$(-6, -3, 10, 11)$. Table II provides the necessary data. $EM_4^+ - EM_3^+ = 3 = S_4/4$. The computation for the right side of (4.2) gives $(1/4)\,(21/1 + 33/2 + 36/3 + 48/4) \neq 63/4 = EM_4^+$.

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DETERMINING BOUNDS ON EXPECTED VALUES OF CERTAIN FUNCTIONS

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1. Introduction and summary. Let $\mathfrak{F}$ be the collection of cumulative distribution functions on $(-\infty, \infty)$ and $\mathfrak{F}_{[a,b]}$ that subset of $\mathfrak{F}$ all of whose elements have $F(a-0) = 0$ and $F(b) = 1$.

Let $\mathfrak{F}^{(\mu_1,\mu_2,\cdots,\mu_k)}$ $(\mathfrak{F}_{[a,b]}^{(\mu_1,\mu_2,\cdots,\mu_k)})$ be the class of cumulative distribution functions on $(-\infty, \infty)$ $([a, b])$ whose first k moments are $\mu_1, \mu_2, \cdots, \mu_k$ respectively. We will suppose throughout that $\mu_1, \mu_2, \cdots, \mu_k$ is a legitimate moment sequence, i.e., that there exists a cumulative distribution function $F(x) \, \varepsilon \, \mathfrak{F}(\mathfrak{F}_{[a,b]})$ whose first k moments are $\mu_1, \mu_2, \cdots, \mu_k$.

Let $g(x)$ be a continuous and bounded function on $[a, b]$. Then, we wish to determine $F^*(x) \, \varepsilon \, \mathfrak{F}_{[a,b]}^{(\mu_1,\mu_2,\cdots,\mu_k)}$ with

$$\int_a^b g(x)\, dF^*(x) = \min_{F \varepsilon \mathfrak{F}_{[a,b]}^{(\mu_1,\mu_2,\cdots,\mu_k)}} (\max) \int_a^b g(x)\, dF(x). \tag{1}$$

Any $F^*(x)$ satisfying (1) will be called an extremal distribution with respect to $g(x)$.

Let $\mathcal{G}_{[a,b]}^{(k)}$ be the set of continuous, bounded, and monotonic functions on $[a, b]$, whose first k derivatives exist and are monotonic in (a, b). In addition, we further require that $\mathcal{G}_{[a,b]}^{(k)}$ contain only functions not linearly dependent on the monomials $1, x, x^2, \cdots, x^k$.

This paper characterizes the extremal distributions for $g(x) \, \varepsilon \, \mathcal{G}_{[a,b]}^{(k)}$. The results are extended to $\mathfrak{F}_{[0,\infty)}^{(\mu_1,\mu_2,\cdots,\mu_k)}$ and $\mathfrak{F}^{(\mu_1,\mu_2,\cdots,\mu_k)}$, in that we investigate

Received May 16, 1961; revised March 19, 1962.

determining

$$\inf_{F \varepsilon \mathfrak{F}_{[0,\infty)}^{(\mu_1,\mu_2,\cdots,\mu_k)}} (\sup) \int_0^\infty g(x)\, dF(x) \text{ and } \inf_{F \varepsilon \mathfrak{F}^{(\mu_1,\mu_2,\cdots,\mu_k)}} (\sup) \int_{-\infty}^\infty g(x)\, dF(x).$$

These results are then applied to the computation of bounds on the moment generating function, knowing the first k moments, in some specific cases. The methodology is largely a straightforward extension of results in an earlier paper by the author [1].

2. Moment inequalities. In order to apply the results of the author [1] and Wald [2], it is convenient to use the natural isomorphism obtained by replacing $\mathfrak{F}_{[a,b]}^{(\mu_1,\mu_2,\cdots,\mu_k)}$ by $\mathfrak{F}_{[0,b^*]}^{(\mu_1^*,\mu_2^*,\cdots,\mu_k^*)}$, $g(x)$ by $g(x - a)$, and $F(x)$ by $F(x - a)$, where $b^* = b - a$ and

$$\mu_i^* = \sum_{j=0}^{i} (-1)^j \binom{i}{j} a^j \mu_{i-j}, \qquad i = 1, 2, \cdots, k, \tag{2}$$

and μ_0 is taken to be unity.

