THE TIME DEPENDENCE OF A SINGLE-SERVER QUEUE WITH POISSON INPUT AND GENERAL SERVICE TIMES[1]

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1. Introduction. A single-server queueing process is considered. Denote by $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ the arrival times of the customers and by χ_n the service time of the nth customer. If an arriving customer finds the server idle then his service starts without delay; if the server is busy then he joins the queue and awaits its turn. We speak about queueing process of type $[F(x), H(x), 1]$ if the interarrival times $\tau_{n+1} - \tau_n$ $(n = 0, 1, \cdots; \tau_0 = 0)$ and service times χ_n $(n = 1, 2, \cdots)$ are independent sequences of identically distributed, mutually independent random variables with distribution functions $\mathbf{P}\{\tau_{n+1} - \tau_n \leqq x\} = F(x)$ $(n = 0, 1, \cdots)$ and $\mathbf{P}\{\chi_n \leqq x\} = H(x)$ $(n = 1, 2, \cdots)$ and if there is a single server.

In what follows we shall consider the particular case when the customers are arriving at the counter in accordance with a Poisson process of density λ, i.e.,

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{1}$$

We suppose that the server is idle if and only if there is no customer in the system, otherwise the order of services is irrelevant.

Denote by $\eta(t)$ the *occupation time* of the server at time t, i.e., $\eta(t)$ is the time needed to complete the service of all those customers who are present in the system at time t. $\eta(0)$ is the initial occupation time of the server. Let $W(t, x) = \mathbf{P}\{\eta(t) \leqq x\}$. If $\eta(t) > 0$ then the server is busy at time t and if $\eta(t) = 0$ then the server is idle at time t. Let $P_0(t) = \mathbf{P}\{\eta(t) = 0\}$, i.e., $P_0(t)$ is the probability that the server is idle at time t. If, in particular, the customers are served in the order of arrival, then $\eta(t)$ can be interpreted as the *virtual waiting time* at time t, i.e., the time that a customer would have to wait if he arrived at the instant t.

The *busy period* is defined as the time interval during which the server is continuously busy. If $\eta(0) > 0$, i.e., the server is busy at time $t = 0$, then there is an initial busy period which ends when $\eta(t)$ vanishes for the first time. Denote by $\hat{G}(x)$ the probability that the length of the initial busy period is $\leqq x$. Following the initial busy period (if any) idle periods and busy periods alternate. The lengths of the busy periods following the initial busy period are identically distributed, mutually independent random variables. Denote by $G(x)$ the probability that the length of a busy period other than the initial is $\leqq x$. The idle

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periods are also identically distributed, mutually independent random variables with distribution function $F(x)$ defined by (1).

In this paper we shall prove a simple geometrical lemma and by using this lemma we shall find explicit formulas for the probabilities $P_0(t)$, $\hat{G}(x)$, $G(x)$, and $W(t, x)$.

We introduce the following notation: $H_n(x)$ is the nth iterated convolution of $H(x)$ with itself; $H_0(x) = 1$ if $x \geqq 0$, and $H_0(x) = 0$ if $x < 0$. Let

$$\psi(s) = \int_0^\infty e^{-sx}\, dH(x)$$

and

$$\Omega(t, s) = \int_0^\infty e^{-sx}\, d_x W(t, x).$$

2. Auxiliary theorems.

LEMMA 1. *Let $\chi_1, \chi_2, \cdots, \chi_n$ be non-negative, interchangeable random variables with sum $\chi_1 + \chi_2 + \cdots + \chi_n = y$. Let $\tau_1, \tau_2, \cdots, \tau_n$ be the coordinates arranged in increasing order of n points distributed uniformly and independently of each other in the interval $(0, t)$. If $\{\chi_k\}$ and $\{\tau_k\}$ are independent sequences, then*

$$\mathbf{P}\{\chi_1 + \cdots + \chi_k \leqq \tau_k \quad for \quad k = 1, 2, \cdots, n\} = \begin{cases} 1 - (y/t) & \text{if } 0 \leqq y \leqq t, \\ 0 & \text{if } y > t. \end{cases} \tag{2}$$

