CONDITIONS FOR WISHARTNESS AND INDEPENDENCE OF SECOND DEGREE POLYNOMIALS IN A NORMAL VECTOR

By C. G. Khatri

University of Baroda, India[1]

1. Introduction. We define a matrix, whose elements are second degree polynomials in a normal vector, as $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$, where $\mathbf{L}$ is a matrix with p rows and n columns (denoted as $\mathbf{L}$: $p \times n$), $\mathbf{A}:n \times n$ and $\mathbf{C}:p \times p$ are symmetric matrices, and the columns of $\mathbf{X}:p \times n$ are independent p-variate normals with means as columns of $\boldsymbol{\mu}:p \times n$ and covariance matrix $\mathbf{V}:p \times p$. In this paper, we establish the necessary and sufficient conditions for Wishartness and independence of such matrices. The results for $\mathbf{C} = \mathbf{0}$, $\mathbf{L} = \mathbf{0}$ have been established in [1, 3] and for $p = 1$ by R. G. Laha [4].

2. Certain lemmas.

Lemma 1. *Let* $\mathbf{A}:n \times n$, $\mathbf{B}:n \times n$ *be symmetric matrices, and suppose that* $\mathbf{L}:p \times n$ *and* $\mathbf{M}:p \times n$ *are matrices such that* t = *rank of* $(\mathbf{A}\ \mathbf{L}')$, u = *rank of* $(\mathbf{B}\ \mathbf{M}')$, $\mathbf{AB} = \mathbf{0}$, $\mathbf{LB} = \mathbf{MA} = \mathbf{0}$ *and* $\mathbf{LM}' = \mathbf{0}$. *Then, there exists a semi-orthogonal matrix* $\mathbf{Q}:n \times (t+u)$, $(t + u \leqq n)$, *such that* $\mathbf{L} = (\mathbf{T}\ \mathbf{0})\mathbf{Q}'$, $\mathbf{M} = (\mathbf{0}\ \mathbf{U})\mathbf{Q}'$,

$$\mathbf{A} = \mathbf{Q}\begin{pmatrix}\mathbf{E} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{Q}' \quad \textit{and} \quad \mathbf{B} = \mathbf{Q}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{pmatrix}\mathbf{Q}',$$

where $\mathbf{E}:t \times t$, $\mathbf{F}:u \times u$ *are symmetric matrices,* $\mathbf{T}:p \times t$, $\mathbf{U}:p \times u$ *and the form of the null matrix* $\mathbf{0}$ *is understood by its context.*

Proof. Using the result (A.3.11) of [5], we can write

$$(2.1) \qquad (\mathbf{A}\ \mathbf{L}') = \mathbf{Q}_1\mathbf{T}_1 \quad \text{and} \quad (\mathbf{B}\ \mathbf{M}') = \mathbf{Q}_2\mathbf{T}_2,$$

where $\mathbf{Q}_1:n \times t(t < n)$, $\mathbf{Q}_2:n \times u(u < n)$ are semi-orthogonal matrices, $\mathbf{T}_1 = (\mathbf{T}_{11}\ \mathbf{T}_{12})$ and $\mathbf{T}_2 = (\mathbf{T}_{21}\ \mathbf{T}_{22})$ are of ranks t and u respectively, $\mathbf{T}_{11}:t \times n$, $\mathbf{T}_{12}:t \times p$, $\mathbf{T}_{21}:u \times n$ and $\mathbf{T}_{22}:u \times p$. Now by the given conditions, we have $\mathbf{T}_1'\mathbf{Q}_1'\mathbf{Q}_2\mathbf{T}_2 = \mathbf{0}$ and so

$$(2.2) \qquad \mathbf{Q}_1'\mathbf{Q}_2 = \mathbf{0}.$$

Hence $\mathbf{Q} = (\mathbf{Q}_1\ \mathbf{Q}_2)$ is a semi-orthogonal matrix, and we can find $\mathbf{Q}_3:n \times (n - t - u)$ such that $(\mathbf{Q}\ \mathbf{Q}_3)$ is an orthogonal matrix [5, (A.1.7)]. Using these results, we have from (2.1),

$$(2.3) \qquad (\mathbf{Q}_2\ \mathbf{Q}_3)'(\mathbf{A}\ \mathbf{L}') = \mathbf{0} \quad \text{and} \quad (\mathbf{Q}_1\ \mathbf{Q}_3)'(\mathbf{B}\ \mathbf{M}') = \mathbf{0}.$$

