PROBABILITY CONTENT OF REGIONS UNDER SPHERICAL NORMAL DISTRIBUTIONS, IV: THE DISTRIBUTION OF HOMOGENEOUS AND NON-HOMOGENEOUS QUADRATIC FUNCTIONS OF NORMAL VARIABLES[1]

By Harold Ruben

Boeing Scientific Research Laboratories and University of Sheffield

0. Synopsis. The distribution function of a non-negative quadratic form, both homogeneous and non-homogeneous, of a finite number of correlated normal random variables is expressed as an infinite linear combination of chi-square distribution functions with arbitrary scale parameter. An alternative representation of the distribution function of the non-homogeneous form in terms of *non-central* chi-square distribution functions with arbitrary scale parameter is also derived.

The nature and accuracy of the above series representations is discussed in detail. It is shown that these reduce to *mixture* representations for certain values of the scale parameter.

1. Introduction and summary. Let $\mathbf{y}$ denote an n-dimensional vector which has a multivariate normal distribution with zero expectation and non-singular variance-covariance matrix $\mathbf{V}$. We shall consider the distribution of the quadratic form $(\mathbf{y} - \boldsymbol{\alpha})'\mathbf{C}(\mathbf{y} - \boldsymbol{\alpha})$ for a given vector $\boldsymbol{\alpha}$ and a given symmetric positive definite matrix $\mathbf{C}$. In geometrical terms, we wish to evaluate the probability content of a fixed ellipsoid R^* of arbitrary size, location and orientation under an underlying mutlivariate normal distribution. This probability content is equal to

$$(2\pi)^{-\frac{1}{2}n} \mid \mathbf{V} \mid^{-\frac{1}{2}} \int_{R^*} e^{-\frac{1}{2}\mathbf{y}'\mathbf{V}^{-1}\mathbf{y}} \, d\mathbf{y}, \tag{1.1}$$

where $R^* = \{\mathbf{y}: (\mathbf{y} - \boldsymbol{\alpha})'\mathbf{C}(\mathbf{y} - \boldsymbol{\alpha}) \leqq t\}$.

Without loss of generality, the required distribution may be expressed in canonical form as the distribution of $(\mathbf{x} - \mathbf{b})'\mathbf{A}(\mathbf{x} - \mathbf{b})$, where $\mathbf{x}$ has a centered, standardized spherical normal distribution of dimensionality n, $\mathbf{b}$ is a fixed n-dimensional vector and $\mathbf{A}$ is a diagonal matrix of size $n \times n$ with diagonal elements $a_1, \cdots, a_n (a_i > 0)$. This is achieved by the linear transformations

$$\mathbf{y} = \mathbf{LKx}, \qquad \boldsymbol{\alpha} = \mathbf{LKb}, \tag{1.2}$$

where $\mathbf{L}$ is a lower triangular matrix defined by $\mathbf{V} = \mathbf{LL}'$ and $\mathbf{K}$ is the orthogonal

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matrix of the eigenvectors of $\mathbf{L'CL}$(cf.[19], p. 610). We then find that (1.1) reduces to

$$H_{n;\mathbf{A},\mathbf{b}}(t) = (2\pi)^{-\frac{1}{2}n} \int_R e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\, d\mathbf{x} = P\left[\sum_1^n a_i(x_i - b_i)^2 \leqq t\right], \tag{1.3}$$

where $R = \{\mathbf{x}: (\mathbf{x} - \mathbf{b})'\mathbf{A}(\mathbf{x} - \mathbf{b}) \leqq t\}$, $\mathbf{A} = \mathbf{K'L'CLK}$, while the a_i are the eigenvalues of $\mathbf{L'CL}$ and therefore of $\mathbf{VC}$. Thus, for $t > 0$, $H_{n;\mathbf{A},\mathbf{b}}(t)$ is the probability content, under a centered spherical normal distribution with unit standard deviation in any direction, of an ellipsoid centered at the point $(b_1, \cdots, b_n)$, with fixed orientation and with lengths of semi-axes $(t/a_1)^{\frac{1}{2}}, \cdots, (t/a_n)^{\frac{1}{2}}$.

We shall assume for convenience, again without suffering any loss of generality, that $a_1 \leqq a_2 \leqq \cdots \leqq a_n$.

An equivalent geometrical interpretation of $H_{n;\mathbf{A},\mathbf{b}}(t)$ is worth noting. On replacing $\mathbf{x}$ by $\mathbf{A}^{-\frac{1}{2}}\mathbf{z}$ in (1.3), we obtain

$$H_{n;\mathbf{A},\mathbf{b}}(t) = (2\pi)^{-\frac{1}{2}n} A^{-\frac{1}{2}} \int_{R^{**}} e^{-\frac{1}{2}\mathbf{z}'\mathbf{A}^{-1}\mathbf{z}}\, d\mathbf{z}, \tag{1.4}$$

where $A = |\mathbf{A}| = a_1 \cdots a_n$ and $R^{**} = \{\mathbf{z}: (\mathbf{z} - \mathbf{A}^{\frac{1}{2}}\mathbf{b})'(\mathbf{z} - \mathbf{A}^{\frac{1}{2}}\mathbf{b}) \leqq t\}$; that is, $H_{n;\mathbf{A},\mathbf{b}}(t)$ is the probability content of an offset sphere under a centered uncorrelated multivariate normal distribution with standard deviations $a_1^{\frac{1}{2}}, \cdots, a_n^{\frac{1}{2}}$.

Clearly, the previous discussion can be generalized to include the case where $\mathbf{C}$ is semi-definite positive. Here at least one of the a_i is zero, R is an elliptic cylinder (rather than an ellipsoid) for $t > 0$, and (1.1) reduces to

$$(2\pi)^{-\frac{1}{2}n} \int_R e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\, d\mathbf{x} = P\left[\sum_1^{n'} a_i(x_i - b_i)^2 \leqq t\right],$$

where n' is the rank of $\mathbf{C}$, i.e., the number of positive (non-zero) a_i. In brief, the results obtained in the sequel are valid, with a suitable interpretation of n, for all *non-negative* quadratic forms of normal variables.

It will be observed that the central and non-central chi-square distribution functions are subsumed under the H-distribution function as very special and relatively trivial cases. Let $F_n(\cdot)$ and $G_{n;\kappa}(\cdot)$ denote the distribution functions of χ_n^2, a (central) chi-square with n degrees of freedom, and of a non-central chi-square with n degrees of freedom and non-centrality parameter κ, respectively, i.e.,

$$F_n(t) \equiv P\left[\sum_1^n x_i^2 \leqq t\right], \tag{1.5}$$

so that

$$F_n(t) = \{2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)\}^{-1} \cdot \int_0^t e^{-\frac{1}{2}y} y^{\frac{1}{2}n-1}\, dy \qquad (t > 0), \tag{1.5'}$$

$$= 0 \qquad (t \leqq 0),$$

while

$$G_{n;\kappa}(t) \equiv P\left[\sum_1^n (x_i - b_i)^2 \leqq t\right], \tag{1.6}$$

so that ([14], [19])

$$\begin{aligned} G_{n;\kappa}(t) &= \tfrac{1}{2}\kappa^{-(\frac{1}{2}n-1)}e^{-\frac{1}{2}\kappa^2}\cdot \int_0^t e^{-\frac{1}{2}y}y^{\frac{1}{2}n-\frac{1}{2}}I_{\frac{1}{2}n-1}(\kappa y^{\frac{1}{2}})\,dy \qquad (t > 0), \\ &= 0 \qquad (t \leqq 0), \end{aligned} \tag{1.6'}$$

where $I_m(z) = i^{-m}J_m(iz)$ is the Bessel function of the first kind with purely imaginary argument. Then

$$F_n(t) = H_{n;\mathbf{I},\mathbf{0}}(t) \tag{1.7}$$

and

$$G_{n;\kappa}(t) = H_{n;\mathbf{I};\mathbf{b}}(t) \qquad \left(\kappa^2 = \sum_1^n b_i^2\right), \tag{1.8}$$

where $\mathbf{I}$ is the unit $n \times n$ matrix. Geometrically, for $t > 0$, $F_n(t)$ is the probability content, under an n-dimensional spherical distribution, of a sphere of radius $t^{\frac{1}{2}}$ whose center coincides with the center of the distribution, while $G_{n;\kappa}(t)$ is the probability content, under the same distribution, of a sphere of radius $t^{\frac{1}{2}}$ whose center is at a distance κ from the center of the distribution.

A rather extensive literature is in existence on the special case $H_{n;\mathbf{A},\mathbf{0}}(\cdot)$, the distribution function of a homogeneous quadratic form,

$$H_{n;\mathbf{A};\mathbf{0}}(t) = P\left[\sum_1^n a_i x_i^2 \leqq t\right].$$

(See [19] and [22] for brief reviews with some applications, and also [1], [2], [5], [6], [7] and [15], pp. 19–24. For tables of the distribution functions and/or percentile points of $\sum_1^2 a_i x_i^2$ and $\sum_1^3 a_i x_i^2$, additional to those listed in [19], see [4], [11] and [22].) In particular, from the point of point of view of this paper the results which are most directly relevant are due to Robbins [16] and Robbins and Pitman [17]. These express the distribution function of $\sum_1^n a_i x_i^2$ as a linear combination of Gamma distribution functions: more specifically, Robbins's result is of the form

$$H_{n;\mathbf{A};\mathbf{0}}(t) = \sum_{j=0}^{\infty} \omega_j F_{n+2j}(t/a_g), \tag{1.9}$$

where a_g is the geometric mean of the a_i, while Robbins and Pitman's result is of the form

$$H_{n;\mathbf{A};\mathbf{0}}(t) = \sum_{j=0}^{\infty} w_j F_{n+2j}(t/a_1), \tag{1.10}$$

where the w_j form a probability sequence, i.e.

$$w_j \geqq 0 (j = 0, 1, \cdots), \qquad \sum_0^\infty w_j = 1. \tag{1.11}$$

The w_j are, in fact, the probabilities in the convolution of n negative binomial distributions; in the terminology of Robbins and Pitman, the distribution of $\sum_1^n a_i x_i^2$ as represented in (1.10) is a mixture representation. We shall generalize (1.9) and (1.10) (Section 3) in the sense of establishing the relationship

$$H_{n;\mathbf{A};\mathbf{0}}(t) = \sum_{j=0}^\infty c_j F_{n+2j}(t/p), \tag{1.12}$$

where p is an arbitrary positive constant and the c_j involve p (as well as n and **A**). Thus (1.12) expresses the distribution function of $\sum_1^n a_i x_i^2$ as a linear combination of Gamma distribution functions with arbitrary scale parameter p ($F_{n+2j}(t/p)$ is, for varying t, the distribution function of $p\chi^2_{n+2j}$, where χ^2_{n+2j} is a chi-square with $n + 2j$ degrees of freedom). The arbitrary nature of p in (1.12) is an advantage inasmuch as it allows of greater flexibility and comprehensiveness, and, moreover, for certain values of p, (1.12) is a mixture representation. The significance of the two scale constants $p = a_0$, $p = a_1$ in (1.9) and (1.10), respectively, will be discussed in Section 5.

Reverting to the general case of $H_{n;\mathbf{A},\mathbf{b}}(t)$, where $\mathbf{b} \neq \mathbf{0}$, very few, if any, theoretical results appear to have been obtained so far[2] (though tables of $H_{2;\mathbf{A},\mathbf{b}}(t)$, presented in a form suitable for the evaluation of the probability of offset circles, as in (1.4), have recently been given in [10] and [18], while the efficient numerical evaluation of this function is discussed in [3] and, finally, some further, unpublished, numerical results have been obtained by Marsaglia for $H_{2;\mathbf{A},\mathbf{b}}(t)$ and $H_{3;\mathbf{A},\mathbf{b}}(t)$). In order to fill this theoretical gap, we shall show that the distribution function of $\sum_1^n a_i(x_i - b_i)^2$ may be expressed as a linear combination of infinitely many chi-square distribution functions with arbitrary scale parameter p, analogously to (1.12), and, furthermore, this distribution function may also be expressed as a linear combination of infinitely many *non-central* chi-square distribution functions with arbitrary scale parameter p. More specifically,

$$H_{n;\mathbf{A},\mathbf{b}}(t) = \sum_{j=0}^\infty c_j F_{n+2j}(t/p), \tag{1.13}$$

[2] A representation of $H_{n;\mathbf{A},\mathbf{b}}(t)$ as a power series in t has recently been given in [21]. However, from a practical point of view, this series, like the corresponding power series for $H_{n;\mathbf{A},\mathbf{0}}(t)$ in [13] of which it is a generalization, is of limited value, except perhaps for quite low values of t.