PROOF. First we note that Lemma 1 can be interpreted as follows: the probability that in Figure 1 the step function lies underneath the 45° line is $1 - (y/t)$ if $0 \leqq y \leqq t$. We prove (2) by induction. If $n = 1$ then (2) is evidently true. Supposing that it is true for $n - 1$ we shall prove that it is also true for n. If $y > t$ then (2) is trivially true. Let $0 \leqq y \leqq t$. Suppose that $\chi_1 + \cdots + \chi_{n-1} = z$ and $\tau_n = u$. Under this condition the random variables $\chi_1, \cdots, \chi_{n-1}$ are also

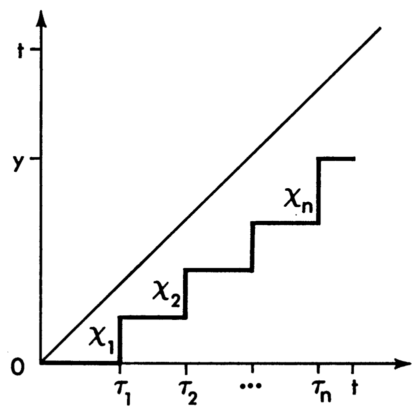

FIGURE 1

non-negative, interchangeable random variables and $\tau_1, \cdots, \tau_{n-1}$ can be considered as the coordinates arranged in increasing order of $n - 1$ points distributed uniformly and independently of each other in the interval $(0, u)$. Now, by assumption,

$$\mathbf{P}\{\chi_1 + \cdots + \chi_k \leqq \tau_k \text{ for } k = 1, 2, \cdots, n \mid \chi_1 + \cdots + \chi_{n-1} = z, \tau_n = u\} = \tag{3} \begin{cases} 1 - (z/u) & \text{if } 0 \leqq z \leqq u \text{ and } y \leqq u \leqq t \\ 0 & \text{otherwise.} \end{cases}$$

Since $\chi_1, \cdots, \chi_n$ are interchangeable random variables we have

$$\mathbf{E}\{\chi_1 + \cdots + \chi_{n-1}\} = [(n - 1)y]/n$$

and thus dropping the condition $\chi_1 + \cdots + \chi_{n-1} = z$ in (3) we get

$$\mathbf{P}\{\chi_1 + \cdots + \chi_k \leqq \tau_k \quad \text{for} \quad k = 1, 2, \cdots, n \mid \tau_n = u\} = 1 - \{[(n - 1)y]/nu\}, \quad \text{if } 0 \leqq y \leqq u.$$

Since $\mathbf{P}\{\tau_n \leqq u\} = (u/t)^n$ if $0 \leqq u \leqq t$, we get finally

$$\mathbf{P}\{\chi_1 + \cdots + \chi_k \leqq \tau_k \quad \text{for} \quad k = 1, 2, \cdots, n\} = n \int_y^t \left[1 - \frac{(n - 1)y}{nu}\right]\left(\frac{u}{t}\right)^{n-1} \frac{du}{t} = 1 - \frac{y}{t},$$

which was to be proved.

Lemma 2. *Let* $\chi_1, \chi_2, \cdots, \chi_n$ *be non-negative, interchangeable random variables with sum* $\chi_1 + \chi_2 + \cdots + \chi_n = t$ *and let* $\tau_1, \tau_2, \cdots, \tau_{n-1}$ *be the coordinates arranged in increasing order of* $n - 1$ *points distributed uniformly and independently of each other in the interval* $(0, t)$. *If* $\{\chi_k\}$ *and* $\{\tau_k\}$ *are independent sequences then*

$$\mathbf{P}\{\chi_1 + \cdots + \chi_k \leqq \tau_k \quad \text{for} \quad k = 1, 2, \cdots, n - 1\} = 1/n. \tag{4}$$

Proof. By Lemma 1 we have

$$\mathbf{P}\{\chi_1 + \cdots + \chi_k \leqq \tau_k \quad \text{for} \quad k = 1, 2, \cdots, n - 1 \mid \chi_1 + \cdots + \chi_{n-1} = y\} = 1 - (y/t)$$

if $0 \leqq y \leqq t$. Since in this case $\mathbf{E}\{\chi_1 + \cdots + \chi_{n-1}\} = [(n - 1)t]/n$, we get unconditionally that

$$\mathbf{P}\{\chi_1 + \cdots + \chi_k \leqq \tau_k \quad \text{for} \quad k = 1, 2, \cdots, n - 1\} = 1/n.$$

