TESTING THE HYPOTHESIS OF NO FIXED MAIN-EFFECTS IN SCHEFFÉ'S MIXED MODEL

BY J. P. IMHOF

University of Geneva

1. Introduction and Summary. Various approaches are available for the formulation of linear models for the analysis of variance. The mixed model in which one factor is a fixed-effects factor, and one factor is a random-effects factor can be obtained for instance as a limiting case of the general models of Cornfield and Tukey [3], and of Wilk and Kempthorne [8]. An entirely different approach is that given by Scheffé [6], and is the one considered in the present paper. Whatever the approach used, a hypothesis of interest will usually be the hypothesis of no fixed-effects. Consider a two-way layout in which A denotes a fixed-effects factor and B a random-effects factor. Let I and J be the numbers of levels of factors A and B respectively at which measurements are taken ($I > 1$, $J > 1$). Let K be the number of replications performed in each cell ($K > 1$). In the light of Table 1 of [3], Table 3 of [8], and formulas (46) and (54) of [6], an adequate F-type statistic to use for testing the hypothesis H_A that all main-effects corresponding to the levels of factor A are zero appears to be

$$\mathfrak{F} = (\mathrm{MS})_A/(\mathrm{MS})_{AB}. \tag{1.1}$$

The usual mean squares $(\mathrm{MS})_A$ and $(\mathrm{MS})_{AB}$ corresponding to factor A and to $A \times B$ interactions are explicitly defined below. With the normal theory models which are commonly used in the case of two fixed-effects factors and in the case of two random-effects factors (e.g., in [7]), the criterion (1.1) has under the hypothesis H_A the F-distribution with $I - 1$ and $(I - 1)(J - 1)$ d.f. In Scheffé's mixed model [6], this is no longer the case. When $J \geqq I$, a Hotelling T^2 statistic can then be constructed for the test of H_A. When multiplied by a constant factor, this statistic has the F-distribution with $I - 1$ and $J - I + 1$ d.f. While requiring a larger amount of computational work, the T^2 test will have little power when $J - I + 1$ is small. It is therefore tempting to construct a test of H_A, based on the ratio (1.1), by assuming that the law of $\mathfrak{F}$ is not much different under H_A from that of F with $I - 1$ and $(I - 1)(J - 1)$ d.f. In Subsection 4.1, we investigate the possible ill-effects of this assumption. They can be considerable, and remedies are suggested in Subsections 4.2 and 4.3.

2. Notation and preliminaries. We begin by recalling the basic assumptions and some results relative to Scheffé's mixed model. A complete two-way layout is considered in which the subscript $i = 1, \cdots, I$ corresponds to the levels of the fixed-effects factor A, the subscript $j = 1, \cdots, J$ to the levels of the random-effects factor B. Replications are labelled with $k = 1, \cdots, K$. We let $n = IJK$.

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The familiar dot convention is used to denote averages over the permissible values of a subscript. A matrix $\mathbf{A}$ with elements $a_{ii'}$ is written $((a_{ii'}))$, while $((a^{ii'})) = ((a_{ii'}))^{-1}$, and $\mathbf{A}'$ is the transpose of $\mathbf{A}$.

Effects are determined in [6] in terms of "true" cell means m_{ij}, and the general observation can be written

$$y_{ijk} = m_{ij} + e_{ijk} = \mu + \alpha_i + b_j + c_{ij} + e_{ijk}\,. \tag{2.1}$$

The general mean μ and the main-effects $\alpha_1, \cdots, \alpha_I$ are unknown parameters. The main effects b_j, interactions c_{ij} and errors e_{ijk} are random variables. The J vectors $(m_{1j}, \cdots, m_{Ij})'$ are assumed to be independently and identically distributed like a normally distributed vector $\mathbf{m} = (m_1, \cdots, m_I)'$, with covariance matrix $\mathbf{\Sigma} = ((\sigma_{ii'}))$. As a result, the J vectors $(b_j, c_{ij}, \cdots, c_{Ij})'$ are independently and identically distributed like a normal vector $(b, c_1, \cdots, c_I)'$ with zero mean and

$$\begin{aligned} \operatorname{Var}(b) &= \sigma_{..}\,, \qquad \operatorname{Cov}(b, c_i) = \sigma_{i\cdot} - \sigma_{..}\,, \\ \operatorname{Cov}(c_i, c_{i'}) &= \sigma^*_{ii'} = \sigma_{ii'} - \sigma_{i\cdot} - \sigma_{\cdot i'} + \sigma_{..}\,. \end{aligned} \tag{2.2}$$

In addition, the vector $\mathbf{m}$ is assumed to be independent of the errors e_{ijk}, which are independently and normally distributed with mean 0 and variance σ_e^2.