Note added in proof: Since revision of this paper, a futher theoretical discussion of the distribution of the non-homogeneous form has appeared in print (J. P. Imhof, "Computing the distribution of quadratic forms in normal variables," *Biometrika*, Vol. 48, 1961). Imhof's paper appears to have no point of contact with the present paper.

and also

$$H_{n;\mathbf{A},\mathbf{b}}(t) = \sum_{j=0}^{\infty} d_j G_{n+2j;\kappa}(t/p) \qquad (\kappa^2 = \sum b_i^2), \tag{1.14}$$

where $c_j \equiv c_{j,n;\mathbf{A},\mathbf{b}}(p)$, $d_j \equiv d_{j,n;\mathbf{A},\mathbf{b}}(p)$. As in (1.12), (1.13) and (1.14) are mixture representations for suitable choice of p.

Generating functions as well as both explicit and recursive formulae for the coefficients (the c_j and the d_j) in the series (1.12), (1.13) and (1.14) will be derived, and upper bounds for the errors induced by using the various series in truncated form will be deduced. The latter recursive formulae are relatively rather simple; in particular, for the special cases $p = a_0$ and $p = a_1$ in (1.9) and (1.10) for $H_{n;\mathbf{A},\mathbf{0}}(t)$, they appear to be much more effective than the recursion relationships given previously in [16] and [17].

Notation and terminology.

(i) $\mathbf{x} = (x_1, \cdots, x_n)$ will denote a random n-dimensional vector or point in Euclidean n-space E_n, and $\mathbf{l} = (l_1, \cdots, l_n)$ a random vector with terminal point on the surface, Ω_n, of the unit n-sphere centered at the origin $(\sum_1^n l_i^2 = 1)$. In accordance with custom, $\mathbf{x}$ and $\mathbf{l}$ will also be used in two additional senses, namely, as open vector variables and as dummy integration variables. (The inconsistency of this notation has proved harmless in the past and will do so again at present.)

(ii) $|\mathbf{x}|$ will denote the norm of $\mathbf{x}(|\mathbf{x}| = +(\sum_1^n x_i^2)^{\frac{1}{2}})$.

(iii) $d\mathbf{x}$ will represent the volume-content of an infinitesimal n-dimensional element at the point $\mathbf{x}$ and $d\mathbf{l}$ the surface-content of an infinitesimal $(n-1)$-dimensional element on Ω_n at the point $\mathbf{l}$.

(iv) S_n will denote the (surface) content of Ω_n,

$$S_n = 2\pi^{\frac{1}{2}n}/\Gamma(\tfrac{1}{2}n). \tag{1.15}$$

(v) For the special case where $\mathbf{l}$ is uniformly distributed on Ω_n, $E\phi(\mathbf{l})$ will be written as $M\phi(\mathbf{l})$, i.e.,

$$M\phi(\mathbf{l}) = S_n^{-1} \cdot \int_{\Omega_n} \phi(\mathbf{l})\, d\mathbf{l}. \tag{1.16}$$

In connection with the M-operator, we note that

$$M \sum_0^{\infty} \phi_j(\mathbf{l}) = \sum_0^{\infty} M\phi_j(\mathbf{l}) \tag{1.17}$$

if $\sum_0^\infty \phi_j(\mathbf{l})$ converges uniformly on Ω_n.

(vi) Whenever convenient, $M\phi(\mathbf{l})$ will henceforth be written as $M\phi$, and $E\phi(\mathbf{x})$ as $E\phi$. Thus the argument of ϕ will generally be suppressed and supplied by the context ($\mathbf{l}$ if the expectation operator is M and $\mathbf{x}$ if the expectation operator is E). Furthermore, even in the absence of an expectation operator, we shall in general designate both $\phi(\mathbf{l})$ and $\phi(\mathbf{x})$ indifferently by ϕ, where the argument in ϕ is to be supplied by the context.

(vii) We shall say that $\mathbf{l}$ is induced by $\mathbf{x}$ if $\mathbf{l} = \mathbf{x}/|\mathbf{x}|$ for $\mathbf{x} \neq \mathbf{0}$, and if $\mathbf{x} = \mathbf{0}$, $\mathbf{l}$ is determined by a random mechanism which assigns a probability measure $d\mathbf{l}/S_n$ to an infinitesimal element at $\mathbf{l}$(on Ω_n) of surface-content $d\mathbf{l}$. Note that this is equivalent to generating a probability-mass distribution on Ω_n from that in E_n by radial projection of the mass in E_n on to Ω_n, with the additional stipulation that any concentration of mass at the origin is to be spread uniformly on Ω_n.

(viii) We shall say that $\mathbf{x}$ has a centered spherical distribution (or has centered spherical symmetry) if the distribution of $\mathbf{Tx}$ is the same as that of $\mathbf{x}$ for every orthogonal matrix $\mathbf{T}$: equivalently, if the probability measure of every Borel set in E_n is invariant with respect to all rotations about the origin.

(ix) In all summations, and elsewhere in the text, the range of i will be from 1 to n and that of j, unless otherwise specified, from 0 to ∞.

REMARK. If $\mathbf{x}$ has a centered spherical distribution and $\mathbf{l}$ is induced by $\mathbf{x}$, then $\mathbf{l}$ is uniform on Ω_n and, furthermore, $\mathbf{l}$ and $|\mathbf{x}|$ are independently distributed.

These are immediate consequences of the definition of centered spherical symmetry. Thus, to prove the first part of the assertion, consider two arbitrary congruent half-cones in E_n with vertices at the origin. To prove the second part, consider the portions of the two latter half-cones which are interior to an arbitrary sphere with center at the origin.

2. Preliminary lemmas.

LEMMA 1. *If* $\mathbf{x}$ *has a centered spherical distribution,* $\mathbf{l}$ *is induced by* $\mathbf{x}$, $\phi(\mathbf{x})$ *is a homogeneous scalar function in the* x_i *of degree* k *and* $E\,|\phi(\mathbf{x})| < \infty$, *then*

$$E\phi(\mathbf{x}) = E|\mathbf{x}|^k \cdot M\phi(\mathbf{l}). \tag{2.1}$$

PROOF. On using the defining property of a homogeneous function together with the property of stochastic independence of $\mathbf{l}$ and $|\mathbf{x}|$, we find

$$\begin{aligned} E\phi(\mathbf{x}) &= E[\phi(|\mathbf{x}|\mathbf{l})] \\ &= E[|\mathbf{x}|^k\phi(\mathbf{l})] \\ &= E|\mathbf{x}|^k \cdot E\phi(\mathbf{l}). \end{aligned}$$

Since $\mathbf{l}$ is uniform on Ω_n, $E\phi(\mathbf{l})$ can be replaced by $M\phi(\mathbf{l})$ and the lemma is proved.

COROLLARY. *If* $\mathbf{x}$ *has a centered standardized spherical normal distribution with density function*

$$(2\pi)^{-\frac{1}{2}n}e^{-\frac{1}{2}|\mathbf{x}|^2}, \tag{2.2}$$

then

$$M\phi = [\Gamma(\tfrac{1}{2}n)/\{2^{\frac{1}{2}k}\Gamma(\tfrac{1}{2}(n+k))\}]\cdot E\phi. \tag{2.3}$$

Equation (2.3) follows on using (2.2) in (2.1) (or, equivalently, on noting that $|\mathbf{x}|^2$ is a χ_n^2).

LEMMA 2. *Define*

$$L(1) \equiv L = \sum p_i l_i, \qquad Q(1) \equiv Q = \sum q_i l_i^2 \tag{2.4}$$

on Ω_n, *where* p_i *and* q_i *are real and* $q_i > 0$. *Then*

$$L^2/Q \leqq \sum p_i^2/q_i. \tag{2.5}$$

PROOF. Set $h_i = q_i^{\frac{1}{2}} l_i$ and use the Schwarz inequality. Thus

$$\begin{aligned} L^2/Q &= \sum \{(p_i/q_i^{\frac{1}{2}})h_i\}^2 / \sum h_i^2 \\ &\leqq \sum (p_i/q_i^{\frac{1}{2}})^2 \sum h_i^2 / \sum h_i^2 \\ &= \sum p_i^2/q_i. \end{aligned}$$

LEMMA 3. *Define*

$$L(\mathbf{x}) \equiv L = \sum p_i x_i, \qquad Q(\mathbf{x}) \equiv Q = \sum q_i x_i^2 \tag{2.6}$$

on E_n, *where the* p_i *and* q_i *are real, and let* $H_k(t)$ *denote the Hermite polynomial of degree* k *in* t:

$$H_k(t) = \sum_{r=0}^{[k/2]} (-1)^r \frac{k!}{(k-2r)!\,2^r r!} t^{k-2r} \tag{2.7}$$

($[k/2]$ *denotes, as usual, the integral part of* $k/2$). *Then if the distribution of* $\mathbf{x}$ *is spherical normal as in* (2.2),

$$|E[Q^j H_{2j}(L/Q^{\frac{1}{2}})]/(2j)!| \leqq (\tfrac{1}{2}n)_j (q_0 + \tfrac{1}{2} \sum p_i^2)^j / j!, \tag{2.8}$$

where $(\frac{1}{2}n)_j = \Gamma(\frac{1}{2}n + j)/\Gamma(\frac{1}{2}n)$ *and* $q_0 = \max_i |q_i|$.

PROOF. From (2.7),

$$Q^j H_{2j}(L/Q^{\frac{1}{2}}) = \sum_{r=0}^{j} (-1)^r \frac{(2j)!}{(2j-2r)!\,2^r r!} L^{2j-2r} Q^r. \tag{2.9}$$

Since $|Q| \leqq q_0 \sum x_i^2$ and $\sum x_i^2$ is a χ_n^2,

$$|EQ^r| \leqq q_0^r \cdot 2^r (\tfrac{1}{2}n)_r. \tag{2.10}$$

Again, since L is normal with mean zero and variance $\sum p_i^2$,

$$EL^{2j-2r} = (\sum p_i^2)^{j-r} \cdot (2j-2r)!/\{2^{j-r}(j-r)!\}. \tag{2.11}$$

Applying (2.10) and (2.11) in (2.9) and replacing $(\frac{1}{2}n)_r$ by $(\frac{1}{2}n)_j$, we obtain

$$\begin{aligned} |E[Q^j H_{2j}(L/Q^{\frac{1}{2}})]| &\leqq \sum_{r=0}^{j} \frac{(2j)!}{(2j-2r)!\,2^r r!} \cdot \left(\sum p_i^2\right)^{j-r} \frac{(2j-2r)!}{2^{j-r}(j-r)!} \\ &\cdot (2q_0)^r (\tfrac{1}{2}n)_r \leqq \{(\tfrac{1}{2}n)_j (2j)!/j!\} \cdot \sum_{r=0}^{j} \frac{j!}{r!(j-r)!} q_0^r (\tfrac{1}{2} \sum p_i^2)^{j-r} \\ &= \{(\tfrac{1}{2}n)_j (2j)!/j!\} \cdot (q_0 + \tfrac{1}{2} \sum p_i^2)^j. \end{aligned}$$

LEMMA 4. *The series*

$$\sum \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{\mu^j}{j!} F_{n+2j}(x) \qquad (\mu \geqq 0) \tag{2.12}$$

converges uniformly on $-\infty \leqq x \leqq X < \infty$ *for every non-negative* μ *and on* $-\infty \leqq x \leqq \infty$ *when* $0 \leqq \mu < 1$.

If $0 \leqq \mu < 1$ *an upper bound to the remainder after* k *terms is*

$$(1-\mu)^{-(\frac{1}{2}n+k)} \cdot \frac{\Gamma(\frac{1}{2}n + k)}{\Gamma(\frac{1}{2}n)} \frac{\mu^k}{k!} F_{n+2k}[(1-\mu)x] \qquad (0 \leqq \mu < 1) \tag{2.13}$$

for $-\infty \leqq x \leqq \infty$.