Then, if $(\mu_1^*, \mu_2^*, \cdots, \mu_k^*)$ is non-degenerate (see [1] or [2]), there are exactly two extremal distributions with respect to $g(x)$, for $g(x) \varepsilon \mathcal{G}_{[a,b]}^{(k)}$, which are elements of $\mathfrak{F}_{[a,b]}^{(\mu_1,\mu_2,\cdots,\mu_k)}$. This statement is an immediate consequence of Theorem 5 in [1]. By a straightforward extension of Theorem 7 in [1], the two extremal distributions $F_1^*(x)$ and $F_2^*(x)$ are characterized by

(a) if $k = 2q + 1$, q a non-negative integer, then $F_1^*(x)$ has a saltus at each of $(k + 1)/2$ points in (a, b); $F_2^*(x)$ has a saltus at each of $(k - 1)/2$ points in (a, b) and at both a and b;

(b) if $k = 2q$, $q > 0$, then $F_1^*(x)$ has a saltus at each of $k/2$ points in (a, b) and at a; $F_2^*(x)$ has a saltus at each of $k/2$ points in (a, b) and at b.

One of these two cumulative distribution functions will be the maximizing distribution, the other the minimizing distribution. We note that whether $F_1^*(x)$ will be a maximizing or minimizing distribution depends on the particular choice of $g(x) \varepsilon \mathcal{G}_{[a,b]}^{(k)}$.

We now show how the above characterization can be extended to $\mathfrak{F}^{(\mu_1,\mu_2,\cdots,\mu_k)}$. The comparable result for $\mathfrak{F}_{[a,\infty)}^{(\mu_1,\mu_2,\cdots,\mu_k)}$ is a straightforward extension of Theorems 8 and 9 in [1].

THEOREM 1. *Let $\{a_n, b_n\}$ be any sequence with $a_n \to -\infty$ and $b_n \to \infty$, and let $F_{1n}^*(x)$ and $F_{2n}^*(x)$ be the two extremal distributions on $[a_n, b_n]$, then there exists a distribution $F_1^*(x)$ on $(-\infty, \infty)$ and for $k = 2q + 1$, $q \geqq 1$, $\{F_{2n}^*(x)\}$ converges in distribution to $F_1^*(x)$ and $\int_{-\infty}^{\infty} x^j\, dF_1^*(x) = \mu_j$, $j = 1, 2, \cdots, 2q - 1$; $k = 2q$, $q \geqq 1$, $\{F_{1n}^*(x)\}$ and $\{F_{2n}^*(x)\}$ converge in distribution to $F_1^*(x)$ and $\int_{-\infty}^{\infty} x^j\, dF_1^*(x) = \mu_j$, $j = 1, 2, \cdots, 2q - 1$.*

PROOF. $\mu_k < \infty$ implies $\alpha_k = E|X|^k < \infty$. Then if there is a saltus at a or b, we have $[F(b) - F(b - 0)]|b|^k \leqq \alpha_k$ and $[F(a) - F(a - 0)]|a|^k \leqq \alpha_k$. Hence $F(b) - F(b - 0) = O(b^{-k})$ and $[F(a) - F(a - 0)] = O(a^{-k})$, and for any

$\epsilon > 0$, there is an n sufficiently large with $F_{in}^*(b_n - 0) - F_{in}^*(a_n) > 1 - \epsilon$, for all $n > N$, $i = 1, 2$. In addition $\int x^j \, dF_{in}^*(x) = \mu_j$, $j = 1, 2, \cdots, k$. Let $\gamma_n = F_{in}^*(b_n - 0) - F_{in}^*(a_n)$ and

$$G_{in}(x) = \begin{cases} 0, & x \leqq a_n, \\ \gamma_n^{-1} F_{in}^*(x), & a_n < x < b_n, \\ 1, & x \geqq b_n. \end{cases}$$

Then, by employing Theorems 5 and 7 in [1] and Proposition 12 in [2], it can be shown that for n sufficiently large, $|G_{in}(x) - F_{in}^*(x)| < \epsilon$ and $\{G_{in}(x)\}$ converges in distribution to $F_1^*(x)$ computed for $2q - 1$ moment constraints, whenever $F_{in}^*(x)$ has a saltus at a_n or b_n or both.

Note that as $a \to -\infty$, $b \to \infty$, it is impossible for $g(x) \in \mathcal{G}_{(-\infty,\infty)}^{(k)}$ to remain bounded, however, in many cases of interest, $\sup(\inf)_{F \in \mathfrak{F}^{(\mu_1,\mu_2,\cdots,\mu_k)}} E\{g(X)\}$ exist. This is summarized in the following theorem.