The following variance components are defined

$$\begin{aligned} \sigma_A^2 &= (I-1)^{-1}\sum_i \alpha_i^2\,, \qquad \sigma_B^2 = \operatorname{Var}(b), \\ \sigma_{AB}^2 &= (I-1)^{-1}\sum_i \operatorname{Var}(c_i). \end{aligned} \tag{2.3}$$

It is shown in [6] that the mean squares $(\mathrm{MS})_A = (I-1)^{-1}JK \sum_i (y_{i..} - y_{...})^2$ and $(\mathrm{MS})_{AB} = (I-1)^{-1}(J-1)^{-1}K \sum_i \sum_j (y_{ij\cdot} - y_{i..} - y_{\cdot j\cdot} + y_{...})^2$ are independent. Their expected values are

$$E(\mathrm{MS})_A = JK\sigma_A^2 + K\sigma_{AB}^2 + \sigma_e^2\,, \qquad E(\mathrm{MS})_{AB} = K\sigma_{AB}^2 + \sigma_e^2\,. \tag{2.4}$$

For the test of H_A, a test criterion with known distribution is obtained in terms of the differences $d_{rj} = y_{rj\cdot} - y_{Ij\cdot}$ and sums of products $a_{rr'} = \sum_j (d_{rj} - d_{r\cdot})(d_{r'j} - d_{r'\cdot})$, $r, r' = 1, \cdots, I-1$. From (2.1) and (2.2) one finds

$$\gamma_{rr'} = \operatorname{Cov}(d_{r\cdot}, d_{r'\cdot}) = J^{-1}[\sigma_{rr'} - \sigma_{rI} - \sigma_{Ir'} + \sigma_{II} + K^{-1}(1 + \delta_{rr'})\sigma_e^2], \tag{2.5}$$

where $\delta_{rr'}$ is the Kronecker δ. The statistic

$$\mathfrak{F}^* = J(J - I + 1)(I-1)^{-1} \sum_r \sum_{r'} a^{rr'} d_{r\cdot} d_{r'\cdot}\,, \tag{2.6}$$

has the F-distribution with $I-1$ and $J-I+1$ d.f. and noncentrality parameter

$$\delta^2 = \sum_r \sum_{r'} (\alpha_r - \alpha_I)(\alpha_{r'} - \alpha_I)\gamma^{rr'}. \tag{2.7}$$

The subscripts r, r' are used instead of i, i' whenever the values taken range only from 1 to $I - 1$.

Consider a fixed level of significance ϵ. We call T_ϵ^* the test which rejects H_A if $\mathfrak{F}^* > F_\epsilon^*$, where F_ϵ^* is given by $P[F_{I-1,J-I+1} > F_\epsilon^*] = \epsilon$. We also want to consider tests of H_A based on the statistic $\mathfrak{F}$. Write $F_\epsilon(h)$ for the ϵ-quantile determined by $P[F_{h,(J-1)h} > F_\epsilon(h)] = \epsilon$. Various choices of h will be considered, all satisfying $1 \leqq h \leqq I - 1$. We call then $T_\epsilon(h)$ the test which rejects H_A if $\mathfrak{F} > F_\epsilon(h)$.

3. An expression for $\mathfrak{F}$. The numerator and denominator of $\mathfrak{F} = (\text{MS})_A/(\text{MS})_{AB}$ are independent quadratic forms in normal variables, and can be expressed each as a linear combination of independent χ^2 variables. Consider first $(\text{MS})_A$. Let $u_i = c_{i\cdot} + e_{i\cdot\cdot}$, $z_i = u_i - u_{\cdot\cdot}$. Then $(\text{MS})_A = (I-1)^{-1}JK(\mathbf{z} + \boldsymbol{\alpha})'(\mathbf{z} + \boldsymbol{\alpha})$, where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_I)'$ and the vector $\mathbf{z} = (z_1, \cdots, z_I)'$ has covariance matrix

$$\mathbf{\Sigma}_z = ((J^{-1}\sigma_{ii'}^* + n^{-1}(I\delta_{ii'} - 1)\sigma_e^2)). \tag{3.1}$$

Because $\sigma_{i\cdot}^* = \sigma_{\cdot i'}^* = 0$ for all i, i', row and column sums in $\mathbf{\Sigma}_z$ are zero.

LEMMA 1. *The rank of* $\mathbf{\Sigma}_z$ *is* $I - 1$. *The nonzero characteristic roots* $\lambda_1, \cdots, \lambda_{I-1}$ *of* $\mathbf{\Sigma}_z$ *are the characteristic roots of the* $I - 1 \times I - 1$ *matrix*

$$\mathbf{M} = ((J^{-1}(\sigma_{rr'}^* - \sigma_{rI}^*) + J^{-1}K^{-1}\delta_{rr'}\sigma_e^2)). \tag{3.2}$$

One has $\lambda_r \geqq J^{-1}K^{-1}\sigma_e^2$, $JK\sum_r \lambda_r = (I-1)(K\sigma_{AB}^2 + \sigma_e^2)$.

The proof is similar to that of Lemma 3 of [4].