PROOF. The first part of the lemma is trivial for $X \leqq 0$, while the second part is trivial for $x \leqq 0$. Assume, then, that $X > 0$ and $x > 0$, respectively. From (1.5′),

$$F_{n+2j}(x) < \{2^{\frac{1}{2}n+j}\Gamma(\tfrac{1}{2}n + j)\}^{-1} x^{\frac{1}{2}n+j-1} \int_0^\infty e^{-\frac{1}{2}y}\, dy$$
$$= 2\cdot\{2^{\frac{1}{2}n+j}\Gamma(\tfrac{1}{2}n + j)\}^{-1} x^{\frac{1}{2}n+j-1} \qquad (x > 0).[3]$$

Hence the series (2.12) is equal to zero for $x \leqq 0$ and is majorized for $0 < x \leqq X$ by

$$2\cdot\{2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)\}^{-1} x^{\frac{1}{2}n-1} \sum_0^\infty \frac{(\frac{1}{2}\mu x)^j}{j!} \leqq 2\cdot\{2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)\}^{-1} X^{\frac{1}{2}n-1} e^{\frac{1}{2}\mu X}.$$

The uniform convergence of the series on every finite x-interval when $\mu \geqq 0$ is thus established. To prove uniformity of convergence for all x (finite or infinite) when $0 \leqq \mu < 1$, observe that the series is then dominated by

$$\sum_0^\infty \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{\mu^j}{j!} = (1-\mu)^{-\frac{1}{2}n} \qquad (0 \leqq \mu < 1).$$

Finally, to prove the last part of the lemma, observe that (for $x > 0$)

$$\sum_{j=k}^\infty \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{\mu^j}{j!} F_{n+2j}(x) = \frac{1}{\Gamma(\frac{1}{2}n)\mu^{\frac{1}{2}n}} \int_0^{\mu x/2} e^{-v/\mu} \sum_{j=k}^\infty \frac{v^{\frac{1}{2}n+j-1}}{j!}\, dv,$$

after using an obvious scale transformation in the integral defining $F_{n+2j}(x)$, interchange of summation and integration being allowed by the uniformity of convergence. But

$$\sum_{j=k}^\infty \frac{v^{\frac{1}{2}n+j-1}}{j!} = \frac{v^{\frac{1}{2}n+k-1}}{k!} \sum_{r=0}^\infty \frac{v^r}{(k+1)(k+2)\cdots(k+r)} \leqq \frac{v^{\frac{1}{2}n+k-1}}{k!} e^v.$$

Hence, for $0 \leqq \mu < 1$,

$$\sum_{j=k}^\infty \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{\mu^j}{j!} F_{n+2j}(x) \leqq \frac{1}{k!\Gamma(\frac{1}{2}n)\mu^{\frac{1}{2}n}} \int_0^{\mu x/2} e^{-v(1-\mu)/\mu} v^{\frac{1}{2}n+k-1}\, dv$$
$$= (1-\mu)^{-(\frac{1}{2}n+k)} \cdot \frac{\Gamma(\frac{1}{2}n + k)}{\Gamma(\frac{1}{2}n)} \frac{\mu^k}{k!} F_{n+2k}[(1-\mu)x].$$

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[3] This inequality is valid for $n = 2, 3, \cdots$ and $j = 0, 1, \cdots$. The subsequent modification of the proof for the special case $n = 1, j = 0$ is both obvious and trivial (the first term in the series (2.12) is then $F_1(x) \leqq F_1(X)$).

LEMMA 5. *The series*

$$\sum \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{\mu^j}{j!} G_{n+2j;\kappa}(x) \qquad (\mu \geqq 0) \tag{2.14}$$

converges uniformly on $-\infty \leqq x \leqq X < \infty$ *for every non-negative* μ *and on* $-\infty \leqq x \leqq \infty$ *when* $0 \leqq \mu < 1$.

If $0 \leqq \mu < 1$ *an upper bound to the remainder after* k *terms is*

$$(1-\mu)^{-(\frac{1}{2}n+k)} \cdot \frac{\Gamma(\frac{1}{2}n + k)}{\Gamma(\frac{1}{2}n)} \frac{\mu^k}{k!} F_{n+2k}[(1-\mu)x] \qquad (0 \leqq \mu < 1) \tag{2.15}$$

for $-\infty \leqq x \leqq \infty$.

PROOF. We have

$$G_{m;\kappa}(x) \leqq F_m(x)$$

(the equality sign holds, trivially, only for $\kappa = 0$ or for $x \leqq 0$). This inequality follows from the well-known fact of the decreasing character of the G-function with respect to κ and on recalling that $G_{m;0}(x) = F_m(x)$. Hence the series (2.14) is dominated by the series (2.12) and Lemma 5 follows directly from Lemma 4.

3. Evaluation of $H_{n;\mathbf{A},\mathbf{b}}(t)$ as an infinite linear combination of χ^2 distribution functions.

In this section we prove two fundamental theorems. The second theorem, which deals with the central case $\mathbf{b} = \mathbf{0}$, may, of course, be regarded as a quite special case of the first theorem. However, because of the great importance of this special case, and also because it will serve subsequently to motivate a fresh theorem (Theorem 3, Section 4), in which $H_{n;\mathbf{A},\mathbf{b}}$ is represented for $\mathbf{b} \neq \mathbf{0}$ as an infinite linear combination of non-central χ^2 distribution functions, Theorem 2 will be stated explicitly.

THEOREM 1.

(i)

$$H_{n;\mathbf{A},\mathbf{b}}(t) = \sum_0^\infty c_j F_{n+2j}(t/p), \tag{3.1}$$

where p *is an arbitrary positive constant,*

$$\begin{aligned} c_j &\equiv c_{j,n;\mathbf{A},\mathbf{b}}(p) \\ &= A^{-\frac{1}{2}} e^{-\frac{1}{2}\Sigma b_i^2} p^{\frac{1}{2}n+j} \cdot E[Q^j H_{2j}(L/Q^{\frac{1}{2}})]/(2j)!, \end{aligned} \tag{3.2}$$

L *and* Q *are defined by*

$$L \equiv L(\mathbf{x}) = \sum (b_i/a_i^{\frac{1}{2}}) x_i, \qquad Q \equiv Q(\mathbf{x}) = \sum (1/a_i - 1/p) x_i^2, \tag{3.3}$$

and the x_i *are independent normal variables with zero means and unit variances. Further, the series in* (3.1) *converges uniformly on every finite interval of* t.

(ii)

$$\begin{aligned} &\exp\left[-\frac{1}{2}\sum b_i^2 \frac{1-z}{1-(1-p/a_i)z}\right] \cdot \prod \left\{ \left(\frac{p}{a_i}\right)^{\frac{1}{2}} \left[1 - \left(1 - \frac{p}{a_i}\right) z\right]^{-\frac{1}{2}} \right\} \\ &\qquad = \sum c_j z^j \quad (|z| < \min_i |1 - p/a_i|^{-1}). \end{aligned} \tag{3.4}$$

(iii) *The c_j satisfy the recursion relationship*

$$(3.5)\qquad \begin{aligned} c_0 &= e^{-\frac{1}{2}\Sigma b_i} \prod (p/a_i)^{\frac{1}{2}}, \\ c_j &= (2j)^{-1} \sum_{r=0}^{j-1} g_{j-r} c_r, \qquad j = 1, 2, \cdots, \end{aligned}$$

where

$$(3.6)\qquad g_m = \sum (1 - p/a_i)^m + mp \sum (b_i^2/a_i)(1 - p/a_i)^{m-1}, \quad m = 1, 2, \cdots$$

PROOF.

(i) Set

$$(3.7)\qquad \boldsymbol{\xi} = \mathbf{A}^{\frac{1}{2}}(\mathbf{x} - \mathbf{b})$$

in (1.3). Then

$$(3.8)\qquad H_{n;\mathbf{A},\mathbf{b}}(t) = A^{-\frac{1}{2}} e^{-\frac{1}{2}\mathbf{b}'\mathbf{b}} \cdot (2\pi)^{-\frac{1}{2}n} \int_{\{\boldsymbol{\xi}'\boldsymbol{\xi} \leq t\}} e^{-\frac{1}{2}\boldsymbol{\xi}'\mathbf{A}^{-1}\boldsymbol{\xi} - \mathbf{b}'\mathbf{A}^{-\frac{1}{2}}\boldsymbol{\xi}}\, d\boldsymbol{\xi}.$$

On substituting

$$(3.9)\qquad \boldsymbol{\xi} = r\mathbf{l} \quad (r = |\boldsymbol{\xi}|)$$

in (3.8),

$$(3.10)\qquad \begin{aligned} H_{n;\mathbf{A},\mathbf{b}}(t) &= A^{-\frac{1}{2}} e^{-\frac{1}{2}\mathbf{b}'\mathbf{b}} \cdot (2\pi)^{-\frac{1}{2}n} \int_0^{t^{\frac{1}{2}}} \int_{\Omega_n} e^{-\frac{1}{2}(\mathbf{l}'\mathbf{A}^{-1}\mathbf{l})r^2 - (\mathbf{b}\,\mathbf{A}^{-\frac{1}{2}}\mathbf{l})r} r^{n-1}\, dr d\mathbf{l} \\ &= 2A^{-\frac{1}{2}} e^{-\frac{1}{2}\mathbf{b}'\mathbf{b}} \{2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)\}^{-1} \int_0^{t^{\frac{1}{2}}} M[e^{-\frac{1}{2}Qr^2 - Lr}] e^{-\frac{1}{2}r^2/p}\, r^{n-1}\, dr, \end{aligned}$$

where L and Q are linear and quadratic functions defined (in matrix notation) on Ω_n as in (3.3) by

$$(3.11)\qquad L \equiv L(\mathbf{l}) = \mathbf{b}'\mathbf{A}^{-\frac{1}{2}}\mathbf{l}, \qquad Q \equiv Q(\mathbf{l}) = \mathbf{l}'(\mathbf{A}^{-1} - p^{-1}\mathbf{I})\mathbf{l}.$$

We now expand $\exp(-Qr^2/2 - Lr)$ as a power series in r. Recall that

$$(3.12)\qquad e^{-\frac{1}{2}u^2 + vu} = \sum_{m=0}^{\infty} H_m(v) u^m/m!,$$

whence

$$(3.13)\qquad e^{-\frac{1}{2}Qr^2 - Lr} = \sum_{m=0}^{\infty} (-1)^m Q^{\frac{1}{2}m} H_m(L/Q^{\frac{1}{2}}) r^m/m!.$$

In order to evaluate the mean of $\exp(-Qr^2/2 - Lr)$ on Ω_n (required in 3.10), note that the series in (3.13), regarded as a series of functions in $\mathbf{l}$ for fixed $r \leqq t^{\frac{1}{2}} < \infty$, converges uniformly on Ω_n. For, since by the Schwarz inequality,

$$(3.14)\qquad |L| \leqq \sum (b_i^2/a_i)^{\frac{1}{2}},$$

we have

$$
\begin{aligned}
\mid Q^{\frac{1}{2}m}H_m(L/Q^{\frac{1}{2}}) \mid &= \left| \sum_{r=0}^{[m/2]} (-1)^r \frac{m!}{(m-2r)!2^r r!} L^{m-2r}Q^r \right| \\
&\leqq \sum_{r=0}^{[m/2]} \frac{m!}{(m-2r)!2^r r!} (\textstyle\sum(b_i^2/a_i)^{\frac{1}{2}})^{m-2r} q_0^r \qquad (3.15)\\
&= (-i)^m q_0^{\frac{1}{2}m} H_m(i(\textstyle\sum b_i^2/a_i)^{\frac{1}{2}}/q_0^{\frac{1}{2}})
\end{aligned}
$$

($q_0 = \max_i |1/p - 1/a_i|$), whence the series on the right of (3.13) is majorized by

$$\sum_{m=0}^{\infty} (-i)^m q_0^{\frac{1}{2}m} H_m(i(\textstyle\sum b_i^2/a_i)^{\frac{1}{2}}/q_0^{\frac{1}{2}}) r^m/m! = \exp\{\tfrac{1}{2}q_0 r^2 + (\textstyle\sum b_i^2/a_i)^{\frac{1}{2}} r\}, \tag{3.16}$$

on using (3.12). This establishes the uniform convergence of the series in (3.13) with respect to l. Consequently (refer to (1.17)),

$$M[e^{-\frac{1}{2}Qr^2 - Lr}] = \sum_{m=0}^{\infty} (-1)^m M[Q^{\frac{1}{2}m}H_m(L/Q^{\frac{1}{2}})] r^m/m!. \tag{3.17}$$

Next, we observe that for odd m, $Q^{\frac{1}{2}m}H_m(L/Q^{\frac{1}{2}})$ is changed in sign when l is replaced by $-$l (this transformation changes L to $-L$ and leaves Q unchanged). Therefore, by symmetry,

$$M[Q^{\frac{1}{2}m}H_m(L/Q^{\frac{1}{2}})] = 0, \qquad m = 1, 3, \cdots, \tag{3.18}$$

and (3.17) reduces to

$$M[e^{-\frac{1}{2}Qr^2 - Lr}] = \sum M[Q^j H_{2j}(L/Q^{\frac{1}{2}})] r^{2j}/(2j)!. \tag{3.19}$$