Moreover, from (2.1) we can write $\mathbf{L} = (\mathbf{T}_{12}'\ \mathbf{0})\mathbf{Q}'$, $\mathbf{M} = (\mathbf{0}\ \mathbf{T}_{22}')\mathbf{Q}'$ and $\mathbf{Q}_1'\mathbf{AQ}_1 = \mathbf{T}_{11}\mathbf{Q}_1 = \mathbf{E}$ (say), $\mathbf{Q}_2'\mathbf{BQ}_2 = \mathbf{T}_{21}\mathbf{Q}_2 = \mathbf{F}$ (say) as symmetric matrices. With the help of (2.3), we can write $\mathbf{A}$ and $\mathbf{B}$ as mentioned in Lemma 1.

Lemma 2. *If the columns of* $\mathbf{X}$: $p \times n$ *are independent p-variate normals with*

Received April 5, 1960; revised December 8, 1961.

[1]Present address: School of Social Sciences, Grujarat University, Ahmedabad 9, India.

means as columns of $\boldsymbol{\mu}: p \times n$ *and covariance matrix* $\mathbf{V}: p \times p$, *then the cumulant generating function of* $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ *is*

$$\sum_{s=1}^{\infty} 2^{s-1}s^{-1} \operatorname{tr} \mathbf{A}^s \operatorname{tr} (\mathbf{V}\boldsymbol{\theta})^s + \operatorname{tr} \boldsymbol{\theta}(\mathbf{C} + \mathbf{L}\boldsymbol{\mu}') + \tfrac{1}{2} \operatorname{tr} \boldsymbol{\theta}\mathbf{V}\boldsymbol{\theta}\mathbf{LL}' +$$

$$\sum_{s=0}^{\infty} 2^s \operatorname{tr} (\boldsymbol{\mu} + \mathbf{V}\boldsymbol{\theta}\mathbf{L})\mathbf{A}^{s+1}(\boldsymbol{\mu} + \mathbf{V}\boldsymbol{\theta}\mathbf{L})'\boldsymbol{\theta}(\mathbf{V}\boldsymbol{\theta})^s,$$

where $\boldsymbol{\theta}: p \times p$ *is a symmetric matrix and* $\mathbf{P}^0 = \mathbf{I}$ *for any matrix* $\mathbf{P}$.

PROOF. By the definition of the moment generating function, we have

$$M(\boldsymbol{\theta}) = g \int \cdots \int \exp [\operatorname{tr} \{\boldsymbol{\theta}(\mathbf{XAX}' + \mathbf{LX}') - \tfrac{1}{2} \mathbf{V}^{-1}(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})'\}] \, d\mathbf{X},$$

where $\boldsymbol{\theta}: p \times p$ is a symmetric matrix, $d\mathbf{X} = \prod dx_{ij}$ and

$$g = (2\pi)^{-\frac{1}{2}pn}|\mathbf{V}|^{-\frac{1}{2}n} \exp(\operatorname{tr} \boldsymbol{\theta}\mathbf{C}).$$

(Note: Use of the result $\operatorname{tr} \mathbf{PQ} = \operatorname{tr} \mathbf{QP}$ is made above and it will often be used in the sequel.) The above expression can be rewritten as

$$\begin{aligned} M(\boldsymbol{\theta}) = h \int \cdots \int \exp [\operatorname{tr} \{\boldsymbol{\theta}\mathbf{XAX}' - \tfrac{1}{2}\mathbf{V}^{-1}(\mathbf{X} - \boldsymbol{\mu} - \mathbf{V}\boldsymbol{\theta}\mathbf{L}) \\ \cdot(\mathbf{X} - \boldsymbol{\mu} - \mathbf{V}\boldsymbol{\theta}\mathbf{L})'\}] \, d\mathbf{X}, \end{aligned} \tag{2.4}$$

where $h = g \exp(\operatorname{tr} \boldsymbol{\theta}\mathbf{L}\boldsymbol{\mu}' + \frac{1}{2} \operatorname{tr} \boldsymbol{\theta}\mathbf{V}\boldsymbol{\theta}\mathbf{LL}')$.