THEOREM 2. *If $\mu_1, \mu_2, \cdots, \mu_k$ is non-degenerate and $k = 2q + 1$, $q > 0$, one extremum of $g(x)$ over $F \in \mathfrak{F}^{(\mu_1,\mu_2,\cdots,\mu_k)}$ is given by $\int_{-\infty}^{\infty} g(x) \, dF_1^*(x)$; if $g(x) = O(x^k)$ as $x \to \pm\infty$, the other extremum exists and is given by*

$$\lim_{a \to -\infty, b \to \infty} \int_a^b g(x) \, dF_2^*(x).$$

If $k = 2q$, and $g(x) = O(x^k)$ as $x \to -\infty$, an extremum exists and is given by $\lim_{a \to -\infty} \int_a^{\infty} g(x) \, dF_{1a}^*(x)$, $F_{1a}^*(x) \in \mathfrak{F}_{[a,\infty)}^{(\mu_1,\mu_2,\cdots,\mu_k)}$; and if $g(x) = O(x^k)$ as $x \to \infty$, the other extremum exists and is given by $\lim_{b \to \infty} \int_{-\infty}^{b} g(x) \, dF_{2b}^*(x)$, $F_{2b}^*(x) \in \mathfrak{F}_{(-\infty,b)}^{(\mu_1,\mu_2,\cdots,\mu_k)}$.

Using the natural isomorphism described earlier, and the methodology of Section 5 in [1], the extremal distributions with respect to $g(x) \in \mathcal{G}_{[a,b)}^{(k)}$, $\mathcal{G}_{[a,\infty)}^{(k)}$, and $\mathcal{G}_{(-\infty,\infty)}^{(k)}$ can be computed. The reader is referred to [1] for details.

3. Some examples of extrema of the moment generating function.

1. Let $\mu_1 = 0$, $\mu_2 = 1$, $\mu_3 = 0$, $\mu_4 = 3$, i.e., the first four moments of the standard normal distribution. Then $\inf_{F \in \mathfrak{F}^{(0,1,0,3)}} E\{e^{tX}\} = \cosh t$. The supremum does not exist. It is readily verified that $\cosh t \leqq \exp\{\frac{1}{2}t^2\}$ for all t.

2. Let $\mu_1 = 0$, $\mu_2 = \frac{1}{3}$, $\mu_3 = 0$; i.e., the first three moments of the rectangular distribution on $[-1, 1]$. Then

$$F_1^*(x) = \begin{cases} 0, & x < -\sqrt{3}/3, \\ \frac{1}{2}, & -\sqrt{3}/3 \leqq x < \sqrt{3}/3, \\ 1, & \sqrt{3}/3 \leqq x, \end{cases}$$

$$F_2^*(x) = \begin{cases} 0, & x < -1, \\ \frac{1}{6}, & -1 \leqq x < 0, \\ \frac{5}{6}, & 0 \leqq x < 1, \\ 1, & 1 \leqq x, \end{cases}$$

and

$$\sup_{F\in\mathfrak{F}^{(0,1/3,0)}_{[-1,1]}} E\{e^{tX}\} = \tfrac{2}{3} + (\cosh t)/3 \quad \text{and} \quad \inf_{F\in\mathfrak{F}^{(0,1/3,0)}_{[-1,1]}} E\{e^{tX}\} = (\cosh \sqrt{3}t)/3.$$

It is readily verified that $(\cosh \sqrt{3}\, t)/3 \leqq (\sinh t)/t \leqq \frac{2}{3} + (\cosh t)/3$, where $(\sinh t)/t$ is the moment generating function of the rectangular distribution on $[-1, 1]$.

3. Let $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 6$, i.e., the first three moments of the exponential distribution with mean unity. Then

$$\inf_{F\in\mathfrak{F}^{(1,2,6)}_{[0,\infty]}} E\{e^{tX}\} = (3 + 2\sqrt{2})(4 + 2\sqrt{2})^{-1} \exp\{(\sqrt{2}t)(1 + \sqrt{2})^{-1}\} + (4 + 2\sqrt{2})^{-1} \exp\{2 + \sqrt{2}t\}.$$

$$F_1^*(x) = \begin{cases} 0, & x < \sqrt{2}(1 + \sqrt{2})^{-1}, \\ (3 + 2\sqrt{2})(4 + 2\sqrt{2})^{-1}, & \sqrt{2}(1 + \sqrt{2})^{-1} \leqq x < 2 + \sqrt{2}, \\ 1, & 2 + \sqrt{2} \leqq x. \end{cases}$$

The supremum does not exist.

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ON BOUNDS OF SERIAL CORRELATIONS

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1. Introduction and summary. The role of serial correlations in time series analysis is well known. Considerable attention has been given to the derivation of their sampling properties when the sample size is both small and large. In all these discussions it has been tacitly assumed that these correlations are bounded between -1 and 1. At least, no literature exists which considers it otherwise. Whereas it is true that the serial correlations are all bounded it is not true that the bounds are -1 and 1. In fact, in small samples these bounds may very well be lower than -1 and higher than 1. To the best of the author's knowledge, this fact has not been mentioned anywhere. The purpose of this note is to discuss this particular aspect.

Received January 2, 1962; revised June 14, 1962.