Again, from (2.9) we note that $Q^j H_{2j}(L/Q^{\frac{1}{2}})$ is a homogeneous function in the l_i of degree $2j$, so that (3.19) may be expressed with the aid of (2.3) in the form

$$M[e^{-\frac{1}{2}Qr^2 - Lr}] = \sum \frac{\Gamma(\frac{1}{2}n)}{2^j \Gamma(\frac{1}{2}n + j)} \{E[Q^j H_{2j}(L/Q^{\frac{1}{2}})]/(2j)!\} r^{2j}. \tag{3.20}$$

Substituting (3.20) in (3.10),

$$
\begin{aligned}
H_{n;\mathbf{A},\mathbf{b}}(t) &= 2A^{-\frac{1}{2}} e^{-\frac{1}{2}\mathbf{b}'\mathbf{b}} \{2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)\}^{-1} \\
&\quad \cdot \int_0^{t^{\frac{1}{2}}} \sum \frac{\Gamma(\frac{1}{2}n)}{2^j \Gamma(\frac{1}{2}n + j)} \{E[Q^j H_{2j}(L/Q^{\frac{1}{2}})]/(2j)!\} e^{-\frac{1}{2}r^2/p} r^{n-1+2j}\, dr.
\end{aligned} \tag{3.21}
$$

The series under the integral sign in (3.21) is uniformly convergent. For, on using (2.8), this series is majorized by

$$
\begin{aligned}
&\sum \frac{\Gamma(\frac{1}{2}n)}{2^j \Gamma(\frac{1}{2}n + j)} \cdot \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{(q_0 + \frac{1}{2}\sum b_i^2/a_i)^j}{j!} \cdot e^{-\frac{1}{2}r^2/p} r^{n-1+2j} \\
&\qquad = e^{-\frac{1}{2}r^2/p} r^{n-1} \cdot \sum \frac{(\frac{1}{2}(q_0 + \frac{1}{2}\sum b_i^2/a_i) r^2)^j}{j!} \\
&\qquad = e^{-\frac{1}{2}r^2(1/p - q_0 - \frac{1}{2}\sum b_i^2/a_i)} r^{n-1},
\end{aligned} \tag{3.22}
$$

thereby establishing the uniform convergence of the series for $r \leqq t^{\frac{1}{2}} < \infty$. Consequently, term by term integration in (3.21) is permissible, and, moreover, such integration will yield a series which is likewise uniformly convergent[4] for $r \leqq t^{\frac{1}{2}} < \infty$. Formula (3.1) now follows immediately from (3.21) with the aid of the formula

$$(3.23) \qquad \int_0^{t^{\frac{1}{2}}} e^{-\frac{1}{2}r^2/p} r^{n-1+2j}\, dr = \tfrac{1}{2}p^{\frac{1}{2}n+j}\cdot 2^{\frac{1}{2}n+j}\Gamma(\tfrac{1}{2}n+j)F_{n+2j}(t/p).$$

(ii) Denote the left-hand member of (3.4) by $\psi(z)$. We have

$$(3.24) \qquad \psi(z) = \prod \left\{ (p/a_i)^{\frac{1}{2}} e^{-\frac{1}{2}b_i^2}\, [1-(1-p/a_i)z]^{-\frac{1}{2}} \cdot \exp\left[\frac{1}{2}\frac{b_i^2}{a_i}\frac{pz}{1-(1-p/a_i)z}\right]\right\}.$$

On recalling that

$$(3.25) \qquad \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}\theta x^2 - rx}\, dx = \theta^{-\frac{1}{2}} e^{\frac{1}{2}r^2/\theta}, \qquad Rl\theta > 0,$$

$\psi(z)$ may be expressed in the integral form

$$(3.26) \qquad \begin{aligned} \psi(z) &= \prod \left\{ (p/a_i)^{\frac{1}{2}} e^{-\frac{1}{2}b_i^2} \cdot \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}\{1-(1-p/a_i)z\}x_i^2 - (pz)^{\frac{1}{2}}(b_i/a_i^{\frac{1}{2}})x_i\right] dx_i \right\} \\ &= A^{-\frac{1}{2}} e^{-\frac{1}{2}\Sigma b_i^2}\, p^{\frac{1}{2}n} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}Qpz - L(pz)^{\frac{1}{2}}} \cdot (2\pi)^{-\frac{1}{2}n} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\, d\mathbf{x} \end{aligned}$$

$$(|z| < \min_i |1 - p/a_i|^{-1}),$$

or, on using (3.13) to express $\exp(-Qpz/2 - L(pz)^{\frac{1}{2}})$ as a power series in $(pz)^{\frac{1}{2}}$,

$$(3.27) \qquad \begin{aligned} \psi(z) &= A^{-\frac{1}{2}} e^{-\frac{1}{2}\Sigma b_i^2}\, p^{\frac{1}{2}n} \cdot \int_{-\infty}^{\infty} \left[\sum_{m=0}^{\infty} (-1)^m Q^{\frac{1}{2}m} H_m(L/Q^{\frac{1}{2}}) \frac{(pz)^{\frac{1}{2}m}}{m!}\right] (2\pi)^{-\frac{1}{2}n} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\, d\mathbf{x} \\ &= A^{-\frac{1}{2}} e^{-\frac{1}{2}\Sigma b_i^2}\, p^{\frac{1}{2}n} \cdot \int_{E_n} \left[\sum_{m=0}^{\infty} (-1)^m Q^{\frac{1}{2}m} H_m(L/Q^{\frac{1}{2}}) \frac{(pz)^{\frac{1}{2}m}}{m!}\right] dP, \end{aligned}$$

where P is the probability measure (defined over the Borel field of subsets in E_n) induced by the standardized n-dimensional spherical normal density func-

[4] The uniform convergence of the series in (3.1) on every finite t-interval follows also from Lemma 4, after majorization of the series through the upper bounds for the $|c_j|$ implied by (2.8) in Lemma 3 (see (4.14)).

tion (2.2). From (3.16)[5] (replace r^2 in (3.16) by $p|z|\mathbf{x}'\mathbf{x}$), the partial sums of the series under the second integral sign in (3.27) are dominated by

$$\exp\{\tfrac{1}{2}q_0(p|z|)\mathbf{x}'\mathbf{x} + (\textstyle\sum b_i^2/a_i)^{\frac{1}{2}}(p|z|\mathbf{x}'\mathbf{x})^{\frac{1}{2}}\}, \tag{3.28}$$

and

$$\begin{aligned}\int_{E_n} \exp\{\tfrac{1}{2}q_0(p\mid z\mid)\mathbf{x}'\mathbf{x} + (\textstyle\sum b_i^2/a_i)^{\frac{1}{2}}(p\mid z\mid \mathbf{x}'\mathbf{x})^{\frac{1}{2}}\}\, dP \\ = (2\pi)^{-\frac{1}{2}n}\int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}(1 - pq_0\mid z\mid)\mathbf{x}'\mathbf{x} \\ + (\textstyle\sum b_i^2/a_i)^{\frac{1}{2}}(p\mid z\mid \mathbf{x}'\mathbf{x})^{\frac{1}{2}}\}\, d\mathbf{x} < \infty,\end{aligned} \tag{3.29}$$

since

$$1 - pq_0|z| = 1 - \max_i|1 - p/a_i|\,|z| > 0 \quad (|z| < \min_i|1 - p/a_i|^{-1}). \tag{3.30}$$

Thus, by Lebesgue dominated convergence, integration term by term over E_n is permissible in (3.27); and, since from (3.18) and the corollary of Lemma 1,

$$E[Q^{\frac{1}{2}m}H_m(L/Q^{\frac{1}{2}})] = 0, \qquad m = 1, 3, \cdots, \tag{3.31}$$

(3.27) reduces to

$$\begin{aligned}\psi(z) &= A^{-\frac{1}{2}}e^{-\frac{1}{2}\Sigma b_i^2}\, p^{\frac{1}{2}n}\cdot\sum_0^\infty E[Q^jH_{2j}(L/Q^{\frac{1}{2}})]\frac{(pz)^j}{(2j)!} \\ &= \sum c_jz^j,\end{aligned}$$

thereby demonstrating that $\psi(\cdot)$ is a generating function for the c_j in the sense of (3.4).

(iii) From (3.4),

$$\psi'(z) = K(z)\psi(z), \tag{3.32}$$

where

$$\begin{aligned}K(z) = \tfrac{1}{2}\{\textstyle\sum(1 - p/a_i)[1 - (1 - p/a_i)z]^{-1} \\ + p\textstyle\sum(b_i^2/a_i)[1 - (1 - p/a_i)z]^{-2}\}.\end{aligned} \tag{3.33}$$

We obtain after some simplification

$$\begin{aligned}K^{(s)}(0) &\equiv (d/dz)^sK(z)|_{z=0} \\ &= s!\cdot\tfrac{1}{2}\{\textstyle\sum(1 - p/a_i)^{s+1} + (s + 1)p\textstyle\sum(b_i^2/a_i)(1 - p/a_i)^s\}, \\ &\qquad s = 0, 1, \cdots.\end{aligned} \tag{3.34}$$

[5] To avoid confusion, recall that $L = L(1)$, $Q = Q(1)$ in (3.13), whereas $L = L(\mathbf{x})$, $Q = Q(\mathbf{x})$ in (3.27).

Therefore, on differentiating (3.32) $j - 1$ times at $z = 0$ and replacing $\psi^{(s)}(0)$ by $s!c_s$, we obtain

$$j!c_j = \sum_{r=0}^{j-1} \binom{j-1}{r} K^{(j-1-r)}(0) \cdot r!c_r, \qquad j = 1, 2, \cdots, \tag{3.35}$$

which reduces to (3.5) and (3.6) with the aid of (3.34).

THEOREM 2.

(i)

$$H_{n;\mathbf{A},\mathbf{0}}(t) = \sum_0^\infty c_j F_{n+2j}(t/p), \tag{3.36}$$

where p *is an arbitrary positive constant,*

$$c_j \equiv c_{j,n;\mathbf{A},\mathbf{0}}(p) = A^{-\frac{1}{2}} p^{\frac{1}{2}n+j} \cdot E[(-Q)^j]/(2^j j!), \tag{3.37}$$

and Q *is defined in* (3.3). *Further, the series in* (3.36) *converges uniformly on every finite interval of* t.

(ii)

$$\prod \{(p/a_i)^{\frac{1}{2}}[1 - (1 - p/a_i)z]^{-\frac{1}{2}}\} = \sum c_j z^j \quad (|z| < \min_i |1 - p/a_i|^{-1}). \tag{3.38}$$

(iii) *The* c_j *satisfy the recursion relationship*

$$c_j = (2j)^{-1} \sum_{r=0}^{j-1} g_{j-r} c_r \; (j = 1, 2, \cdots), \qquad c_0 = \prod (p/a_i)^{\frac{1}{2}}, \tag{3.39}$$

where

$$g_m = \sum (1 - p/a_i)^m, \qquad m = 1, 2, \cdots. \tag{3.40}$$

PROOF. Set $\mathbf{b} = \mathbf{0}$ in Theorem 1.

We conclude this Section by deriving two rather special and known results from Theorems 1 and 2. At the same time this will illustrate the use of the latter Theorems in more general situations.

First, consider the case $\mathbf{A} = \mathbf{I}$ and let $p = 1$. Here

$$L = \sum b_i x_i, \qquad Q = 0, \tag{3.41}$$

so that (3.2) and (2.11) give

$$\begin{aligned} c_j &= e^{-\frac{1}{2}\Sigma b_i^2} EL^{2j}/(2j)! \\ &= e^{-\frac{1}{2}\Sigma b_i^2} (\tfrac{1}{2} \textstyle\sum b_i^2)^j / j!, \end{aligned} \tag{3.42}$$

since L is normal with zero mean and variance $\sum b_i^2$. Thus (3.1) yields

$$\begin{aligned} H_{n;\mathbf{I};\mathbf{b}}(t) &\equiv G_{n;\kappa}(t) \\ &= e^{-\frac{1}{2}\kappa^2} \sum_0^\infty \frac{(\frac{1}{2}\kappa^2)^j}{j!} F_{n+2j}(t) \qquad (\kappa^2 = \textstyle\sum b_i^2), \end{aligned} \tag{3.43}$$

i.e., the non-central χ^2 distribution function is expressible as a mixture of (central) χ^2 distribution functions in which the coefficients are Poisson probabilities.[6] This result appears to have been first proved by Robbins and Pitman [17].