Consider an orthogonal matrix $\mathbf{P} = ((p_{ii'}))$ such that $\mathbf{P}'\mathbf{\Sigma}_z\mathbf{P} = \mathbf{\Lambda}$, the diagonal matrix with the characteristic roots $\lambda_1, \cdots, \lambda_{I-1}, 0$ down the main diagonal. Let $\mathbf{q} = (q_1, \cdots, q_{I-1}, q_I)' = \mathbf{P}'\boldsymbol{\alpha}$. The last column of $\mathbf{P}$, which is the normalized characteristic vector of $\mathbf{\Sigma}_z$ corresponding to the characteristic root 0, has all its elements equal to $I^{-\frac{1}{2}}$, hence

$$q_I = 0, \qquad \sum_{i=1}^{I} p_{ir} = 0, \qquad r = 1, \cdots, I-1. \tag{3.3}$$

Now let $\mathbf{z} = \mathbf{P}\mathbf{w}$. The vector $\mathbf{w}$ is $N(\mathbf{0}, \mathbf{\Lambda})$, and

$$(\text{MS})_A = (I-1)^{-1}JK(\mathbf{w} + \mathbf{q})'(\mathbf{w} + \mathbf{q}).$$

Writing $\delta_r^2 = \lambda_r^{-1}q_r^2$, one obtains

LEMMA 2. $(\text{MS})_A = (I-1)^{-1}JK\sum_r \lambda_r\chi^2_{(r);\delta_r^2}$, *where the noncentral* χ^2 *variables* $\chi^2_{(r);\delta_r^2}$ *with one d.f. each and noncentrality parameter* δ_r^2 *are independent.*

Next, consider $(\text{MS})_{AB}$. Writing $l_{ij} = c_{ij} - c_{i\cdot} + e_{ij\cdot} - e_{i\cdot\cdot} - e_{\cdot j\cdot} + e_{\cdots}$, one has

$$(\text{MS})_{AB} = (I-1)^{-1}(J-1)^{-1}K\sum_i\sum_j l_{ij}^2$$

and one finds $\text{Cov}(l_{ij}, l_{i'j'}) = (J\delta_{jj'} - 1)[J^{-1}\sigma_{ii'}^* + n^{-1}(I\delta_{ii'} - 1)\sigma_e^2]$. Let $\mathbf{l} = (l_{11}, \cdots, l_{1I}, \cdots, l_{J1}, \cdots, l_{JI})'$. Then $\mathbf{l}$ is $N(\mathbf{0}, \mathbf{\Sigma}_l)$ and $\mathbf{\Sigma}_l$ is the Kronecker direct product of the $J \times J$ matrix $((J\delta_{jj'} - 1))$ with $\mathbf{\Sigma}_z$. The characteristic roots of $\mathbf{\Sigma}_l$ are therefore $J\lambda_1, \cdots, J\lambda_{I-1}$, each with multiplicity $J - 1$. It follows that $(\text{MS})_{AB} = (I - 1)^{-1}(J - 1)^{-1}JK \sum_r \lambda_r \chi^2_{(r),J-1}$, where the central χ^2 variables $\chi^2_{(r),J-1}$ with $J - 1$ d.f. each are independent. This, together with Lemma 2, shows that

$$\mathfrak{F} = (J - 1)\{\sum_r \lambda_r \chi^2_{(r);\delta_r^2} / \sum_r \lambda_r \chi^2_{(r),J-1}\}, \tag{3.4}$$

where the $2(I - 1)\chi^2$ variables are independent.

According to (3.3), $\delta_1, \cdots, \delta_{I-1}$ are $I - 1$ orthogonal contrasts among the parameters $\alpha_1, \cdots, \alpha_I$. One has,

Lemma 3. *Let* $\Delta^2 = \sum_{r=1}^{I-1} \delta_r^2$ *and let* δ^2 *be the non-centrality parameter* (2.7). *Then* $\Delta^2 = \delta^2$.

Proof. Let $\mathbf{P}^*$ be the $I - 1 \times I - 1$ upper left corner of $\mathbf{P}$. Let Λ^* be the $I - 1 \times I - 1$ upper left corner of Λ. Let $\mathbf{L}$ be the $I - 1 \times I - 1$ diagonal matrix with elements $\lambda_1^{-\frac{1}{2}}, \cdots, \lambda_{I-1}^{-\frac{1}{2}}$ down the diagonal. Let

$$\boldsymbol{\alpha}^{*\prime} = (\alpha_1 - \alpha_I, \cdots, \alpha_{I-1} - \alpha_I).$$

Because of (3.3) one has $\mathbf{P}^{*\prime}\boldsymbol{\alpha}^* = (q_1, \cdots, q_{I-1})'$. Therefore $\Delta^2 = \boldsymbol{\alpha}^{*\prime}\mathbf{QQ}'\boldsymbol{\alpha}^*$, where $\mathbf{Q} = \mathbf{P}^*\mathbf{L}$. Let $v_{ii'}$ denote the i, i'-element of $\mathbf{\Sigma}_z$. Let $s_{rr'} = v_{rr'} - v_{rI} - v_{Ir'} + v_{II}$, $r, r' = 1, \cdots, I - 1$, and let $\mathbf{S} = ((s_{rr'}))$. One verifies by direct computation, using (3.3), that $\mathbf{P}'\mathbf{\Sigma}_z\mathbf{P} = \mathbf{\Lambda}$ implies $\mathbf{P}^{*\prime}\mathbf{SP}^* = \mathbf{\Lambda}^*$. Therefore $\mathbf{Q}'\mathbf{SQ} = \mathbf{U}$, the identity matrix, from which it follows that $(\mathbf{QQ}')^{-1} = \mathbf{S}$. Thus $\Delta^2 = \boldsymbol{\alpha}^{*\prime}\mathbf{S}^{-1}\boldsymbol{\alpha}^*$. But one verifies at once that $s_{rr'} = \gamma_{rr'}$ as given in (2.5). The conclusion follows.