3. The probability that the server is idle. This problem has been investigated earlier by V. E. Beneš [3], E. Reich [16], and the author [19]. Now we shall prove

Theorem 1. *If the initial occupation time* $\eta(0) = c$ *(constant), then the probability that the server is idle at time t is given by*

$$P_0(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \int_0^{t-c} \left(1 - \frac{y}{t}\right) dH_n(y) \tag{5}$$

if $t \geqq c$ *and* $P_0(t) = 0$ *if* $t < c$. *In* (5) *we can use the identity*

$$\int_0^{t-c} \left(1 - \frac{y}{t}\right) dH_n(y) = \frac{1}{t} \int_0^{t-c} H_n(y)\, dy - \frac{c}{t} H_n(t - c). \tag{6}$$

PROOF. The case $t \leqq c$ is trivial. Suppose that $t > c$. The event $\eta(t) = 0$ can occur in several mutually exclusive ways: In the time interval $(0, t)$ exactly n ($n = 0, 1, \cdots$) customers arrive. This event has probability $e^{-\lambda t}(\lambda t)^n/n!$. Denote by $\tau_1, \tau_2, \cdots, \tau_n$ the arrival times and by $\chi_1, \chi_2, \cdots, \chi_n$ the service times of these customers. Let $\chi_1 + \chi_2 + \cdots + \chi_n = y$. Under these conditions $\eta(t) = 0$ is satisfied if and only if $0 \leqq y \leqq t - c$ and

$$\tau_j \leqq \chi_1 + \cdots + \chi_{j-1} + t - y \qquad \text{for } j = 1, 2, \cdots, n, \tag{7}$$

where the empty sum is equal to zero. If we know that in the time interval $(0, t)$ exactly n customers arrive then the arrival instants can be considered as the coordinates arranged in increasing order of n points distributed uniformly and independently of each other in the interval $(0, t)$. Further if $\chi_1 + \cdots + \chi_n = y$ then $\chi_1, \chi_2, \cdots, \chi_n$ are non-negative, interchangeable random variables. If $\chi_1 + \cdots + \chi_n = y$ then the event (7) has the same probability as

$$\chi_1 + \cdots + \chi_k \leqq \tau_k \qquad \text{for } k = 1, 2, \cdots, n. \tag{8}$$

For, in (7) we can replace χ_j by χ_{n+1-j} ($j = 1, 2, \cdots, n$) and τ_j by $t - \tau_{n+1-j}$ ($j = 1, 2, \cdots, n$) without a change in the probability of the event. By Lemma 1 the probability of the event (8) or (7) is $1 - (y/t)$ if $0 \leqq y \leqq t$. Since $\mathbf{P}\{\chi_1 + \cdots + \chi_n \leqq y\} = H_n(y)$, finally, by the theorem of total probability we get (5).

4. The probability law of the busy period. This problem has been investigated earlier by J. Gani [4], J. Gani and N. U. Prabhu [5], D. P. Gaver [7], S. Karlin, R. G. Miller, and N. U. Prabhu [8], D. G. Kendall [10], [11], B. McMillan and J. Riordan [12], F. Pollaczek [13], N. U. Prabhu [15], and the author [19], [20]. Now we shall prove

THEOREM 2. *If* $\eta(0) = c$ *(positive constant), then the probability that the initial busy period has length* $\leqq x$ *is given by*

$$\hat{G}(x) = \sum_{n=0}^{\infty} (c\lambda^n/n!) \int_0^{x-c} e^{-\lambda(c+y)}(c + y)^{n-1}\, dH_n(y) \tag{9}$$

if $x \geqq c$ *and* $\hat{G}(x) = 0$ *if* $x < c$.