Next, consider the case $\mathbf{A} = \mathbf{I}$, $\mathbf{b} = \mathbf{0}$. Theorem 2 gives through either (3.37) or (3.38),

$$H_{n;\mathbf{I},\mathbf{0}}(t) \equiv F_n(t) \tag{3.44}$$
$$= p^{\frac{1}{2}n} \sum \frac{(\frac{1}{2}n)_j}{j!} (1-p)^j F_{n+2j}(t/p).$$

This result was first proved in [17] for $p \leqq 1$ and stated (without proof) for $p > 1$. Formula (3.44) demonstrates that the χ^2 distribution function may be expressed as a mixture of scaled χ^2 distribution functions in which the coefficients are the probabilities in a negative binomial distribution ($p \leqq 1$).

REMARK. The series in (3.1) and (3.36) have been shown to converge uniformly on every *finite* t-interval for each $p > 0$. The domain of uniform convergence may, however, be extended to the entire t-axis by suitable restriction of p. In fact from the upper bound subsequently established (Equ. (4.14)) for $|c_j|$, we find that the series in (3.1) is majorized by

$$c_0 \cdot \sum \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{\mu^j}{j!} F_{n+2j}(t/p),$$

where $\mu = \max_i |1 - p/a_i| + (p/2) \sum b_i^2/a_i$. It follows from Lemma 4 that the series in (3.1) converges uniformly on the entire (extended) t-axis for each p satisfying $\max_i |1 - p/a_i| + (p/2) \sum b_i^2/a_i < 1$. In particular, the series in (3.36) has the same uniform convergence property for each p satisfying

$$\max_i |1 - p/a_i| < 1,$$

i.e., for $p < 2a_1$.

Observe that uniform convergence over the extended t-axis implies that t can be replaced by $+\infty$ in (3.1) and (3.36), i.e., it implies that $\sum c_j = 1$. It will in fact be shown subsequently (Section 5) that $\sum c_j = 1$ if and only if $p < 2a_1$, whatever the value of $\mathbf{b}$.

3.1. *Further explicit formulae for the* c_j. In addition to the recursion formula (3.5) for the c_j of Theorem 1, two (equivalent) *explicit* formulae for these coefficients have been given, namely, c_j was expressed as the expectation of a certain homogeneous function of degree $2j$ in independent standardized normal variables $x_1, \cdots, x_n$ (formula (3.2)), and $j!c_j$ was expressed as the jth derivative at the origin of a certain generating function (formula (3.4)). Exactly similar remarks apply to the c_j of Theorem 2 (Theorem 2 is a special case of Theorem 1).

---

[6] (3.43) may also be derived directly by expanding the Bessel function occurring in the density function of the non-central χ^2 (see, for instance, [14] and [19]) in its usual power series form and integrating term by term.

We shall now obtain an additional explicit formula for the c_j as polynomials in the g_m. Recall that the g_m occur in the recursion formulae for the c_j.

To obtain the desired formula, consider first the special case of Theorem 2 ($\mathbf{b} = \mathbf{0}$). On referring to (3.37), c_j can be expressed in terms of the jth moment, μ_j', of a distribution by

$$c_j = c_0 \cdot E\left[\sum (1 - p/a_i)(x_i^2/2)\right]^j/j!. \tag{3.1.1}$$

From the aditive property of cumulants, the βth cumulant, λ_β, of the variate within square brackets is $\{(\beta - 1)!/2\} \cdot \sum (1 - p/a_i)^\beta = \{(\beta - 1)!/2\} g_\beta$. On using the well-known formula ([8], pp. 68–69)

$$\frac{\mu_j'}{j!} = \sum_{i_1+2i_2+\cdots+ji_j=j} \frac{(\lambda_1/1!)^{i_1}(\lambda_2/2!)^{i_2} \cdots (\lambda_j/j!)^{i_j}}{i_1!i_2! \cdots i_j!}$$

for moments in terms of cumulants, we find

$$c_j = c_0 \cdot \sum_{i_1+2i_2+\cdots+ji_j=j} \frac{(g_1/2)^{i_1}(g_2/4)^{i_2}(g_3/6)^{i_3} \cdots (g_j/2j)^{i_j}}{i_1!\, i_2! \cdots i_j!}. \tag{3.1.2}$$

Consider now the c_j in the general case (Theorem 1). On comparing the recursion formulae for the c_j in Theorems 1 and 2 (Equs. (3.5) and (3.39), respectively), we find that these are of precisely the same form. It follows that the c_j of Theorem 1 are likewise given by (3.1.2) with the appropriate values of the g_m as given in (3.6). In brief, (3.1.2) is valid quite generally.

On using (3.1.2), or alternatively through repeated application of the recursion formulae, the first few c's are found to have the following values:

$$\begin{aligned} c_1 &= \tfrac{1}{2} g_1 c_0, \\ c_2 &= \tfrac{1}{4}(g_2 + \tfrac{1}{2} g_1^2) c_0, \\ c_3 &= \tfrac{1}{6}(g_3 + \tfrac{3}{4} g_2 g_1 + \tfrac{1}{8} g_1^3) c_0, \\ c_4 &= \tfrac{1}{8}(g_4 + \tfrac{2}{3} g_3 g_1 + \tfrac{1}{4} g_2^2 + \tfrac{1}{4} g_1^2 g_2 + \tfrac{1}{48} g_1^4). \end{aligned} \tag{3.1.3}$$

It is of some interest to note that the c_j of Theorem 2 may be expressed alternatively in terms of powers and products of powers of the n quantities

$$1 - p/a_i.$$

Thus (3.1.1) yields

$$\begin{aligned} c_j &= c_0 \cdot E\left[\sum_{i_1+\cdots+i_n=j} \frac{j!}{i_1! \cdots i_n!} \cdot (1 - p/a_1)^{i_1} x_1^{2i_1} \cdots (1 - p/a_n)^{i_n} x_n^{2i_n}\right] \Big/ (2^j j!) \\ &= c_0 \cdot \left[\sum_{i_1+\cdots+i_n=j} \frac{(2i_1)! \cdots (2i_n)!}{(i_1!)^2 \cdots (i_n!)^2} \cdot (1 - p/a_1)^{i_1} \cdots (1 - p/a_n)^{i_n}\right] \Big/ 2^{2j}. \end{aligned} \tag{3.1.4}$$

We remark finally that the recursion formulae appear to be best adapted for purposes of computation.

4. Evaluation of $H_{n;\mathbf{A},\mathbf{b}}(t)$ as an infinite linear combination of non-central χ^2 distributions. In Theorem 2, the distribution function of the homogeneous quadratic form $\sum a_i x_i^2$ was represented as an infinite linear combination of central χ^2 distribution functions, i.e., the probability content of the ellipsoid $\sum a_i x_i^2 \leqq t$ was expressed as an infinite linear combination of the probability contents of spheres, under spherical normal distributions with unit standard deviation in any direction, immersed in Euclidean spaces of dimensionalities $n, n + 2, \cdots$ and with centers at the centers of the corresponding distributions. This result suggests by analogy that the probability content of the offset ellipsoid $\sum a_i(x_i - b_i)^2 \leqq t$ may be expressed as an infinite linear combination of the probability contents of *offset* spheres, under spherical normal distributions with unit standard deviation in any direction, immersed in Euclidean spaces of dimensionalities $n, n + 2, \cdots$; i.e., it suggests that the distribution function of the non-homogeneous quadratic form $\sum a_i(x_i - b_i)^2$ may be represented as an infinite linear combination of *non-central* χ^2 distribution functions. This conjecture will be proved in Theorem 3. We then have a result which provides an interesting contrast to Theorem 1: whereas Theorem 1 represents the distribution function of the non-homogeneous quadratic form in terms of central χ^2 distribution functions, Theorem 3 represents this distribution function in terms of non-central χ^2 distribution functions.

The form of the coefficients in the required expansion is *indicated* by the following consideration: *Assume* that

$$H_{n;\mathbf{A},\mathbf{b}}(t) = \sum_{j=0}^{\infty} d_j G_{n+2j;\kappa}(t/p) \qquad (\kappa^2 = \sum b_i^2), \tag{4.1}$$

where $d_j \equiv d_{j,n;\mathbf{A},\mathbf{b}}(p)$, for arbitrary positive p. According to (3.43),

$$G_{n+2j;\kappa}(t/p) = e^{-\frac{1}{2}\kappa^2} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}\kappa^2)^r}{r!} F_{n+2j+2r}(t/p). \tag{4.2}$$

On substituting for $G_{n+2j;\kappa}(t/p)$, as given by (4.2), in (4.1), and proceeding quite formally (without any attempt at rigour), we obtain plausibly

$$H_{n;\mathbf{A},\mathbf{b}}(t) = e^{-\frac{1}{2}\kappa^2} \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} d_j \frac{(\frac{1}{2}\kappa^2)^r}{r!} F_{n+2j+2r}(t/p). \tag{4.3}$$

On comparing (4.3) with (3.1), we have the tentative result

$$c_s = e^{-\frac{1}{2}\kappa^2} \sum_{l=0}^{s} d_l \frac{(\frac{1}{2}\kappa^2)^{s-l}}{(s-l)!}, \qquad s = 0, 1, 2, \cdots, \tag{4.4}$$

or, equivalently,

$$\{c_j\} = \{d_j\} * \{e^{-\frac{1}{2}\kappa^2}(\tfrac{1}{2}\kappa^2)^j/j!\}, \tag{4.5}$$

where $\{u_j\} * \{v_j\}$ is the sequence formed by the Cauchy product of the sequences

$\{u_j\}$, $\{v_j\}$. (In the special case when $\{u_j\}$ and $\{v_j\}$ are probability sequences, $*$ represents the convolution operator.)

Equation (4.5) suggests a further result concerning the tie-up between the generating function $\psi(\cdot)$ of the c_j (given by the left-hand member of (3.4)) and the generating function, say $\Psi(\cdot)$, of the d_j. The probability generating function of the Poisson distribution with mean $\kappa^2/2$ is

$$e^{-\frac{1}{2}\kappa^2(1-z)}. \tag{4.6}$$

Therefore, (4.5) indicates that

$$\Psi(z) = e^{\frac{1}{2}\kappa^2(1-z)}\psi(z), \tag{4.7}$$

from which the possible relationship between the present d_j and the c_j of Theorem 1 is easily obtained by equating the coefficients of z^j for $j = 0, 1, \cdots$.

We now proceed to prove these results rigorously.

THEOREM 3.

(i)

$$H_{n;\mathbf{A},\mathbf{b}}(t) = \sum_0^\infty d_j G_{n+2j;\kappa}(t/p) \qquad (\kappa^2 = \sum b_i^2), \tag{4.8}$$

where p is an arbitrary positive constant, and the d_j ($d_j \equiv d_{j,n;\mathbf{A},\mathbf{b}}(p)$) and the c_j, defined in Theorem 1, *are related reciprocally by*

$$d_j = e^{\frac{1}{2}\kappa^2} \sum_{l=0}^{j} \frac{(-\frac{1}{2}\kappa^2)^{j-l}}{(j-l)!} c_l, \tag{4.9}$$

$$c_j = e^{-\frac{1}{2}\kappa^2} \sum_{l=0}^{j} \frac{(\frac{1}{2}\kappa^2)^{j-l}}{(j-l)!} d_l. \tag{4.9$'$}$$

Further, the series in (4.8) *converges uniformly on every finite interval of t.*

(ii)

$$\exp\left[-\frac{1}{2}\sum b_i^2(1-p/a_i)\frac{z(1-z)}{1-(1-p/a_i)z}\right] \cdot\prod \{(p/a_i)^{\frac{1}{2}}[1-(1-p/a_i)z]^{-\frac{1}{2}}\} = \sum d_j z^j \left(|z| < \min_i |1-p/a_i|^{-1}\right). \tag{4.10}$$

(iii) *The d_j satisfy the recursion relationship*

$$\begin{aligned} d_0 &= \prod (p/a_i)^{\frac{1}{2}}, \\ d_j &= (2j)^{-1}\sum_{r=0}^{j-1} h_{j-r}\, d_r, \qquad j = 1, 2, \cdots, \end{aligned} \tag{4.11}$$

where[7]

$$\begin{aligned} h_1 &= \sum (1-b_i^2)(1-p/a_i), \\ h_m &= \sum (1-p/a_i)^m + mp\sum (b_i^2/a_i)(1-p/a_i)^{m-1}, \quad m = 2, 3, \cdots. \end{aligned} \tag{4.12}$$

[7] It will be observed that in terms of the g_m of (3.6), $h_1 = g_1 - \kappa^2$ while $h_m = g_m (m = 2, 3, \cdots)$.

