.2}$$

For instance in the case $n = 2m$, from (2.3) and (3.1) we have

$$P[H_{2m} \leqq 2r] = \sum_{l=-r}^{r} P_{2m}(r + 1 - l, r + 1 + l, 2r + 2)$$

$$= \sum_{l=-r}^{r} \frac{2}{r+1} \sum_{k=1}^{r} \sin \frac{k\pi(r+1-l)}{2(r+1)} \sin \frac{k\pi(r+1+l)}{2(r+1)} \left[\cos \frac{k\pi}{2(r+1)}\right]^{2m}$$

$$= \frac{1}{r+1} \sum_{k=1}^{r} \left[\cos \frac{k\pi}{2(r+1)}\right]^{2m} \left[(-1)^{k+1}(2r+1) + \sum_{l=-r}^{+r} \cos \frac{k\pi}{r+1} l\right].$$

Using the identity $\frac{1}{2} + \sum_{l=1}^{r} \cos 2\pi l\alpha = [\sin \pi\alpha(2r + 1)]/[2 \sin \pi\alpha]$ we obtain easily the first formula of (3.2).

Now it is a trivial matter to derive the limiting distribution of $H_n/n^{\frac{1}{2}}$.

$$\text{(3.3)} \qquad \begin{aligned} \lim_{n\to\infty} P[H_n/n^{\frac{1}{2}} \leqq t] &= 2\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k^2(\pi^2/2t^2)} \\ &= 1 - 4t\sum_{k=1}^{\infty} \varphi((2k-1)t) \end{aligned}$$

where φ is the standard normal density.

The last equality is a well known fact from the theory of theta functions. While working on this note the paper of Hill [4] came to the attention of the authors. In this paper Hill gives a relation between the tests of Hodges and Daniels [2]. Daniels has derived the limiting form (3.3) and the following exact expression

$$\text{(3.4)} \qquad P[H_n \geqq h] = \frac{h}{2^{n-1}} \sum_{j \geqq 0} \binom{n}{m + jh}$$

where the series terminates at $j = [k/h]$ with $h = n - 2k = 2m - n$. This expression can easily be seen to be equivalent to (3.2). (3.2) seems very useful for computations, since only the first few terms in the sum give important contributions.

Another way of deriving (3.3) is to approximate the random walk by a Wiener process and then use formula (14) of Lévy [8] page 213. It may be worthwhile to note that Lévy's formula may be obtained as a limiting case of (3.1). The moments of the limiting distribution obtained from (3.3) are

$$\text{(3.5)} \qquad \begin{aligned} \mu_{2p+1} &= 2^{p+1}(2p+1)p!(2/\pi)^{\frac{1}{2}}\zeta(2p+2)[1 - 2^{-(2p+2)}] \\ \mu_{2p} &= \{2p!/[2^{p-2}(p-1)!]\}\zeta(2p+1)[1 - 2^{-(2p+1)}] \end{aligned}$$

where ζ is the Riemann zeta-function.

4. Limiting efficiency. It is of interest to compare the test of Hodges with its parametric competitor, the Hotelling T^2-test, for bivariate normal alternatives. Let $(X_i - X_i', Y_i - Y_i')$ have a bivariate normal distribution with mean

$m = (\mu, \nu)$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma^2 & \rho\sigma\tau \\ \rho\sigma\tau & \tau^2 \end{pmatrix}.$$