Using the method given in [3], we can show that the cumulant generating function reduces to the form mentioned in Lemma 2.

COROLLARY 1. *If the distribution of* $\mathbf{X}$ *is*

$$(2\pi)^{-\frac{1}{2}pn}|\mathbf{V}|^{-\frac{1}{2}n}|\mathbf{W}|^{-\frac{1}{2}p} \exp [- \tfrac{1}{2} \operatorname{tr} \mathbf{V}^{-1}(\mathbf{X} - \boldsymbol{\mu})\mathbf{W}^{-1}(\mathbf{X} - \boldsymbol{\mu})'] \, d\mathbf{X}, \tag{2.5}$$

where $\mathbf{V}: p \times p$, $\mathbf{W}: n \times n$ *are symmetric positive definite and* $\boldsymbol{\mu}: p \times n$, *then the cumulant generating function of* $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ *is*

$$\sum_{s=1}^{\infty} 2^{s-1}s^{-1} \operatorname{tr}(\mathbf{WA})^s \operatorname{tr}(\mathbf{V}\boldsymbol{\theta})^s + \operatorname{tr} \boldsymbol{\theta}(\mathbf{C} + \mathbf{L}\boldsymbol{\mu}') + \tfrac{1}{2} \operatorname{tr} \boldsymbol{\theta}\mathbf{V}\boldsymbol{\theta}\mathbf{LWL}' +$$

$$\sum_{s=0}^{\infty} 2^s \operatorname{tr}(\boldsymbol{\mu} + \mathbf{V}\boldsymbol{\theta}\mathbf{LW})\mathbf{A}(\mathbf{WA})^s(\boldsymbol{\mu} + \mathbf{V}\boldsymbol{\theta}\mathbf{LW})'\boldsymbol{\theta}(\mathbf{V}\boldsymbol{\theta})^s,$$

where $\boldsymbol{\theta}: p \times p$ *is a symmetric matrix and* $\mathbf{P}^0 = \mathbf{I}$ *for any matrix* $\mathbf{P}$.

PROOF. This follows from Lemma 2 by noting that $\mathbf{W} = \tilde{\mathbf{T}}' \tilde{\mathbf{T}}$ [5 (A.3.9)] where $\tilde{\mathbf{T}}: n \times n$ is a non-singular triangular matrix and the columns of $\mathbf{Y} = \mathbf{X}\tilde{\mathbf{T}}^{-1}$ are independent p-variate normals with means as columns of $\boldsymbol{\mu}\tilde{\mathbf{T}}^{-1}$ and covariance matrix $\mathbf{V}$.

COROLLARY 2. *If in Corollary* 1, $\mathbf{AWA} = \mathbf{A}$ *and the rank of* $\mathbf{A}$ *is* r, *then the moment generating function of* $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ *is*

$$\begin{aligned} M(\boldsymbol{\theta}) = |\mathbf{I} - 2\mathbf{V}\boldsymbol{\theta}|^{-\frac{1}{2}r} \exp [\operatorname{tr} \boldsymbol{\theta}(\mathbf{C} + \mathbf{L}\boldsymbol{\mu}') + \tfrac{1}{2} \operatorname{tr} \boldsymbol{\theta}\mathbf{V}\boldsymbol{\theta}\mathbf{LWL}'] \\ \cdot\exp [\operatorname{tr} \boldsymbol{\theta}(\mathbf{I} - 2\mathbf{V}\boldsymbol{\theta})^{-1}(\boldsymbol{\mu} + \mathbf{V}\boldsymbol{\theta}\mathbf{LW})\mathbf{A}(\boldsymbol{\mu} + \mathbf{V}\boldsymbol{\theta}\mathbf{LW})']. \end{aligned}$$