Equation (3.4) shows that if $\lambda_1 = \cdots = \lambda_{I-1} = \lambda$, then

$$\mathfrak{F} = F_{I-1,(I-1)(J-1);\delta^2}, \tag{3.5}$$

an F-variable with $I - 1$ and $(I - 1)(J - 1)$ d.f. and noncentrality parameter δ^2. This is in particular the case when $I = 2$. When $I > 2$, it is easily verified that (3.5) holds, if and only if all variances are equal and all covariances are equal in the basic matrix $\mathbf{\Sigma}$, so that

$$\sigma_{ii'} = \sigma^2[\rho + (1 - \rho)\delta_{ii'}], \qquad i, i' = 1, \cdots, I, |\rho| < 1. \tag{3.6}$$

The noncentrality parameter $\delta^2 = \lambda^{-1} \sum_r q_r^2$ can then be written

$$\delta^2 = (K\sigma_{AB}^2 + \sigma_e^2)^{-1}JK \sum_i \alpha_i^2.$$

4. Testing H_A with the criterion $\mathfrak{F}$. Although the distribution of $\mathfrak{F}$ reduces to that of an F-variable only when (3.6) holds, it is tempting to assume that under H_A, it never departs much from the central F distribution with $I - 1$ and $(I - 1)(J - 1)$ d.f. Then, one would use for testing the hypothesis H_A the test $T_\epsilon(I - 1)$ described in Section 3. There is however no justification in Scheffé's mixed model for assumption (3.6); a more careful investigation of its effects will therefore be made.

4.1. *Effect on the level of significance of using* $T_\epsilon(I - 1)$. Departure from the assumption (3.6), under which $T_\epsilon(I - 1)$ is a test of H_A with level of significance ϵ, results in $\mathfrak{F}$ having the distribution (3.4) instead of (3.5). The true level of significance for the test $T_\epsilon(I - 1)$ then equals

$$(4.1) \qquad P[(J - 1) \sum_r \lambda_r \chi^2_{(r)} / \sum_r \lambda_r \chi^2_{(r),J-1} > F_\epsilon(I - 1)].$$

When computing values of (4.1), there is no loss of generality in assuming $\sum_r \lambda_r = I - 1$. Then, according to Lemma 1,

$$(4.2) \qquad \lambda_r \geqq \varphi, \qquad \varphi = (K\sigma_{AB}^2 + \sigma_e^2)^{-1}\sigma_e^2 .$$

If $K\sigma_{AB}^2$ is small compared to σ_e^2, the lower bound φ will be little different from unity and so will the λ_r, irrespective of the structure of the basic covariance matrix $\mathbf{\Sigma}$. This means that the distribution of $\mathfrak{F}$ will be close to that of F with $I - 1$ and $(I - 1)(J - 1)$ d.f. Our interest, however, lies mainly in the case where interactions are not negligible. Then, φ will often be close to zero. If for instance $K = 3$ and $\sigma_{AB}^2 = 3\sigma_e^2$, one has $\varphi = .1$. This means that if the covariance matrix $\mathbf{\Sigma}$ departs sufficiently from condition (3.6) (examples will be given), the λ_r can be highly unequal. Little information is available concerning the effect of inequality of the values of $\lambda_1, \cdots, \lambda_{I-1}$ on the probability (4.1) (see Box [2], Table 4 and page 301). Some results are given in Table 1 for the case $I = 5$. They show that with the "usual F-test" $T_\epsilon(I - 1)$, the probability of type I error can be considerably higher than the nominal level ϵ. The trend found in Table 1, namely that the true level of significance always exceeds the nominal one, only reverses itself for a value of ϵ larger than .3. That the same trend will prevail for small values of ϵ, whatever the values of I and J, can be shown by a heuristic argument similar to the one used at the end of Section 5 of [4]. An example might be in order to show that with a matrix $\mathbf{\Sigma}$ that does not appear exceptional, the ratio of largest to smallest of the λ_r's can already

TABLE 1

Exact probability that $\mathfrak{F}$ exceeds the upper ϵ-quantile of $F_{4,4(J-1)}$ when H_A holds

100ϵ	$J - 1$	$(\lambda_1, \cdots, \lambda_4)$				
		(1) (3.7, .1, .1, .1)	(2) (1.9, 1.9, .1, .1)	(3) (2.2, .6, .6, .6)	(4) (1.8, 1, .6, .6)	(5) (1, 1, 1, 1)
1	4	.078	.036	.023	.016	.010
	8	.070	.034	.024	.017	.010
5	4	.140	.096	.073	.063	.050
	8	.129	.092	.072	.062	.050
10	4	.186	.149	—	—	.100
25	4	.280	.274	—	—	.250

be considerable. Let $I = 4$ and let $\mathbf{\Sigma}$ be the circular symmetric matrix

$$\mathbf{\Sigma} = \begin{bmatrix} u & v & w & v \\ v & u & v & w \\ w & v & u & v \\ v & w & v & u \end{bmatrix}.$$

Then $\lambda_1 = \lambda_3 = J^{-1}(u - w + K^{-1}\sigma_e^2)$, $\lambda_2 = J^{-1}(u + w - 2v + K^{-1}\sigma_e^2)$, $\lambda_4 = 0$. Thus, if $(u, v, w) = (1, -.4, .4)$, which gives $\sigma_{AB}^2 = 1.13$ and if $K^{-1}\sigma_e^2 = .2$, one has $\lambda_2/\lambda_1 = 3$. The situation is then comparable to that in column (4) of Table 1. More extreme examples are contained in Lemma 4 below.