A measure of efficiency between tests has been suggested by Bahadur [1]. Following his methods and notation we compute slopes for the two tests. Let $T_n^{(1)} = T$ where T^2 is the Hotelling bivariate statistic. Under the hypothesis $m = (0,0)$, $T_n^{(1)}$ has a limiting χ_2 distribution. Thus for a chi distribution we have $\log(1 - F(x)) = (-ax^2/2)(1 + o(1))$ as $x \to \infty$ with $a = 1$. Under the alternative $m = (\mu,\nu)$, we have $T_n^{(1)}/n^{\frac{1}{2}} \xrightarrow{P} \Delta$ where $\Delta^2 = m\Sigma^{-1}m'$. Thus the slope of Hotelling's test is given by $c_1(m, \Sigma) = \Delta^2$. Similarly let $T_n^{(2)} = H_n/n^{\frac{1}{2}}$ for the test of Hodges. From (3.3) the limiting null distribution again satisfies the condition on the tail with $a = 1$. To compute the limit of $T_n^{(2)}/n^{\frac{1}{2}}$ as $n \to \infty$ under the alternative, we consider the statistic M where $H_n = 2M - n$. Since M is the maximum number of vectors with positive projections on some line through the origin we can write $M = \sup S_n(\theta)$, $0 \leqq \theta < 2\pi$, where $S_n(\theta)$ is simply the statistic of the sign test for the projections $(X_i - X_i') \cos\theta + (Y_i - Y_i') \sin\theta$ on the line with direction θ. $S_n(\theta)$ is thus a binomial random variable and

$$S_n(\theta)/n \xrightarrow{\text{a.s.}} P(\theta) = P[(X_i - X_i')\cos\theta + (Y_i - Y_i')\sin\theta > 0].$$

Using a theorem of Parzen ([9] Theorem 15A, p. 43), it follows that the convergence is uniform in θ, and thus

$$\sup_{0 \leqq \theta < 2\pi} S_n(\theta)/n \xrightarrow{\text{a.s.}} \sup_\theta P(\theta) \tag{4.1}$$

where

$$P(\theta) = \Phi\left(\frac{\mu\cos\theta + \nu\sin\theta}{[\sigma^2\cos^2\theta + 2\rho\sigma\tau\sin\theta\cos\theta + \tau^2\sin^2\theta]^{\frac{1}{2}}}\right) \tag{4.2}$$

with Φ the standard normal distribution. Evaluating the supremum of (4.1) we have $M/n \xrightarrow{\text{a.s.}} \Phi(\Delta)$. Hence

$$T_n^{(2)}/n^{\frac{1}{2}} = H/n = (2M - n)/n \xrightarrow{\text{a.s.}} 2\Phi(\Delta) - 1$$

and the slope of the test of Hodges is thus given by

$$C_2(m, \Sigma) = (2\Phi(\Delta) - 1)^2.$$

The Bahadur efficiency is now computed as

$$e_{21}(\Delta) = C_2(\Delta)/C_1(\Delta) = (2\Phi(\Delta) - 1)^2/\Delta^2. \tag{4.3}$$

As $m \to (0,0)$ we have $\Delta \to 0$ and (4.3) gives the value $2/\pi$. It is of interest to note that (4.3) coincides with the corresponding expression for the one dimensional sign test and the t-test with $\Delta = \mu/\sigma$ for this case

TABLE I

$\lim_{n\to\infty} P[H/n^{\frac{1}{2}} < t]$*

t	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0	.00000	.00000	.00000	.00000	.00000	.00000	.00008	.00090	.00452
1.	.01438	.03387	.06497	.10785	.16120	.22280	.29008	.36046	.43156	.50132
2.	.56807	.63054	.68782	.73939	.78501	.82472	.85874	.88745	.91135	.93095
3.	.94682	.95949	.96948	.97726	.98324	.98778	.99119	.99371	.99556	.99690
4.	.99786	.99854	.99901	.99934	.99956	.99971	.99981	.99988	.99992	.99995

* computed on the McGill I.B.M. 650

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NOTE ON MULTIVARIATE GOODNESS-OF-FIT TESTS[1]

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1. Introduction and summary. Let $X_1, X_2, \cdots, X_n$ be m-dimensional statistically independent random vectors with common distribution function F. It is frequently desirable to test the hypothesis that F is a member of some class of distribution functions $\mathcal{H}_o$. For the scalar case, $(m = 1)$, much research has been done; see for example [1], [2], [3]. For $m > 1$ comparatively little has been accomplished, and a useful extension of the techniques used for $m = 1$ awaits

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