PROOF.

(i) We show first that the series in (4.8) is absolutely convergent by establishing an upper bound to $|d_j|$. We have

$$|d_j| \leqq e^{\frac{1}{2}\kappa^2} \sum_{l=0}^{j} \frac{(\frac{1}{2}\kappa^2)^{j-l}}{(j-l)!} |c_l|. \tag{4.13}$$

Again, from the explicit formula for c_l in (3.2) and the inequality (2.8) (with p_i in the latter inequality replaced by $b_i/a_i^{\frac{1}{2}}$ and q_0 denoting, as before, $\max_i |1/a_i - 1/p|$),

$$|c_l| \leqq c_0 \cdot \frac{\Gamma(\frac{1}{2}n + l)}{\Gamma(\frac{1}{2}n)} \frac{\mu^l}{l!}, \tag{4.14}$$

where

$$\mu = p(q_0 + \tfrac{1}{2}\sum b_i^2/a_i). \tag{4.15}$$

On applying (4.14) in (4.13),

$$\begin{aligned} |d_j| &\leqq c_0 \cdot e^{\frac{1}{2}\kappa^2} \frac{(\frac{1}{2}\kappa^2)^j}{j!} \sum_{l=0}^{j} \binom{j}{l} \frac{\Gamma(\frac{1}{2}n + l)}{\Gamma(\frac{1}{2}n)} \left(\frac{\mu}{\frac{1}{2}\kappa^2}\right)^l \\ &\leqq c_0 e^{\frac{1}{2}\kappa^2} \cdot \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{(\frac{1}{2}\kappa^2)^j}{j!} \left(1 + \frac{\mu}{\frac{1}{2}\kappa^2}\right)^j \\ &= d_0 \cdot \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{(\mu + \frac{1}{2}\kappa^2)^j}{j!}. \end{aligned} \tag{4.16}$$

Therefore, from Lemma 5, the series in (4.8) converges absolutely, and it follows with the aid of (4.2) that the latter series may be expressed as an absolutely convergent repeated series in the following manner:

$$\begin{aligned} \sum_{j=0}^{\infty} d_j G_{n+2j;\kappa}(t/p) &= \sum_{j=0}^{\infty} d_j \sum_{r=0}^{\infty} e^{-\frac{1}{2}\kappa^2} \frac{(\frac{1}{2}\kappa^2)^r}{r!} F_{n+2j+2r}(t/p) \\ &= e^{-\frac{1}{2}\kappa^2} \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} d_j \frac{(\frac{1}{2}\kappa^2)^r}{r!} F_{n+2j+2r}(t/p). \end{aligned} \tag{4.17}$$

The sum of the required series is thus equal to the sum of the convergent double series

$$e^{-\frac{1}{2}\kappa^2} \sum_{j,r=0}^{\infty} d_j \frac{(\frac{1}{2}\kappa^2)^r}{r!} F_{n+2j+2r}(t/p), \tag{4.18}$$

which series is then equivalent to

$$\sum_{m=0}^{\infty} \lambda_m F_{n+2m}(t/p), \tag{4.19}$$

where

$$\lambda_m = e^{-\frac{1}{2}\kappa^2} \sum_{s=0}^{m} d_s \frac{(\frac{1}{2}\kappa^2)^{m-s}}{(m-s)!}, \qquad m = 0, 1, \cdots. \tag{4.20}$$

On substituting for d_s from (4.9) in (4.20),

$$\lambda_m = \sum_{s=0}^{m} \sum_{l=0}^{s} \frac{(-\frac{1}{2}\kappa^2)^{s-l}}{(s-l)!} \frac{(\frac{1}{2}\kappa^2)^{m-s}}{(m-s)!} c_l, \qquad m = 0, 1, \cdots. \tag{4.21}$$

The coefficient of c_l in (4.21) is

$$\sum_{s=l}^{m} \frac{(-\frac{1}{2}\kappa^2)^{s-l}}{(s-l)!} \frac{(\frac{1}{2}\kappa^2)^{m-s}}{(m-s)!} = \sum_{k=0}^{m-l} \frac{(-\frac{1}{2}\kappa^2)^{k}}{k!} \frac{(\frac{1}{2}\kappa^2)^{m-l-k}}{(m-l-k)!}$$
$$= (-\tfrac{1}{2}\kappa^2 + \tfrac{1}{2}\kappa^2)^{m-l}/(m-l)!,$$

i.e., the coefficient of c_l in (4.21) is 1 for $l = m$ and is zero for $l = 0, 1, \cdots, m-1$. In other words, $\lambda_m = c_m$, as required. [Formula (4.9′) of (i) is proved below.]

The *uniform* convergence of the series in (4.8) on every finite t-interval follows from (4.16) and Lemma 5. In fact, the latter series is majorized by

$$d_0 \cdot \sum \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{(\mu + \frac{1}{2}\kappa^2)^j}{j!} G_{n+2j;\kappa}(t/p),$$

and this majorizing series converges uniformly by Lemma 5 on every finite t-interval.

(ii) Denoting the left-hand member in (4.10) by $\Psi(z)$, it is easily verified that $\Psi(z)$ is related to $\psi(z)$ (the left-hand member in (3.4)) by (4.7). Equating coefficients of z^j in (4.7) immediately yields (4.10).

The inverse relationship (4.9′) for the c_j in terms of the d_j is now readily established by expressing $\psi(z)$ in terms of $\Psi(z)$,

$$\psi(z) = e^{-\frac{1}{2}\kappa^2(1-z)}\Psi(z). \tag{4.22}$$

Equating coefficients of z^j in (4.22) immediately yields (4.9′).

(iii) We have

$$\Psi'(z) = N(z)\Psi(z), \tag{4.23}$$

where

$$N(z) = \tfrac{1}{2}\sum (1 - p/a_i)\{[1 - (1 - p/a_i)z]^{-1} - b_i^2[1 - 2z + (1 - p/a_i)z^2][1 - (1 - p/a_i)z]^{-2}\}. \tag{4.24}$$

We obtain after some simplification

$$\begin{aligned} N^{(s)}(0) &\equiv (d/dz)^s N(z)|_{z=0} \\ &= s!\cdot\tfrac{1}{2}\,\{\textstyle\sum (1 - p/a_i)^{s+1} + (s+1)p\sum (b_i^2/a_i)(1 - p/a_i)^s\}, \\ &\qquad s = 1, 2, \cdots, \\ N(0) &= \tfrac{1}{2}\textstyle\sum (1 - b_i^2)(1 - p/a_i). \end{aligned} \tag{4.25}$$

Therefore, on differentiating (4.23) $j - 1$ times at $z = 0$ and replacing $\Psi^{(s)}(0)$ by $s!d_s$, we obtain

$$j!\,d_j = \sum_{r=0}^{j-1} \binom{j-1}{r} N^{(j-1-r)}(0)\cdot r!d_r, \qquad j = 1, 2, \cdots, \tag{4.26}$$

which reduces to (4.11) and (4.12) with the aid of (4.25).

REMARK. The series in (4.8) has been shown to converge uniformly on every *finite* t-interval for each $p > 0$. Just as for the series in (3.1) and (3.36), the domain of uniform convergence may be extended to the entire t-axis by suitably restricting p. Thus, from (4.16) the series in (4.8) is majorized by

$$d_0 \cdot \sum \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{\nu^j}{j!} G_{n+2j;\kappa}(t/p),$$

where $\nu = \max_i |1 - p/a_i| + \sum (b_i^2/2)(1 + p/a_i)$. It follows from Lemma 4 that the series in (4.8) converges uniformly on the entire t-axis for each p satisfying $\max_i |1 - p/a_i| + \sum (b_i^2/2)(1 + p/a_i) < 1$.

Observe that uniform convergence over the extended t-axis implies that t can be replaced by $+\infty$ in (4.8), i.e., it implies that $\sum d_j = 1$. It will in fact be shown subsequently (Section 5) that $\sum d_j = 1$ if and only if $p < 2a_1$.

4.1. *A further explicit formula for the* d_j. On comparing (4.11) with (3.5), we find that the d_j satisfy the same recursion relationship as the c_j, provided g_m is replaced by h_m. It follows from (3.1.2) that

$$d_j = d_0 \cdot \sum_{i_1 + 2i_2 + \cdots + ji_j} \frac{(h_1/2)^{i_1}(h_2/4)^{i_2}(h_3/6)^{i_3} \cdots (h_j/2j)^{i_j}}{i_1!\, i_2! \cdots i_j!}. \tag{4.1.1}$$

In particular, the first few d's are given by

$$\begin{aligned} d_1 &= \tfrac{1}{2}h_1 d_0, \\ d_2 &= \tfrac{1}{4}(h_2 + \tfrac{1}{2}h_1^2) d_0, \\ d_3 &= \tfrac{1}{6}(h_3 + \tfrac{3}{4}h_2 h_1 + \tfrac{1}{8}h_1^3) d_0, \\ d_4 &= \tfrac{1}{8}(h_4 + \tfrac{2}{3}h_3 h_1 + \tfrac{1}{4}h_2^2 + \tfrac{1}{4}h_1^2 h_2 + \tfrac{1}{48}h_1^4). \end{aligned} \tag{4.1.2}$$

We remark finally that in spite of the undoubted theoretical interest of (4.1.1), the d_j (like the c_j) are best computed recursively.

5. The nature and accuracy of the expansions. In this concluding Section, we discuss briefly the significance of the scale factor p in the fundamental expansions (3.1), (3.36) and (4.8), with special reference to the question whether, and under what conditions, these expansions reduce to mixture representations, in the sense of Robbins and Pitman [17], by suitable choice of p. Specifically, we pose the problem[8]: Is there a (non-empty) set of values of p for which

$$c_j(p) \geqq 0, \qquad \sum c_j(p) = 1, \tag{5.1}$$

and

$$d_j(p) \geqq 0, \qquad \sum d_j(p) = 1. \tag{5.2}$$

[8] This problem can be posed in terms of characteristic functions as follows: It has been shown previously that $\psi(z)$ and $\Psi(z)$ are generating functions of the c_j and d_j, respectively. We now require that $\psi(z)$ and $\Psi(z)$ shall represent *probability* generating functions, i.e., that (exp $i\tau$) and Ψ (exp $i\tau$) shall represent characteristic functions (τ real). More precisely, we wish to determine the sets of values of p for which the two latter functions are characteris-

We propose also to obtain various upper bounds to the errors induced by using the fundamental expansions in truncated form.

Let us first consider the following three special values of p: $p = a_1$, $p = a_n$, $p = a_g$, where a_g is (as before) the geometric mean of the a_i ($a_g = (a_1 \cdots a_n)^{1/n}$). Now the n-dimensional spheres of radii $(t/a_n)^{\frac{1}{2}}$ and $(t/a_1)^{\frac{1}{2}}$ with centers at the center of the ellipsoid $\sum a_i(x_i - b_i)^2 \leqq t$ are inscribed and circumscribed spheres, respectively, with respect to this ellipsoid. Therefore, since the probability contents of the latter spheres are $G_{n;\kappa}(t/a_n)$ and $G_{n;\kappa}(t/a_1)$ (as remarked after Equ. (1.8)) while the probability content of the ellipsoid is $H_{n;\mathbf{A};\mathbf{b}}(t)$, we have

$$G_{n;\kappa}(t/a_n) \leqq H_{n;\mathbf{A},\mathbf{b}}(t) \leqq G_{n;\kappa}(t/a_1) \qquad (\kappa^2 = \sum b_i^2), \tag{5.3}$$

i.e., $G_{n;\kappa}(t/a_n)$ underestimates the required probability content while $G_{n;\kappa}(t/a_1)$ overestimates it (the equality signs in (5.3) hold, trivially, only when $a_1 = a_2 = \cdots = a_n$). In particular, on setting $\kappa = 0$, we have

$$F_n(t/a_n) \leqq H_{n;\mathbf{A},\mathbf{0}}(t) \leqq F_n(t/a_1), \tag{5.4}$$

i.e., $F_n(t/a_n)$ and $F_n(t/a_1)$ underestimate and overestimate, respectively, the probability content of the (centrally situated) ellipsoid. Now the first term in the expansions (4.8) and (3.36) are $d_0(p)G_{n;\kappa}(t/p)$ and $c_0(p)F_n(t/p)$, respectively, and it is of some interest to observe that the form of d_0 and c_0 is consistent with these considerations, in the sense that $d_0(a_1), c_0(a_1) \leqq 1$ while

$$d_0(a_n), c_0(a_n) \geqq 1.$$

We remark that $p = a_1$ was used in [17] in developing a special case of Theorem 2.