4.2. *Approximating the distribution of* $\mathfrak{F}$ *under* H_A. When H_A holds, $\delta_1^2 = \cdots = \delta_{I-1}^2 = 0$ in (3.4) and the distribution of $\mathfrak{F}$ can be approximated by using a method first suggested by Satterthwaite [5]: Substituting $g\chi_h^2$ for $\sum_r \lambda_r \chi_{(r)}^2$ and $g'\chi_{h'}^2$ for $\sum_r \lambda_r \chi_{(r),J-1}^2$, where g, g', h, h' are determined so that in each case the first two moments coincide, one obtains

$$\mathfrak{F} \cong F_{h,(J-1)h}, \qquad h = (\sum_r \lambda_r^2)^{-1}(\sum_r \lambda_r)^2. \tag{4.3}$$

Box [2] has given some numerical results which indicate that when h is known, this type of approximation is satisfactory for determining upper 5%-points of $\mathfrak{F}$. Further evidence of this is contained in Table 2, which gives values of the probability $P[\mathfrak{F} > F_\epsilon(h)]$ for $I = 5$, $\epsilon = .01$ and $\epsilon = .05$, $J - 1 = 4$ and $J - 1 = 8$, for five combinations $(\lambda_1, \cdots, \lambda_4)$. The approximation (4.3) is poor, only when both $J - 1$ and h are small (in column (3), $h = 1.58$). Its use then results in the probability of type I error being kept below the nominal value ϵ, which causes no concern.

In practice, the λ_r are unknown and h must be estimated from the data before (4.3) can be used. Lemma 1, (4.3) and the relations $\sigma_{i.}^* = 0$, $\sum_i \sigma_{ii}^* = (I - 1)\sigma_{AB}^2$ give

$$h = (I-1)^2[I - 1 + S(\sigma_{AB}^2 + K^{-1}\sigma_e^2)^{-2}]^{-1}, \qquad S = \sum_i \sum_{i'} \sigma_{ii'}^{*2} + (I-1)\sigma_{AB}^4.$$

An estimator of h is obtained by substituting for $\sigma_{ii'}^*$ the unbiased estimator (63) of [6], namely

$$\tilde{\sigma}_{ii'}^* = (J-1)^{-1} \sum_j l_{ij} l_{i'j} + (IK)^{-1}(I\delta_{ii'} - 1)(\text{MS})_e,$$

TABLE 2

Exact probability that $\mathfrak{F}$ *exceeds the upper* ϵ*-quantile of* $F_{h,(J-1)h}$ *when* H_A *holds*

100ϵ	$J - 1$	$(\lambda_1, \cdots, \lambda_4)$ (1) (.1, .7, 1.6, 1.6)	(2) (.7, .7, 1, 1.6)	(3) (.1, .1, .7, 3.1)	(4) (.8, .5, 1.3, 1.4)	(5) (1, 1, 1, 1)
1	4	.0095	.0096	.0072	.0097	.0100
	8	.0103	.0103	.0096	.0102	.0100
5	4	.049	.048	.043	.049	.050
	8	.049	.049	.047	.049	.050

where the l_{ij} are as in Section 3, and substituting for σ_{AB}^2, σ_e^2 their unbiased estimators $K^{-1}[(\text{MS})_{AB} - (\text{MS})_e]$ and $(\text{MS})_e$. Thus an estimator of h which has the same range as h is

$$\tilde{h} = \max\{1, (I-1)^2[I - 1 + K^2\tilde{S}(\text{MS})_{AB}^{-2}]^{-1}\},$$

where $\tilde{S} = \sum_i \sum_{i'} \tilde{\sigma}_{ii'}^{*2} + (I-1)K^{-2}[(\text{MS})_{AB} - (\text{MS})_e]^2$. The level of significance of the resulting test $T_\epsilon(\tilde{h})$ is unknown. When J is small, it may well differ considerably from ϵ, and the use of this test does not seem advisable.

4.3. *A quick test based on* $\mathfrak{F}$. For fixed values of σ_{AB}^2 and σ_e^2, Lemma 1 and (4.3) show that h is a monotonously decreasing function of $\sum \lambda_r^2$. The minimum of h is achieved when all but one of the λ_r equal the lower bound $(JK)^{-1}\sigma_e^2$, and is

$$h_{\min} = (I-1)[1 + (I-2)(1-\varphi)^2]^{-1}, \qquad h_{\min} > 1. \tag{4.4}$$