PROOF. The number of arrivals during the initial busy period may be $n = 0, 1, 2, \cdots$. If $n = 0$ then the initial busy period has length c and the probability that no customer arrives in the time interval $(0, c)$ is $e^{-\lambda c}$. If $n \geqq 1$ then denote by $\tau_1, \tau_2, \cdots, \tau_n$ the arrival times and by $\chi_1, \chi_2, \cdots, \chi_n$ the service times of these customers. They must satisfy the following conditions

$$\tau_j \leqq \chi_1 + \cdots + \chi_{j-1} + c \qquad \text{for } j = 1, 2, \cdots, n. \tag{10}$$

If $\chi_1 + \cdots + \chi_n = y$ then the length of the initial busy period is $c + y$ and the probability that exactly n customers arrive during the time interval $(0, c + y)$ is $e^{-\lambda(c+y)} [\lambda (c + y)]^n/n!$. The arrival instants can be considered as the coordinates arranged in increasing order of n points distributed uniformly and independently of each other in the interval $(0, c + y)$. Further $\chi_1, \chi_2, \cdots, \chi_n$ are non-negative, interchangeable random variables. Similarly to (7) the event (10) has probability $1 - y/(c + y) = c/(c + y)$. Finally, by the theorem of total probability we get for $x \geqq c$ that

$$\hat{G}(x) = e^{-\lambda c} + \sum_{n=1}^{\infty} \int_0^{x-c} e^{-\lambda(c+y)} \frac{[\lambda(c + y)]^n}{n!} \left(\frac{c}{c + y}\right) dH_n(y) \tag{11}$$

which was to be proved.

THEOREM 3. *The probability that a busy period other than the initial has length* $\leqq x$ *is given by*

$$G(x) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} \int_0^x e^{-\lambda y} y^{n-1}\, dH_n(y). \tag{12}$$

PROOF. If we suppose that a busy period consists of n $(n = 1, 2, \cdots)$ services then its length is $\chi_1 + \chi_2 + \cdots + \chi_n$ where $\chi_1, \chi_2, \cdots, \chi_n$ are identically distributed, mutually independent random variables with the distribution function $\mathbf{P}\{\chi_i \leqq x\} = H(x)$ $(i = 1, 2, \cdots, n)$. In this case exactly $n - 1$ customers arrive during this busy period. Measure time from the starting point of the busy period and denote by $\tau_1, \tau_2, \cdots, \tau_{n-1}$ the arrival times. They must satisfy the conditions

$$\tau_j \leqq \chi_1 + \cdots + \chi_j \qquad j = 1, 2, \cdots, n - 1. \tag{13}$$

If $\chi_1 + \cdots + \chi_n = y$ then the busy period has length y and the arrival instants can be considered as the coordinates arranged in increasing order of n points distributed uniformly and independently of each other in the interval $(0, y)$. Further $\chi_1, \cdots, \chi_n$ are non-negative, interchangeable random variables. If $\chi_1 + \cdots + \chi_n = y$ then (13) has the same probability as

$$\chi_1 + \cdots + \chi_k \leqq \tau_k \qquad k = 1, 2, \cdots, n - 1. \tag{14}$$

For, in (13) we can replace χ_j by χ_{n+1-j} and τ_j by $y - \tau_{n-j}$ $(j = 1, 2, \cdots, n - 1)$ without a change in the probability of the event. By Lemma 2 the probability of the event (14) or (13) is $1/n$. Since $\mathbf{P}\{\chi_1 + \cdots + \chi_n \leqq y\} = H_n(y)$, and the probability that during the time interval $(0, y)$ exactly $n - 1$ customers arrive is $e^{-\lambda y} (\lambda y)^{n-1}/(n - 1)!$, unconditionally we get that

$$G(x) = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^x e^{-\lambda y} \frac{(\lambda y)^{n-1}}{(n - 1)!}\, dH_n(y) \tag{15}$$

which was to be proved.

REMARK 1. Let us consider the dual process $[H(x), F(x), 1]$ of the queueing process $[F(x), H(x), 1]$ considered so far, supposing that the interarrival times have the distribution function $H(x)$ and the service times have the distribution function $F(x)$ defined by (1); i.e., the interarrival times and service times are interchanged. If at the dual process $G^*(t)$ denotes the probability that a busy period has length $\leqq t$ then we have

$$G^*(t) = \lambda \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{n-1}}{n!} \int_0^t [1 - H_n(y)]\, dy. \tag{16}$$

For, if we suppose that a busy period starts at $t = 0$ in the dual process, then in the initial busy period τ_k is the kth departure time and χ_k is the interarrival time between the kth and the $(k + 1)$st arrivals. Therefore the probability that the busy period exceeds t is

$$1 - G^*(t) = \mathbf{P}\{\chi_1 + \chi_2 + \cdots + \chi_k \leqq \tau_k\,;\, k = 1, 2, \cdots, n\}, \tag{17}$$

where n is the number of departures in the interval $(0, t]$. By (8), the right side of (17) is $P_0(t)$ with $c = 0$; this proves (16).