Consider now the choice $p = a_g$. It is easily verified that the volume-content of the n-dimensional sphere with radius $(t/a_g)^{\frac{1}{2}}$ is equal to the volume-contents of the ellipsoids $\sum a_i(x_i - b_i)^2 \leqq t$ and $\sum a_i x_i^2 \leqq t$. Call the n-dimensional sphere of radius $(t/a_g)^{\frac{1}{2}}$ and with center at the center of the ellipsoid

$$\sum a_i(x_i - b_i)^2 \leqq t$$

the "equivalent" sphere (equivalent to the latter ellipsoid in the sense that the centroids and volume-contents of the two bodies coincide). It appears reasonable to approximate the probability content of the ellipsoid by the probability content of the "equivalent" sphere, i.e., to approximate $H_{n;\mathbf{A},\mathbf{b}}(t)$ by $G_{n;\kappa}(t/a_g)$ and, concomitantly, to approximate $H_{n;\mathbf{A};\mathbf{0}}(t)$ by $F_n(t/a_g)$[9]. It is of considerable in-

tic functions. In order to ensure boundedness and continuity of these functions the inequalities $|1 - p/a_i| < 1$, for $i = 1, \cdots, n$, must be satisfied, and these inequalities are equivalent to $p < 2a_1$ (as determined subsequently by other considerations). Subject to this inequality, the required sets then contain precisely those values of p for which $\psi(\exp i\tau)$ and $\Psi(\exp i\tau)$ are non-negative definite. (We appeal here to one of the possible sets of conditions characterizing the class of characteristic functions, viz. Bochner's theorem [9], p. 207.) However, this characterization of the required sets, though compact, is of little value in actually deriving the sets for specified a_i and b_i.

[9] From the fact that the density in a spherical normal distribution decreases with distance from the center of the distribution it follows that $H_{n;\mathbf{A},\mathbf{0}}(t) \leqq F_n(t/a_g)$ (see [12]). This gives a sharper upper bound to $H_{n;\mathbf{A},\mathbf{0}}(t)$ than (5.4).

terest that the form of d_0 in (4.8) and of c_0 in (3.36) is consistent with these considerations, in the sense that $d_0(a_g) = 1$ and $c_0(a_g) = 1$ in these expansions.[10] We remark that $p = a_g$ was used in [16] in developing a special case of Theorem 2.

One other special value of p, namely $p = a_h$, where a_h denotes the harmonic mean of the $a_i(a_h^{-1} = \sum a_i^{-1}/n)$ is of some interest. For this value of p, we find in Theorem 2 that $c_1 = 0$, so that the first correcting term in the expansion of $H_{n;\mathbf{A},\mathbf{0}}(t)$ when only the first term, $c_0(p_h)F_n(t/p_h)$, is used as an approximant involves the χ^2 distribution function with $n + 4$ (*not* $n + 2$) degrees of freedom.

We now consider the nature of our expansions from the point of view of mixtures of distributions. Denote the sets of values of p for which the basic expansions (3.1) and (3.36) of Theorems 1 and 2 are mixture representations (in the sense of (5.1) and (5.2)) by $C \equiv C_{n;\mathbf{A},\mathbf{b}}$ and $C_0 \equiv C_{n,\mathbf{A},\mathbf{0}}$, respectively. As a preliminary, we may note that $p \not\in C_0$ if $p > a_n$, for with such a value of p, $Q > 0$ with probability 1, and therefore, on referring to (3.37), the c_j oscillate in sign. (Indeed this statement may be strengthened by the assertion that $p \not\in C_0$ if $p > a_h$, since $p > a_h$ implies $c_1 < 0(\mathbf{b} = \mathbf{0})$.)

For a mixture representation, the coefficients must (i) be non-negative and (ii) sum to 1. From the form of the c_j in (3.2), together with (2.9), we find that (i) is satisfied if, and only if,

$$\sum_{r=0}^{j} (-1)^r \frac{(2j)!}{(2j-2r)!2^r r!} E[L^{2j-2r}Q^r] \geqq 0, \qquad j = 0, 1, \cdots. \tag{5.5}$$

As for (ii), consider the generating function $\psi(z)$ on the left of (3.4). This function is regular and analytic within the circle of convergence,

$$|z| < \min_i |1 - p/a_i|^{-1},$$

of its power series expansion as given on the right of (3.4). Therefore $z = 1$ may be substituted in (3.4) if, and only if, the latter point is interior to the circle of convergence, i.e. $1 < \min_i |1 - p/a_i|^{-1}$; and this inequality is satisfied if, and only if,

$$p < 2a_1. \tag{5.6}$$

Thus C is the intersection of the two sets (of p) defined by (5.5) and (5.6). In particular, on setting $\mathbf{b} = \mathbf{0}$ (for which $L = 0$), C_0 is the intersection of the set of p satisfying $E[(-Q)^j] \geqq 0$ $(j = 0, 1, \cdots)$, which is equivalent to

$$E[Q^j] \leqq 0, \qquad j = 1, 3, \cdots, \tag{5.7}$$

and the set (5.6).

From these considerations, it follows immediately that

$$\{p:0 < p \leqq a_1\} \,\varepsilon\, C, \qquad \{p:0 < p \leqq a_1\} \,\varepsilon\, C_0. \tag{5.8}$$

[10] Note also that $c_1(a_g) = (n/2)\,(1 - a_g/a_h) \leqq 0$, where a_h is the harmonic mean of the a_i (cf., [16]), so that the second term is the expansion (3.36) when $p = a_g$ is non-positive. This is consistent with the inequality in footnote 9.

For if $0 < p \leqq a_1$, then $Q \leqq 0$, and so each of the terms on the left of (5.5) is non-negative, while (5.6) is, of course, satisfied trivially. The second part of (5.8) is proved similarly. However, it is important to note that $\{p:0 < p \leqq a_1\}$ is, in general, only a proper subset of C or of C_0, i.e., there will, in general, be values of $p > a_1$ which yield mixture representations. We shall exhibit this by a specific example when $\mathbf{b} = \mathbf{0}$. Choose $p = a_h$ and consider the special case where n is even and the $1/a_i$ are distributed symmetrically around $1/a_h$ in the sense that

$$a_{n-s+1}^{-1} - a_h^{-1} = -(a_s^{-1} - a_h^{-1}) \qquad (s = 1, 2, \cdots n/2; n \text{ even}). \tag{5.9}$$

Then by symmetry

$$E[Q^j] = 0 \qquad (j = 1, 3, \cdots; n \text{ even}; 1/a_i \text{ symmetrically distributed around } 1/a_h; p = a_h), \tag{5.10}$$

so that (5.7) is satisfied for $p = a_h$ with a symmetrical distribution of the $1/a_i$. Moreover, (5.6) is easily seen to hold for $p = a_h$. We have thus established the result that when n is even and the $1/a_i$ are distributed symmetrically around the reciprocal of the harmonic mean of the a_i, then $p = a_h (\geqq a_1)$ yields a mixture representation in (3.36)[11]. Indeed, this result may be strengthened to include values of p in the open interval (a_1, a_h). Thus, since

$$\begin{aligned} \partial(E[Q^j])/\partial p &= (j/p^2)E[(\sum x_i^2)Q^{j-1}] \\ &> 0, \qquad\qquad j = 1, 3, \cdots \end{aligned}$$

we find with the aid of (5.10) that (5.7) holds for $a_1 < p < a_h$. Again, for $p > a_h$, $c_1 < 0$ whatever be the distribution of the $1/a_i$. Thus,

$$C_0 = \{p:0 < p \leqq a_h\} \qquad (n \text{ even}; 1/a_i \text{ symmetrically distributed around } 1/a_h). \tag{5.11}$$

In particular, for $n = 2$, $1/a_1$ and $1/a_2$ are necessarily distributed symmetrically around $a_h^{-1} = (a_1 + a_2)/(2a_1a_2)$, so that

$$C_0 = \{p:0 < p \leqq 2a_1a_2/(a_1 + a_2)\} \qquad (n = 2). \tag{5.12}$$

We have so far discussed the nature of the sets C and C_0. Similar considerations are involved in the determination of the set of values of p, say D, $D \equiv D_{n;\mathbf{A},\mathbf{b}}$, for which (4.8) is a mixture representation. The nature of D will not here be discussed in detail, but we shall content ourselves by pointing out that it follows from (4.1.1) that $h_s \geqq 0$, $s = 1, 2, \cdots$, implies $d_j \geqq 0, j = 0, 1, \cdots$. Again, as before, $\sum d_j = 1$ if, and only if, $p < 2a_1$. In other words, the set of p for which $h_s \geqq 0\ (s = 1, 2, \cdots)$ and $p < 2a_1$ is a subset of D. As a consequence of this result, we may note that if $p \leqq a_1$ and if further[12]

$$h_1 = \sum (1 - b_i^2)(1 - p/a_i) \geqq 0 \text{ (e.g., } |b_i| \leqq 1),$$

[11] Clearly, a similar result holds for symmetrically distributed $1/a_i$ when n is *odd* and $a_h = a_{(n+1)/2}$.

[12] These two inequalities ensure that $h_s \geqq 0$, $s = 1, 2, \cdots$.

then we have a mixture representation in (4.8). Formally, the intersection of the two sets defined by the inequalities $p \leqq a_1$ and $\sum (1 - b_i^2)(1 - p/a_i) \geqq 0$ is a subset of D.

We now obtain upper bounds for the error after k terms in the fundamental expansions of Theorems 1, 2 and 3. First, for a value of p which yields a mixture representation,

$$\begin{aligned} 0 &\leqq H_{n;\mathbf{A},\mathbf{b}}(t) - \sum_0^{k-1} c_j(p) F_{n+2j}(t/p) \\ &= \sum_k^{\infty} c_j(p) F_{n+2j}(t/p) \leqq \left(1 - \sum_0^{k-1} c_j(p)\right) \cdot F_{n+2k}(t/p) \qquad (p \in C_{n;\mathbf{A},\mathbf{b}}), \end{aligned} \tag{5.13}$$

and similarly[13]

$$\begin{aligned} 0 &\leqq H_{n;\mathbf{A},\mathbf{b}}(t) - \sum_0^{k-1} d_j(p) G_{n+2j;\kappa}(t/p) \\ &\leqq \left(1 - \sum_0^{k-1} d_j(p)\right) \cdot G_{n+2k;\kappa}(t/p) \qquad (p \in D_{n;\mathbf{A},\mathbf{b}}). \end{aligned} \tag{5.14}$$

Clearly, the last two inequalities are weakened (cf., [17]) if $F_{n+2k}(t/p)$ and $G_{n+2k;\kappa}(t/p)$ are replaced by 1. The modified inequalities are, however, uniform with respect to t.