LEMMA 3. *If* $\mathbf{x}:1 \times n$ *is normal with mean* $\mathbf{m}:1 \times n$ *and covariance matrix* $\mathbf{I}:n \times n$, *then a set of necessary and sufficient conditions for* $\mathbf{xAx}' + \mathbf{lx}' + c$ *to be distributed as noncentral Chi-Square is* $\mathbf{A}^2 = \mathbf{A}$, $\mathbf{l} = \mathbf{lA}$ *and* $c = \frac{1}{4}\mathbf{lAl}'$, *where* $\mathbf{l}:1 \times n$ *and* c *is a constant.*

PROOF. From Lemma 2 ($p = 1$), we can write the moment generating function of $v = \mathbf{xAx}' + \mathbf{lx}' + c$ as

$$(2.6) \quad |\mathbf{I} - 2\varphi\mathbf{A}|^{-\frac{1}{2}} \exp [\varphi c - \tfrac{1}{2}\mathbf{mm}' + \tfrac{1}{2}(\mathbf{m} + \varphi\mathbf{l})(\mathbf{I} - 2\varphi\mathbf{A})^{-1}(\mathbf{m} + \varphi\mathbf{l})'].$$

Let us suppose that v is distributed as non-central Chi-Square with r degrees of freedom and η^2 as the noncentral parameter. Then its moment generating function is

$$(2.7) \quad (1 - 2\varphi)^{-\frac{1}{2}r} \exp [\varphi\eta^2(1 - 2\varphi)^{-1}].$$

For the necessity of the conditions, we must have (2.6) $=$ (2.7) for any φ. This can be rewritten as

$$(2.8) \quad f_1(\varphi) = \exp [f_2(\varphi)] \qquad \text{for any } \varphi,$$

where $f_1(\varphi) = |\mathbf{I} - 2\varphi\mathbf{A}|\,(1 - 2\varphi)^{-r}$ and $f_2(\varphi) = 2\varphi c - \mathbf{mm}' + (\mathbf{m} + \varphi\mathbf{l})(\mathbf{I} - 2\varphi\mathbf{A})^{-1}(\mathbf{m} + \varphi\mathbf{l})' - 2\varphi\eta^2(1 - 2\varphi)^{-1}$. It is easy to see that $f_1(\varphi)$ and $f_2(\varphi)$ are the ratios of two finite order polynomials in φ; and so, (2.8) is true if and only if $f_1(\varphi) = 1$ and $f_2(\varphi) = 0$ for any φ (cf., [4]). That is,

$$(2.9) \quad |\mathbf{I} - 2\varphi\mathbf{A}| = (1 - 2\varphi)^r,$$

and

$$(2.10) \quad 2(1 - 2\varphi)\varphi c = (1 - 2\varphi)[\mathbf{mm}' - (\mathbf{m} + \varphi\mathbf{l})(\mathbf{I} - 2\varphi\mathbf{A})^{-1}(\mathbf{m} + \varphi\mathbf{l})'] + 2\varphi\eta^2.$$

Now, it is easy to see that from (2.9), we have $r = \operatorname{rank} \mathbf{A}$ and the nonzero latent roots of $\mathbf{A}$ as unity. That is,

$$(2.11) \quad \mathbf{A}^2 = \mathbf{A} \quad \text{and} \quad r = \operatorname{rank} \mathbf{A}.$$

Since $\mathbf{A}$ is a symmetric matrix and satisfies (2.11), we can write $\mathbf{A} = \mathbf{Q}_1\mathbf{Q}_1'$ where $\mathbf{Q}_1:n \times r$ is a semi-orthogonal matrix. Let $(\mathbf{Q}_1 \; \mathbf{Q}_2)$ be an orthogonal matrix. Using this in (2.10) and equating the coefficients of φ's, we have $(\mathbf{lQ}_2)(\mathbf{lQ}_2)' = 0$, $c = \frac{1}{4}\mathbf{ll}' - (\mathbf{lQ}_2)(\mathbf{mQ}_2)'$ and $\eta^2 = (\mathbf{mQ}_1)(\mathbf{mQ}_1)' + (\mathbf{lQ}_1)(\mathbf{mQ}_1)' + c$. Hence,

$$(2.12) \quad \mathbf{l} = \mathbf{lA},\; c = \tfrac{1}{4}\mathbf{ll}' \quad \text{and} \quad \eta^2 = (\mathbf{m} + \tfrac{1}{2}\mathbf{l})\mathbf{A}(\mathbf{m} + \tfrac{1}{2}\mathbf{l})'.$$

The results of (2.11) and (2.12) prove the necessity conditions, while their sufficiency follows immediately from (2.6).