5. The distribution of the occupation time. This problem has been investigated earlier by V. E. Beneš [1], [2], [3], J. Gani and N. U. Prabhu [5], J. Gani and R. Pyke [6], D. P. Gaver [7], J. Keilson and A. Kooharian [9], N. U. Prabhu [15], E. Reich [16], [17], J. Th. Runnenburg [18], and the author [19], [21].

Define the process $\{\xi(t), 0 \leqq t < \infty\}$ as follows

$$\xi(t) = \sum_{0<\tau_n \leqq t} \chi_n\,, \tag{18}$$

i.e., $\xi(t)$ is the total service time of all those customers who arrive in the time interval $(0, t]$. Let $\mathbf{P}\{\xi(t) \leqq x\} = K(t, x)$. We have

$$K(t, x) = \sum_{n=0}^{\infty} e^{-\lambda t}[(\lambda t)^n/n!]H_n(x), \tag{19}$$

whence

$$\int_0^{\infty} e^{-sx}\, d_x K(t, x) = e^{-\lambda t[1-\psi(s)]}. \tag{20}$$

Knowing $P_0(t) = W(t, 0)$ and $K(t, x)$ the distribution function $W(t, x)$ can be obtained by quadratures. The following theorem is a particular case of a more general theorem of V.E. Beneš [2].

THEOREM 4. *If* $\eta(0) = c$ *(constant), then we have*

$$W(t, x) = K(t, t + x - c) - (\partial/\partial x) \int_0^t K(t - u, t - u + x) P_0(u)\, du. \tag{21}$$

where $P_0(u)$ *is defined by* (5).

PROOF. The process $\{\eta(t), 0 \leqq t < \infty\}$ is evidently a Markov process. If $\delta_{\Delta t}$ denotes the number of customers arriving in the time interval $(t, t + \Delta t]$ then we can write that

$$\eta(t + \Delta t) = [\eta(t) - \Delta t]^+ \quad \text{if} \quad \delta_{\Delta t} = 0, \tag{22}$$

where $[a]^+ = \max(0, a)$, and

$$\eta(t + \Delta t) = \eta(t) + \chi_t - \epsilon\Delta t \quad \text{if} \quad \delta_{\Delta t} = 1, \tag{23}$$

where $0 \leqq \epsilon \leqq 1$ and χ_t is the service time of the customer arriving in the time interval $(t, t + \Delta t]$.

Now, by definition $\Omega(t, s) = \mathbf{E}\{e^{-s\eta(t)}\}$ and by using the theorem of total expectation we get

$$\begin{aligned} \Omega(t + \Delta t, s) &= \mathbf{E}\{e^{-s\eta(t+\Delta t)}\} \\ &= \sum_{j=0}^{\infty} \mathbf{P}\{\delta_{\Delta t} = j\}\mathbf{E}\{e^{-s\eta(t+\Delta t)} \mid \delta_{\Delta t} = j\} = (1 - \lambda\Delta t)\mathbf{E}\{e^{-s\eta(t+\Delta t)} \mid \delta_{\Delta t} = 0\} \\ &\quad + \lambda\Delta t\mathbf{E}\{e^{-s\eta(t+\Delta t)} \mid \delta_{\Delta t} = 1\} + o(\Delta t) \end{aligned} \tag{24}$$

because $\mathbf{P}\{\delta_{\Delta t} = j\} = e^{-\lambda\Delta t}(\lambda\Delta t)^j/j!$. Here

$$\mathbf{E}\{e^{-s\eta(t+\Delta t)} \mid \delta_{\Delta t} = 0\} = (1 + s\Delta t)\Omega(t, s) - sP_0(t)\Delta t + o(\Delta t), \tag{25}$$

for, by (22)

$$\begin{aligned} &\mathbf{E}\{e^{-s\eta(t+\Delta t)} \mid \delta_{\Delta t} = 0\} \\ &\quad = e^{s\Delta t}\mathbf{P}\{\eta(t) > \Delta t\}\mathbf{E}\{e^{-s\eta(t)} \mid \eta(t) > \Delta t\} + \mathbf{P}\{\eta(t) \leqq \Delta t\}, \end{aligned} \tag{26}$$

and on the other hand

$$\Omega(t, s) = \mathbf{P}\{\eta(t) > \Delta t\}\mathbf{E}\{e^{-s\eta(t)} \mid \eta(t) > \Delta t\} + \mathbf{P}\{\eta(t) \leqq \Delta t\} + o(\Delta t). \tag{27}$$