Alternative upper bounds which are, in general, considerably sharper than those provided by (5.13) and (5.14) may be derived from Lemmas 4 and 5. On referring to these lemmas, and using the upper bounds for $|c_j|$ and $|d_j|$ given previously (formulae (4.14) and (4.16)), we obtain

$$\begin{aligned} \left|\sum_k^{\infty} c_j(p) F_{n+2j}(t/p)\right| &\leqq c_0(p) \cdot \sum_k^{\infty} \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{\mu^j}{j!} F_{n+2j}(t/p) \\ &\leqq c_0(p) \cdot \frac{\Gamma(\frac{1}{2}n + k)}{\Gamma(\frac{1}{2}n)} \frac{\mu^k}{k!} \\ &\quad \cdot (1 - \mu)^{-(\frac{1}{2}n+k)} F_{n+2k}[(1 - \mu)t/p], \qquad 0 \leqq \mu \leqq 1, \end{aligned} \tag{5.15}$$

and

$$\left|\sum_k^{\infty} d_j(p) G_{n+2j;\kappa}(t/p)\right| \leqq d_0(p) \cdot \sum_k^{\infty} \frac{\Gamma(\frac{1}{2}n + j)}{\Gamma(\frac{1}{2}n)} \frac{(\mu + \frac{1}{2}\kappa^2)^j}{j!} G_{n+2j;\kappa}(t/p)$$

[13] In the derivation of (5.13) and (5.14) the monotonic decreasing character with respect to N of the functions $F_N(\cdot)$ and $G_{N;\kappa}(\cdot)$ has been used. That $G_{N;\kappa}(\cdot)$ is a decreasing function in N follows perhaps most simply from the following considerations: $G_{N+r;\kappa}(y)$ and $G_{N;\kappa}(y)$ are the probability measures of the spheres, in Euclidean spaces of dimensionality $N + r$ and N, respectively, defined by $\sum_{s=1}^{N+r} (x_s - \kappa_s)^2 \leqq y$ and $\sum_{s=1}^{N} (x_s - \kappa_s)^2 \leqq y$, where, without any loss in generality, $\kappa_s = 0$, $s = N + 1, \cdots, N + r$, $\sum_{s=1}^{N} \kappa_s^2 = \kappa^2$ and the x_s, $s = 1, \cdots, N + r$, are independent normal variables with zero means and unit variances. However, $G_{N;\kappa}(y)$ may also be interpreted as the probability measure of the *cylinder set* $\sum_{s=1}^{N} (x_s - \kappa_s)^2 \leqq y$ in $(N + r)$-dimensional Euclidean space. The required result follows on noting that the $(N + r)$-sphere is a subset of the cylinder.

$$\leq d_0(p)\cdot\sum_k^{\infty}\frac{\Gamma(\frac{1}{2}n+k)}{\Gamma(\frac{1}{2}n)}\frac{(\mu+\frac{1}{2}\kappa^2)^k}{k!} \cdot(1-\mu-\tfrac{1}{2}\kappa^2)^{-(\frac{1}{2}n+k)}F_{n+2k}[(1-\mu-\tfrac{1}{2}\kappa^2)t/p], \quad 0\leq\mu+\tfrac{1}{2}\kappa^2<1, \tag{5.16}$$

where $\mu = \max_i |1 - p/a_i| + p\sum b_i^2/(2a_i)$ (Equ. 4.15) and $\kappa^2 = \sum b_i^2$. As a special (but important) case, consider $p \leq a_1$, $\mathbf{b} = \mathbf{0}$. (We recall that every $p \leq a_1$ gives a mixture representation when $\mathbf{b} = \mathbf{0}$.) Here $\mu = 1 - p/a_n$, and the right-hand member of (5.15) reduces to

$$\prod\left(\frac{a_n}{a_i}\right)^{\frac{1}{2}}\cdot\frac{\Gamma(\frac{1}{2}n+k)}{\Gamma(\frac{1}{2}n)}\frac{(a_n/p-1)^k}{k!}F_{n+2k}(t/a_n).$$

This expression is minimized subject to $p \leq a_1$ when $p = a_1$. Hence

$$\left|\sum_k^{\infty}c_j(a_1)F_{n+2j}(t/a_1)\right|\leq\prod\left(\frac{a_n}{a_i}\right)^{\frac{1}{2}}\cdot\frac{\Gamma(\frac{1}{2}n+k)}{\Gamma(\frac{1}{2}n)}\frac{(a_n/a_1-1)^k}{k!}F_{n+2k}(t/a_n) \quad (\mathbf{b}=\mathbf{0}). \tag{5.17}$$

However, it should be noted that the upper bound can be considerably sharpened by suitable choice of $p > a_1$. Consider, for example, the case where $p^{-1} = (a_1^{-1} + a_n^{-1})/2$. This value of p corresponds to the point at which the graph of $\mu = \max_i |1 - p/a_i|$ against p changes slope (μ is equal to $1 - p/a_n$ for

$$0 < p \leq 2/(a_1^{-1} + a_n^{-1})$$

and to $p/a_1 - 1$ for $p \geq 2/(a_1^{-1} + a_n^{-1})$), and does not, in general, give a mixture representation for $H_{n;\mathbf{A},\mathbf{0}}(t)$. The inequality (5.15) reduces for this value of p, say $p = p_0$, to

$$\left|\sum_k^{\infty}c_j(p_0)F_{n+2j}(t/p_0)\right|\leq\prod\left(\frac{a_n}{a_i}\right)^{\frac{1}{2}}\cdot\frac{\Gamma(\frac{1}{2}n+k)}{\Gamma(\frac{1}{2}n)}\frac{(a_n/a_1-1)^k}{2^k k!}F_{n+2k}(t/a_n) \quad (p_0=2a_1a_n/(a_1+a_n),\ \mathbf{b}=\mathbf{0}), \tag{5.18}$$

i.e., on comparing (5.18) with (5.17), the effect of choosing a scale factor p_0 rather than a_1 is to reduce the upper bound for the error after k terms by the factor 2^{-k}. This highlights the fact that from a numerical point of view it is not always advisable to choose a value of p which yields a mixture representation.

We have so far discussed the accuracy of the fundamental expansions for small or moderate p by obtaining upper bounds for the truncation errors in these expansions. We now obtain upper bounds for the truncation errors when p is *large*: specifically, $p > a_n$ in Theorem 1, and $p \geq a_n$ in Theorem 2. (We recall that every $p > a_n$ *fails* to give a mixture representation when $\mathbf{b} = \mathbf{0}$.) At the same time, this will suggest that p_0, which is intermediate in value between a_1 and a_n, is a highly efficient choice for p in the expansion of Theorem 2. Since

$$C_s(r)\equiv\frac{\partial^s}{\partial r^s}e^{-\frac{1}{2}Qr^2-Lr}=e^{\frac{1}{2}L^2/Q}\cdot\frac{\partial^s}{\partial r^s}e^{-\frac{1}{2}Q(r+L/Q)^2} = e^{\frac{1}{2}L^2/Q}\cdot(-1)^sQ^{\frac{1}{2}s}\cdot H_s[Q^{\frac{1}{2}}(r+L/Q)]e^{-\frac{1}{2}Q(r+L/Q)^2}, \tag{5.19}$$

we have (through the general mean-value theorem)

$$e^{-\frac{1}{2}Qr^2-Lr} = \sum_{m=0}^{s-1} (-1)^m Q^{\frac{1}{2}m} H_m(L/Q^{\frac{1}{2}}) \frac{r^m}{m!} + C_s(\zeta)\frac{r^s}{s!}, \qquad 0 < \zeta < r. \tag{5.20}$$

The fundamental expansion of Theorem 1 was derived from (3.10) after representing $\exp(-Qr^2/2 - rL)$ as an *infinite* power series (3.13). If, then, the polynomial of degree $s - 1$ in r on the right of (5.20) is used in (3.10) as an approximant for $\exp(-Qr^2/2 - rL)$, the error is

$$2A^{-\frac{1}{2}}e^{-\frac{1}{2}\mathbf{b}'\mathbf{b}}\{2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)\}^{-1}\cdot\frac{1}{s!}\int_0^{t^{\frac{1}{2}}} M[C_s(\zeta)]e^{-\frac{1}{2}r^2/p}r^{n+s-1}\,dr,$$

or,[14] on setting $s = 2k$ and replacing m in (5.20) by $2j$,

$$\begin{aligned} &H_{n;\mathbf{A},\mathbf{b}}(t) - \sum_0^{k-1} c_j(p)F_{n+2j}(t/p) \\ &\quad = 2A^{-\frac{1}{2}}e^{-\frac{1}{2}\mathbf{b}'\mathbf{b}}\{2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)\}^{-1}\cdot\frac{1}{(2k)!}\int_0^{t^{\frac{1}{2}}} M[C_{2k}(\zeta)]e^{-\frac{1}{2}r^2/p}r^{n+2k-1}\,dr. \end{aligned} \tag{5.21}$$

The required upper bound is now obtained by establishing an upper bound for $M[C_{2k}(\zeta)]$ in (5.21). To achieve this, we establish an upper bound for each of the three terms on the right of (5.19) with $s = 2k$. First, from Lemma 2,

$$e^{\frac{1}{2}L^2/Q} \leqq e^{\frac{1}{2}\Sigma(b_i^2/a_i)(1/a_i-1/p)}, \qquad p > a_n. \tag{5.22}$$

Next, for $k > 0$,

$$Q^k < (1/a_1 - 1/p)^k, \qquad p > a_n. \tag{5.23}$$

Finally, from the identity[15] (see e.g., [8], p. 157)

$$H_{2k}(\eta)e^{-\frac{1}{2}\eta^2} = (-1)^k\int_{-\infty}^{\infty} x^{2k}e^{i\eta x}\cdot(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}\,dx,$$

we have (on replacing $e^{i\eta x}$ by 1)

$$|H_{2k}(\eta)e^{-\frac{1}{2}\eta^2}| \leqq (2k)!/(2^k k!), \qquad \eta \text{ real},$$

and, therefore, since $Q > 0$ for $p > a_n$,

$$|H_{2k}[Q^{\frac{1}{2}}(\zeta + L/Q)]e^{-\frac{1}{2}Q(\zeta+L/Q)^2}| \leqq (2k)!/(2^k k!), \qquad p > a_n. \tag{5.24}$$

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[14] Because of (3.18), the approximation of $\exp(-Qr^2/2 - rL)$ by the polynomial of degree $s - 1$ $(= 2k - 1)$ yields only k terms, namely, $\sum_0^{k-1} c_j(p)F_{n+2j}(t/p)$, after the k zero terms have been deleted.

[15] This identity follows from the well-known formula

$$\int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}}\exp(-x^2/2 + i\eta x)\,dx = \exp(-\eta^2/2)$$

(itself a particular case of (3.25), vis $\theta = 1$, $\tau = -i\eta$) by differentiating $2k$ times with respect to η.

On applying the inequalities (5.22), (5.23) and (5.24) in (5.19) with $s = 2k$, we obtain

$$(5.25) \quad |M[C_{2k}(\zeta)]| < e^{\frac{1}{2}\Sigma(b_i^2/a_i)(1/a_i - 1/p)} \cdot (1/a_1 - 1/p)^k \cdot (2k)!/(2^k k!), \quad p > a_n .$$

The use of (5.25) in (5.21) yields (with the aid of (3.23)) the required upper bound to the error after k terms in the form

$$(5.26) \quad \begin{aligned} &\left| H_{n;\mathbf{A},\mathbf{b}}(t) - \sum_0^{k-1} c_j(p) F_{n+2j}(t/p) \right| \\ &\leqq e^{\frac{1}{2}\Sigma a_i b_i^2/(p-a_i)} \cdot \prod \left(\frac{p}{a_i}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}n + k)}{\Gamma(\frac{1}{2}n)} \frac{(p/a_1 - 1)^k}{k!} F_{n+2k}(t/p), \quad p > a_n. \end{aligned}$$

Examination of the method of derivation of (5.26) shows that if $\mathbf{b} = \mathbf{0}$, then the condition $p > a_n$ may be strengthened to $p \geqq a_n$. Thus (5.26) gives as a special case

$$(5.27) \quad \begin{aligned} &\left| H_{n;\mathbf{A},\mathbf{0}}(t) - \sum_0^{k-1} c_j(p) F_{n+2j}(t/p) \right| \\ &\leqq \prod \left(\frac{p}{a_i}\right)^{\frac{1}{2}} \cdot \frac{\Gamma(\frac{1}{2}n + k)}{\Gamma(\frac{1}{2}n)} \frac{(p/a_1 - 1)^k}{k!} F_{n+2k}(t/p), \quad p \geqq a_n. \end{aligned}$$

Now it may be shown that the right-hand member in this inequality is an increasing function in p, so that the latter expression is minimized, subject to $p \geqq a_n$, when $p = a_n$. We then have

$$(5.28) \quad \begin{aligned} &\left| H_{n;\mathbf{A},\mathbf{0}}(t) - \sum_0^{k-1} c_j(a_n) F_{n+2j}(t/a_n) \right| \\ &\leqq \prod \left(\frac{a_n}{a_i}\right)^{\frac{1}{2}} \cdot \frac{\Gamma(\frac{1}{2}n + k)}{\Gamma(\frac{1}{2}n)} \frac{(a_n/a_1 - 1)^k}{k!} F_{n+2k}(t/a_n). \end{aligned}$$

On comparing (5.28) with (5.17), we find that so far as can be judged from the upper bounds given by these two inequalities no gain or loss in accuracy is obtained in using $p = a_n$ rather than $p = a_1$, provided $\mathbf{b} = \mathbf{0}$. On the other hand, recall that when $\mathbf{b} = \mathbf{0}$, $p = p_0 = 2a_1a_n/(a_1 + a_n)$ gives much greater accuracy than $p = a_1$ (formula (5.18)). Thus the choice of p is crucial from the point of view of accuracy. In the absence of a more detailed analysis, the above discussion suggests that $p = p_0$, which is intermediate between a_1 and a_n, is a highly efficient choice when $\mathbf{b} = \mathbf{0}$ and may be near the optimal value. We have in fact shown that $p = p_0$ is certainly a superior choice than any value of $p \leqq a_1$ or $\geqq a_n$ when $\mathbf{b} = \mathbf{0}$.

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