Note: v will be distributed as central Chi-Square if and only if $\mathbf{A}^2 = \mathbf{A}$ $\mathbf{l} = -2\mathbf{mA}$ and $c = \mathbf{mAm}'$.

3. Theorems on forms of the type: $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$.

THEOREM I. *If the columns of* $\mathbf{X}$ *are independent p-variate normals with means as columns of* $\boldsymbol{\mu}$ *and covariance matrix* $\mathbf{V}$, *then a set of necessary and sufficient condi-*

tions for the form $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ *to be distributed as noncentral Wishart or pseudo-Wishart* [1] *is* $\mathbf{A}^2 = \mathbf{A}$, $\mathbf{L} = \mathbf{LA}$ *and* $\mathbf{C} = \frac{1}{4}\mathbf{LAL}'$.

PROOF. The necessary and sufficient conditions for the form $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ to be distributed as Wishart or pseudo-Wishart is that it must be of form $(\mathbf{Z} + \nu)(\mathbf{Z} + \nu)'$ where the columns of $\mathbf{Z}$ are independent p-variate normals [1]. Hence, if the form is non-central Wishart or pseudo-Wishart, then, in particular, its ith diagonal element will be distributed as non-central Chi-Square ($i = 1, 2, \cdots, p$), and so, applying Lemma 3, we have $\mathbf{A}^2 = \mathbf{A}$, $\mathbf{L} = \mathbf{LA}$ and the diagonal elements of $(\mathbf{C} - \frac{1}{4}\mathbf{LAL}')$ are zero. Under these conditions the form reduces to $(\mathbf{Y} + \frac{1}{2}\mathbf{L}_1)(\mathbf{Y} + \frac{1}{2}\mathbf{L}_1)' + \mathbf{C} - \frac{1}{4}\mathbf{LAL}'$, where $\mathbf{A} = \mathbf{QQ}'$, $\mathbf{Q}:n \times r$ is semi-orthogonal, $\mathbf{L}_1 = \mathbf{LQ}$ and the columns of $\mathbf{Y} = \mathbf{XQ}$ are independent p-variate normals with means as columns of $\boldsymbol{\mu}\mathbf{Q}$ and covariance matrix $\mathbf{V}$. Since the given form is distributed as non-central Wishart or pseudo-Wishart, the constant term in the above expression must vanish. That is, $\mathbf{C} = \frac{1}{4}\mathbf{LAL}'$. Thus the necessity of the conditions is established, while their sufficiency is immediate.

Note: The distribution of the given form will be central Wishart or pseudo-Wishart if and only if $\mathbf{A}^2 = \mathbf{A}$, $\mathbf{L} = -2\boldsymbol{\mu}\mathbf{A}$, and $\mathbf{C} = \boldsymbol{\mu}\mathbf{A}\boldsymbol{\mu}'$.

COROLLARY 3. *If the distribution of* $\mathbf{X}$ *is given by* (2.5), *then a set of necessary and sufficient conditions for the form* $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ *to be distributed as non-central Wishart or pseudo-Wishart is* $\mathbf{AWA} = \mathbf{A}$, $\mathbf{LWA} = \mathbf{L}$ *and* $\frac{1}{4}\mathbf{LWL}' = \mathbf{C}$.

THEOREM II. *If the columns of* $\mathbf{X}$ *are independent p-variate normals with means as columns of* $\boldsymbol{\mu}$ *and covariance matrix* $\mathbf{V}$, *then a set of necessary and sufficient conditions for the two forms* $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ *and* $\mathbf{XBX}' + \frac{1}{2}(\mathbf{MX}' + \mathbf{XM}') + \mathbf{D}$ *to be distributed independently is* $\mathbf{AB} = \mathbf{0}$, $\mathbf{LB} = \mathbf{MA} = \mathbf{0}$ *and* $\mathbf{LM}' = \mathbf{0}$.