This can easily be estimated, using only the mean squares $(\text{MS})_{AB}$ and $(\text{MS})_e$. An estimator of φ which has the same range as φ is

$$\tilde{\varphi} = \inf\{1, (\text{MS})_e/(\text{MS})_{AB}\}.$$

Substituting $\tilde{\varphi}$ for φ in (4.4) yields the estimator

$$\tilde{h}_{\min} = (I-1)[1 + (I-2)(1-\tilde{\varphi})^2]^{-1}.$$

We propose to use for testing H_A the test $T_\epsilon(\tilde{h}_{\min})$, that is reject H_A if $\mathfrak{F}$ exceeds the upper ϵ-quantile $F_\epsilon(\tilde{h}_{\min})$ of F with $\tilde{h}_{\min}$ and $(J-1)\tilde{h}_{\min}$ d.f. Although the true level of significance of this test is again unknown, it should be well below the nominal level ϵ, except when the λ_r are such that h is close to $h_{\min}$. Lemma 4 below gives indications about the possibility of this occurring. One has $\tilde{h}_{\min} \geqq 1$, hence $F_\epsilon(\tilde{h}_{\min}) \leqq F_\epsilon(1)$. A lower bound for the power of the test $T_\epsilon(\tilde{h}_{\min})$ is therefore

$$P[(J-1)\sum_r \lambda_r \chi_{(r);\delta_r^2}^2 / \sum_r \lambda_r \chi_{(r),J-1}^2 > F_\epsilon(1)].$$

TABLE 3

Lower bound for the power of the test $T_\epsilon(\tilde{h}_{\min})$

	100ϵ	$(\delta_1^2, \delta_2^2, \delta_3^2, \delta_4^2)$	$J-1$	$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ (1, .7, 1.6, 1.6)	(.7, .7, 1, 1.6)	(.1, .1, .7, 3.1)	(.8, .5, 1.3, 1.4)	(1, 1, 1, 1)	Power of T_ϵ^*
(1)	1	(100, 25, 0, 0)	4	.04	.61	.01	.65	.88	<.2
(2)	1	(40, 10, 0, 0)	8	.01	.36	.01	.41	.69	.66
(3)	1	(25, 50, 25, 25)	4	.81	.83	.69	.80	.88	<.2
(4)	1	(10, 20, 10, 10)	8	.61	.62	.49	.58	.69	.66
(5)	1	(10, 25, 25, 65)	4	.95	.89	.97	.93	.88	<.2
(6)	1	(4, 10, 10, 26)	8	.84	.71	.91	.79	.69	.66
(7)	5	(36, 9, 0, 0)	4	.09	.64	.05	.67	.85	.19
(8)	5	(20, 5, 0, 0)	8	.06	.50	.04	.54	.74	.72
(9)	5	(9, 18, 9, 9)	4	.79	.81	.66	.79	.85	.19
(10)	5	(5, 10, 5, 5)	8	.67	.69	.53	.66	.74	.72
(11)	5	(4, 9, 9, 23)	4	.92	.86	.94	.90	.85	.19
(12)	5	(2, 5, 5, 13)	8	.84	.75	.88	.80	.74	.72

Table 3 contains values of this probability for $I = 5$, $J = 5$ and $J = 9$, $\epsilon = .01$ and $\epsilon = .05$, and various combinations $(\lambda_1, \cdots, \lambda_4)$ and $(\delta_1^2, \cdots, \delta_4^2)$.

A comparison can be made, between the lower bound of the power of the test $T_\epsilon(\tilde{h}_{\min})$ and the power of the Hotelling T^2-test T_ϵ^*. According to Lemma 3, the values in lines (1), (3) and (5) should be compared with the power of the F-test with 4 and 1 d.f. and noncentrality parameter $\delta^2 = 125$, which is less than .2: Those in lines (2), (4) and (6) with the power for 4 and 5 d.f. and $\delta^2 = 50$, namely .66: Those in lines (7), (9) and (11) with the power for 4 and 1 d.f. and $\delta^2 = 45$, namely .19: Finally those in lines (8), (10) and (12) with the power for 4 and 5 d.f. and $\delta^2 = 25$, namely .72. To facilitate comparison, those values have been reproduced in the last column of Table 3.

It is seen that $T_\epsilon(\tilde{h}_{\min})$ compares very favorably with T_ϵ^*, especially if one takes into account that unless $h_{\min}$ is close to one, the lower bounds tabled will be considerably smaller than the true values of the power. As expected, the gain in power due to the use of $T_\epsilon(\tilde{h}_{\min})$ is most considerable when $J - I + 1$ is small. $T_\epsilon(\tilde{h}_{\min})$ is poor only against the alternatives for which most of δ^2 is contributed by δ_r^2's corresponding to very small values of the λ_r's. Some idea regarding the occurrence of such alternatives can be gained from

LEMMA 4.

(i) *If* (*rank* $\mathbf{\Sigma}$) $= p < I$, *then at least* $I - p - 1$ *of the* λ_r *equal the minimum value* $(JK)^{-1}\sigma_e^2$.