PROOF.

(a) Let the two forms be independently distributed. Then, in particular, ith diagonal elements of the two forms must be independently distributed ($i = 1, 2, \cdots, p$), and hence, by using R. G. Laha's result [4], we must have

$$\mathbf{AB} = \mathbf{0},\ \mathbf{LB} = \mathbf{MA} = \mathbf{0} \tag{3.1}$$

and the diagonal elements of $\mathbf{LM}'$ are zero.

If the two forms are independently distributed, we must have conditions given in (3.1) and, possibly, certain further conditions. Now, since $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices and $\mathbf{AB} = \mathbf{0}$, we can write $\mathbf{A} = \mathbf{Q}_1\mathbf{D}_a\mathbf{Q}_1'$ and $\mathbf{B} = \mathbf{Q}_2\mathbf{D}_b\mathbf{Q}_2'$ where $\mathbf{Q}_1:n \times r$, $\mathbf{Q}_2:n \times s$ and $(\mathbf{Q}_1\ \mathbf{Q}_2)$ are semi-orthogonal matrices, $r = \text{rank } \mathbf{A}$, $s = \text{rank } \mathbf{B}$ and $\mathbf{D}_a:r \times r$, $\mathbf{D}_b:s \times s$ are non-singular diagonal matrices with diagonal elements as the nonzero latent roots of $\mathbf{A}$ and $\mathbf{B}$ respectively. Let $\mathbf{Q} = (\mathbf{Q}_1\ \mathbf{Q}_2\ \mathbf{Q}_3)$ be an orthogonal matrix. Then, by using the transformation $\mathbf{Y}_1 = \mathbf{XQ}_1$, $\mathbf{Y}_2 = \mathbf{XQ}_2$ and $\mathbf{Y}_3 = \mathbf{XQ}_3$, and the conditions $\mathbf{LB} = \mathbf{MA} = \mathbf{0}$, i.e., $\mathbf{LQ}_2 = \mathbf{0}$ and $\mathbf{MQ}_1 = \mathbf{0}$, we can write the given two forms as

(i) $\mathbf{Y}_1\mathbf{D}_a\mathbf{Y}_1' + \frac{1}{2}(\mathbf{LQ}_1\mathbf{Y}_1' + \mathbf{Y}_1\mathbf{Q}_1'\mathbf{L}') + \mathbf{C} + \frac{1}{2}(\mathbf{LQ}_3\mathbf{Y}_3' + \mathbf{Y}_3\mathbf{Q}_3'\mathbf{L}')$,

(ii) $\mathbf{Y}_2\mathbf{D}_b\mathbf{Y}_2' + \frac{1}{2}(\mathbf{MQ}_2\mathbf{Y}_2' + \mathbf{Y}_2\mathbf{Q}_2'\mathbf{M}') + \mathbf{D} + \frac{1}{2}(\mathbf{MQ}_3\mathbf{Y}_3' + \mathbf{Y}_3\mathbf{Q}_3'\mathbf{M}')$, where the columns of $\mathbf{Y} = (\mathbf{Y}_1\ \mathbf{Y}_2\ \mathbf{Y}_3)$ are independent p-variate normals with means as columns of $\boldsymbol{\mu}\mathbf{Q}$ and covariance matrix $\mathbf{V}$. Now, by using Lemma 2 and the definition of the joint moment generating function $M(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, it can be seen that the joint cumulant generating function, $\log M(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, can be written as

$$\log M(\boldsymbol{\theta}_1, \mathbf{0}) + \log M(\mathbf{0}, \boldsymbol{\theta}_2) + 2 \operatorname{tr} \boldsymbol{\theta}_1\mathbf{V}\boldsymbol{\theta}_2\mathbf{L}\mathbf{M}',$$

where $\boldsymbol{\theta}_1 : p \times p$ and $\boldsymbol{\theta}_2 : p \times p$ are any two symmetric matrices. Now, since the two forms are independently distributed, we must have $\log M(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \log M(\boldsymbol{\theta}_1, \mathbf{0}) + \log M(\mathbf{0}, \boldsymbol{\theta}_2)$; i.e., we must have $\operatorname{tr} \boldsymbol{\theta}_1\mathbf{V}\boldsymbol{\theta}_2\mathbf{L}\mathbf{M}' = 0$ for any symmetric matrices $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. This gives us $\mathbf{L}\mathbf{M}' = \mathbf{0}$. Hence, we have the conditions

$$\mathbf{AB} = \mathbf{0}, \mathbf{LB} = \mathbf{0} = \mathbf{MA} \quad \text{and} \quad \mathbf{LM}' = \mathbf{0}$$

if the two forms are independently distributed.

(b) The converse is immediate by using Lemma 1.

The following are immediate consequences of Theorem II.

COROLLARY 4. *If the distribution of* $\mathbf{X}$ *is given by* (2.5), *then a set of necessary and sufficient conditions for the two forms* $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ *and* $\mathbf{XBX}' + \frac{1}{2}(\mathbf{MX}' + \mathbf{XM}') + \mathbf{D}$ *to be independently distributed is* $\mathbf{AWB} = \mathbf{0}$, $\mathbf{LWB} = \mathbf{MWA} = \mathbf{0}$ *and* $\mathbf{LWM}' = \mathbf{0}$.

COROLLARY 5. *If the distribution of* $\mathbf{X}$ *is given by* (2.5), *and the two forms* $\mathbf{P}(\mathbf{X}) = \mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ *and* $\mathbf{Q}(\mathbf{X}) = \mathbf{XBX}' + \frac{1}{2}(\mathbf{MX}' + \mathbf{XM}') + \mathbf{D}$ *are independently distributed, then there exists a non-singular matrix* $\mathbf{R}$, *giving a non-singular transformation* $\mathbf{Y} = \mathbf{XR}$ *such that* $\mathbf{P}(\mathbf{X}) = \mathbf{P}_1(\mathbf{Y}_1)$ *and* $\mathbf{Q}(\mathbf{X}) = \mathbf{Q}_1(\mathbf{Y}_2)$, *where* $\mathbf{R} = (\mathbf{R}_1\ \mathbf{R}_2\ \mathbf{R}_3)$, $\mathbf{R}_1 : n \times t$, $\mathbf{R}_2 : n \times u$, $\mathbf{R}_3 : n \times (n - t - u)$, t $=$ *rank of* $(\mathbf{A}\ \mathbf{L}')$, $u =$ *rank of* $(\mathbf{B}\ \mathbf{M}')$, $\mathbf{Y}_1 = \mathbf{XR}_1$, $\mathbf{Y}_2 = \mathbf{XR}_2$, *and* $\mathbf{R}'\mathbf{WR}$ $= \operatorname{diag.}(\mathbf{R}_1'\mathbf{WR}_1, \mathbf{R}_2'\mathbf{WR}_2, \mathbf{R}_3'\mathbf{WR}_3)$.

This follows from Corollary 4 and Lemma 1.

COROLLARY 6. *If the distribution of* $\mathbf{X}$ *is given by* (2.5), *then a set of necessary and sufficient conditions for the independence of* $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ *and* $\mathbf{MX}' + \mathbf{XM}'$ (*or* $\mathbf{MX}'$) *is* $\mathbf{AWM}' = \mathbf{0}$ *and* $\mathbf{LWM}' = \mathbf{0}$.

COROLLARY 7. *If the distribution of* $\mathbf{X}$ *is given by* (2.5), *then a necessary and sufficient condition for the independence of* $\mathbf{LX}' + \mathbf{XL}'$ (*or* $\mathbf{LX}'$) *and* $\mathbf{MX}' + \mathbf{XM}'$ (*or* $\mathbf{MX}'$) *is* $\mathbf{LWM}' = \mathbf{0}$.

COROLLARY 8. *If the distribution of* $\mathbf{X}$ *is given by* (2.5), *then a set of necessary and sufficient conditions for the mutual independence of* $(\mathbf{X} + \mathbf{L}_i)\mathbf{A}_i(\mathbf{X} + \mathbf{L}_i)'$, $i = 1, 2, \cdots, m$, *is* $\mathbf{A}_i\mathbf{WA}_j = 0$ *for* $i \neq j = 1, 2, \cdots, m$.