(ii) *If* $\mathbf{\Sigma}$ *has* $t + 1$ *identical rows* (*we may assume they are the last* $t + 1$), *then* t *of the* λ_r'*s equal* $(JK)^{-1}\sigma_e^2$ (*we may assume they are* $\lambda_{I-t}, \cdots, \lambda_{I-1}$) *and*

$$\sum_{I-t}^{I-1} \delta_r^2 = (t + 1)^{-1} JK\sigma_e^{-2} \sum_{j=1}^{t} \sum_{j'=1}^{j} (\alpha_{I-j} - \alpha_{I-j-j'})^2. \tag{4.5}$$

PROOF. If $\mathbf{C}$ is the $I \times I$ matrix $((\delta_{ii'} - I^{-1}))$, then $((\sigma_{ii'}^*)) = \mathbf{C\Sigma C}$ implies rank $((\sigma_{ii'}^*)) \leqq \inf\{I - 1, \text{rank } \mathbf{\Sigma}\}$. Similarly one has rank $((\sigma_{rr'}^* - \sigma_{rI}^*)) \leqq$ rank $((\sigma_{ii'}^*))$. From the form of (3.2), i) follows. Next, suppose the last $t + 1$ rows of $\mathbf{\Sigma}$ are identical. Then one verifies that $\mathbf{\Sigma}_z$ has $t + 1$ characteristic roots equal to $(JK)^{-1}\sigma_e^2$, say they are the last $t + 1$, and that one can take for the last $t + 1$ rows of the matrix $\mathbf{P}'$ which diagonalizes $\mathbf{\Sigma}_z$, $(0, \cdots, 0, -k[k(k + 1)]^{-\frac{1}{2}}, [k(k + 1)]^{-\frac{1}{2}}, \cdots, [k(k + 1)]^{-\frac{1}{2}})$, $k = 1, \cdots, t$, and $(I^{-\frac{1}{2}}, \cdots, I^{-\frac{1}{2}})$ for the last row. Formula (4.5) follows.

Lemma 4 shows that whenever $\mathbf{\Sigma}$ is of rank less than $I - 1$ and $K\sigma_{AB}^2/\sigma_e^2$ is large, there are "bad" alternatives against which $T_\epsilon(\tilde{h}_{\min})$ is powerless. If in particular, (corresponding to the hypothesis in (ii)) for several levels of factor A the "true means" m_i are the same random variable except for additive constants, then $T_\epsilon(\tilde{h}_{\min})$ is powerless to detect any nonzero contrast between the expected values of those "true means" (notice such cannot be the case for all m_i, else $\sigma_{AB}^2 = 0$). It should be pointed out that the same conclusions are valid for the test $T_\epsilon(\tilde{h})$ considered in Subsection 4.2.

5. Table of $F_{.05}(h)$ and $F_{.01}(h)$. To carry out the tests described in Section 4, one must determine the upper ϵ-quantile of the F-distribution with h and

TABLE 4

Upper 5%-point of $F_{h,mh}$

h \ m	2	3	4	5	6	7	8
1.00	18.513	10.128	7.709	6.608	5.987	5.591	5.318
1.05	16.936	9.554	7.366	6.358	5.785	5.418	5.163
1.10	15.613	9.053	7.062	6.133	5.603	5.261	5.023
1.15	14.488	8.613	6.790	5.931	5.436	5.117	4.893
1.20	13.523	8.223	6.545	5.746	5.284	4.984	4.774
1.25	12.687	7.875	6.323	5.578	5.145	4.862	4.664
1.30	11.957	7.563	6.121	5.424	5.016	4.749	4.562
1.35	11.314	7.281	5.937	5.282	4.897	4.644	4.466
1.40	10.744	7.025	5.768	5.150	4.786	4.547	4.377
1.45	10.236	6.792	5.612	5.029	4.683	4.455	4.294
1.50	9.780	6.579	5.468	4.915	4.587	4.370	4.216
1.60	8.997	6.202	5.210	4.711	4.413	4.214	4.073
1.70	8.350	5.880	4.986	4.532	4.258	4.076	3.946
1.80	7.806	5.602	4.789	4.373	4.121	3.952	3.831
1.90	7.343	5.358	4.615	4.231	3.997	3.840	3.728
2.00	6.944	5.143	4.459	4.103	3.885	3.739	3.634
2.10	6.597	4.952	4.319	3.987	3.784	3.646	3.548
2.20	6.293	4.781	4.192	3.882	3.691	3.562	3.469
2.30	6.024	4.627	4.077	3.786	3.606	3.484	3.396
2.40	5.784	4.487	3.972	3.697	3.527	3.412	3.328
2.50	5.568	4.360	3.875	3.616	3.455	3.345	3.266
2.60	5.374	4.244	3.786	3.541	3.387	3.283	3.207
2.70	5.198	4.137	3.704	3.471	3.325	3.225	3.153
2.80	5.038	4.038	3.628	3.405	3.266	3.171	3.102
2.90	4.892	3.947	3.557	3.344	3.211	3.120	3.054
3.00	4.757	3.863	3.490	3.287	3.160	3.073	3.009
3.20	4.518	3.710	3.370	3.183	3.066	2.985	2.926
3.40	4.313	3.577	3.264	3.091	2.982	2.907	2.852
3.60	4.134	3.459	3.169	3.008	2.907	2.836	2.785
3.80	3.977	3.354	3.084	2.934	2.838	2.772	2.724
4.00	3.838	3.259	3.007	2.866	2.776	2.714	2.668
4.20	3.714	3.174	2.937	2.804	2.720	2.661	2.618
4.40	3.602	3.096	2.873	2.748	2.667	2.612	2.571
4.60	3.501	3.026	2.815	2.696	2.619	2.566	2.527
4.80	3.410	2.961	2.761	2.648	2.575	2.524	2.487
5.00	3.326	2.901	2.711	2.603	2.534	2.485	2.450
5.20	3.249	2.846	2.665	2.561	2.495	2.449	2.414
5.40	3.178	2.795	2.621	2.523	2.459	2.415	2.382
5.60	3.113	2.747	2.581	2.487	2.425	2.382	2.351
5.80	3.052	2.703	2.544	2.452	2.394	2.352	2.322

TABLE 4—*Continued*

Upper 1%-point of $F_{h,mh}$

h \ m	2	3	4	5	6	7	8
1.00	98.503	34.116	21.198	16.258	13.745	12.246	11.259
1.05	83.954	30.786	19.606	15.236	12.985	11.631	10.733
1.10	72.597	28.025	18.247	14.349	12.317	11.086	10.265
1.15	63.570	25.706	17.075	13.571	11.726	10.600	9.846
1.20	56.278	23.736	16.054	12.884	11.199	10.164	9.467
1.25	50.303	22.045	15.159	12.274	10.726	9.770	9.124
1.30	45.345	20.581	14.367	11.727	10.299	9.413	8.812
1.35	41.183	19.302	13.662	11.236	9.913	9.088	8.526
1.40	37.655	18.178	13.032	10.791	9.561	8.790	8.264
1.45	34.635	17.183	12.464	10.387	9.239	8.516	8.022
1.50	32.029	16.296	11.950	10.017	8.943	8.264	7.798
1.60	27.779	14.788	11.058	9.368	8.419	7.814	7.397
1.70	24.481	13.554	10.310	8.815	7.968	7.425	7.048
1.80	21.862	12.528	9.673	8.339	7.576	7.084	6.741
1.90	19.743	11.663	9.125	7.924	7.231	6.783	6.469
2.00	18.000	10.925	8.649	7.559	6.927	6.515	6.226
2.10	16.545	10.287	8.231	7.236	6.655	6.275	6.008
2.20	15.314	9.732	7.861	6.948	6.411	6.059	5.810
2.30	14.262	9.243	7.532	6.689	6.191	5.863	5.631
2.40	13.354	8.811	7.237	6.455	5.991	5.684	5.467
2.50	12.563	8.425	6.970	6.242	5.808	5.521	5.317
2.60	11.869	8.079	6.729	6.049	5.641	5.371	5.178
2.70	11.255	7.768	6.509	5.871	5.487	5.232	5.050
2.80	10.709	7.485	6.308	5.708	5.345	5.104	4.932
2.90	10.220	7.227	6.123	5.557	5.214	4.985	4.821
3.00	9.780	6.992	5.953	5.417	5.092	4.874	4.718
3.20	9.020	6.577	5.649	5.166	4.872	4.674	4.531
3.40	8.389	6.222	5.386	4.948	4.679	4.497	4.367
3.60	7.856	5.915	5.156	4.755	4.508	4.340	4.220
3.80	7.400	5.648	4.953	4.584	4.355	4.200	4.088
4.00	7.006	5.412	4.773	4.431	4.218	4.074	3.969
4.20	6.662	5.203	4.611	4.293	4.095	3.960	3.862
4.40	6.360	5.015	4.465	4.168	3.982	3.856	3.763
4.60	6.091	4.847	4.333	4.054	3.880	3.760	3.673
4.80	5.852	4.694	4.213	3.950	3.786	3.673	3.590
5.00	5.636	4.556	4.103	3.855	3.699	3.592	3.514
5.20	5.442	4.429	4.001	3.767	3.619	3.517	3.443
5.40	5.265	4.313	3.908	3.685	3.545	3.448	3.377
5.60	5.104	4.205	3.822	3.610	3.476	3.383	3.315
5.80	4.956	4.106	3.741	3.539	3.411	3.322	3.258

$(J - 1)h$ d.f. When both h and $J - 1$ are small, interpolation between integral values of h gives inaccurate results. A table of upper 5% - and upper 1% -quantiles of $F_{h,mh}$, where m is integer but h is not, is therefore included. Other uses of this table can be found, e.g., in [1].

The entries in Table 4 have been computed by successive approximations, using the Newton-Raphson method for solving the equation $P(x) - \epsilon = 0$, where $P(x) = P[F_{h,mh} > x]$. Values of $P(x)$ were obtained by the method of numerical integration described in [4a], and a finite difference approximation was used for dP/dx. Calculations were performed so that an error smaller than 10^{-4} could be guaranteed for the solution. When the fourth decimal was a 4, 5 or 6, calculations were repeated with higher accuracy to make sure the rounding to three decimal places was correct. In the upper-left corner of the table for $\epsilon = .01$, it was necessary to compute $P(x)$ with six decimal accuracy, and a rapidly convergent series for $P(x)$ was used instead of the method of numerical integration. For integral values of h, four decimal agreement was obtained in all cases with existing values of the percentiles. For $h = 1.2$, $m = 5$ and $h = 1.5$, $m = 2(2)8$, agreement exists also with the values of the percentiles obtainable from the tables of Vogler and Norton [7a].

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