THEOREM III. *Let the distribution of* $\mathbf{X}$ *be given by* (2.5), *and*

$$\sum_{i=1}^{m} (\mathbf{X} + \mathbf{L}_i)\mathbf{A}_i(\mathbf{X} + \mathbf{L}_i)' = \mathbf{XAX}' + (\mathbf{LX}' + \mathbf{XL}') + \mathbf{C},$$

where rank of $\mathbf{A} = r$ *and rank of* $\mathbf{A}_i = r_i$, $i = 1, 2, \cdots, m$. *Consider the following conditions.*

a_1: $(\mathbf{X} + \mathbf{L}_i)\mathbf{A}_i(\mathbf{X} + \mathbf{L}_i)'$, $i = 1, 2, \cdots, m$, *are distributed as non-central Wisharts or pseudo-Wisharts*;

a_2: $(\mathbf{X} + \mathbf{L}_i)\mathbf{A}_i(\mathbf{X} + \mathbf{L}_i)'$ *and* $(\mathbf{X} + \mathbf{L}_j)\mathbf{A}_j(\mathbf{X} + \mathbf{L}_j)'$, *for all* $i \neq j$, *are independently distributed;*

a_3: $\mathbf{XAX}' + (\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$ *is distributed as non-central Wishart or pseudo-Wishart,*

$$c_1: \mathbf{A}_i\mathbf{W}\mathbf{A}_i = \mathbf{A}_i \qquad i = 1, 2, \cdots, m,$$

$$c_2: \mathbf{A}_i\mathbf{W}\mathbf{A}_j = \mathbf{0} \qquad i \neq j = 1, 2, \cdots, m,$$

$$c_3: \mathbf{AWA} = \mathbf{A}, \qquad \textit{and}$$

$$c_4: \sum_{i=1}^{m} r_i = r.$$

Then, (a) *any two of the three conditions* a_1, a_2, a_3; *or* (b) *any two of the three conditions* c_1, c_2, c_3; *or* (c) *any one set of* a_i *and* c_j, $i \neq j = 1, 2, 3$; *or* (d) c_4 *and* a_3; *or* (e) c_4 *and* c_3 *are necessary and sufficient for all the remaining conditions.*

PROOF. If $\mathbf{XAX}' + (\mathbf{LX}' + \mathbf{XL}') + \mathbf{C} = \sum_{i=1}^{m} (\mathbf{X} + \mathbf{L}_i)\mathbf{A}_i(\mathbf{X} + \mathbf{L}_i)'$, then we must have

$$\mathbf{A} = \sum_{i=1}^{m} \mathbf{A}_i, \qquad \mathbf{L} = \sum_{i=1}^{m} \mathbf{L}_i\mathbf{A}_i \quad \text{and} \quad \mathbf{C} = \sum_{i=1}^{m} \mathbf{L}_i\mathbf{A}_i\mathbf{L}_i'. \tag{3.2}$$

Moreover, from Graybill and Marsaglia's result [2] or [3, Lemma 5] we have the results:

(3.3) any two of the three conditions c_1, c_2, c_3 imply c_1, c_2, c_3 and c_4,

and

(3.4) c_4 and c_3 imply c_1 and c_2.

Further, by Corollary (3) and Corollary (8), we may note that $a_1 \Leftrightarrow c_1$, $a_2 \Leftrightarrow c_2$ and $a_3 \Leftrightarrow c_3$, $\mathbf{LWA} = \mathbf{L}$ and $\mathbf{C} = \mathbf{LWL}'$. Also, with the help of c_1 and c_2, it is easy to show c_3, $\mathbf{LWA} = \mathbf{L}$ and $\mathbf{C} = \mathbf{LWL}'$, i.e., a_3. By (3.3), we get c_1, c_2, c_3 and c_4 if any two of c_1, c_2 and c_3 are given, and thus, we get the results for (a), (b) and (c). By (3.4), we get c_1, c_2, c_3 and c_4 if c_3 and c_4 are given, and so, we get the results for (d) and (e). Thus, Theorem III is proved.

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