Comparing (26) and (27) we get (25). Further we have

$$\mathbf{E}\{e^{-s\eta(t+\Delta t)} \mid \delta_{\Delta t} = 1\} = \psi(s)\Omega(t, s) + O(\Delta t). \tag{28}$$

Putting (25) and (28) into (24) we get

$$\Omega(t + \Delta t, s) = \Omega(t, s)[1 + s\Delta t - \lambda\Delta t(1 - \psi(s))] - sP_0(t)\Delta t + o(\Delta t), \tag{29}$$

whence

$$\partial\Omega(t, s)/\partial t = \{s - \lambda[1 - \psi(s)]\}\Omega(t, s) - sP_0(t). \tag{30}$$

The solution of this differential equation is

$$\Omega(t, s) = e^{st-\lambda t[1-\psi(s)]}\left\{\Omega(0, s) - s\int_0^t e^{-su+\lambda u[1-\psi(s)]}P_0(u)\,du\right\}. \tag{31}$$

If we suppose that $\eta(0) = c$ (constant), then $\Omega(0, s) = e^{-sc}$ and by inverting (31) we get (21).

THEOREM 5. *If we suppose that $\eta(0) = 0$, then we can write that*

$$W(t, x) = \sum_{n=0}^{\infty} e^{-\lambda t}[(\lambda t)^n/n!]V_n(t, x), \tag{32}$$

where $V_0(t, x) \equiv 1$ and $V_n(t, x)$ can be obtained step by step by the following recurrence formula

$$(33)\quad V_{n+1}(t, x) = [(n+1)/t] \int_0^t \int_0^{x+u} V_n(t-u, x+u-y)\, dH(y)\, du.$$

PROOF. If $\eta(0) = 0$, then we can write that

$$(34)\qquad W(t, x) = \mathbf{P}\{\eta(t) \leqq x\} = \mathbf{P}\{\sup_{0 \leqq u \leqq t} [\xi(u) - u] \leqq x\}.$$

To prove (34) we note that

$$(35)\qquad \mathbf{P}\{\eta(t) \leqq x\} = \mathbf{P}\{\xi(t) - t - \inf_{0 \leqq u \leqq t} [\xi(u) - u] \leqq x\}.$$

Since the process $\{\xi(t), 0 \leqq t < \infty\}$ has independent increments and $\xi(0) = 0$ we can replace $\xi(u)$ by $\xi(t) - \xi(t-u)$ in (35). Thus we get (34). If τ_1 denotes the arrival time and χ_1 the service time of the first customer then

$$(36)\qquad \mathbf{P}\{\sup_{0 \leqq u \leqq t} [\xi(u) - u] \leqq x \mid \tau_1 = u, \chi_1 = y\} = \begin{cases} W(t-u, x+u-y) & \text{if } u \leqq t \\ 1 & \text{if } u > t \end{cases}$$

and thus unconditionally

$$(37)\quad W(t, x) = e^{-\lambda t} + \lambda \int_0^t \int_0^{x+u} W(t-u, x+u-y) e^{-\lambda u}\, dH(y)\, du.$$

On the other hand we can write that

$$(38)\qquad W(t, x) = \sum_{n=0}^{\infty} e^{-\lambda t} [(\lambda t)^n/n!] V_n(t, x)$$

where $V_n(t, x)$ denotes the probability that $\eta(t) \leqq x$ given that in the time interval $(0, t]$ exactly n customers arrive. Putting (38) into (37) we get the recurrence relation (33).

We note that if $\eta(0)$ is arbitrary, then (34) is to be replaced by

$$(39)\qquad \mathbf{P}\{\eta(t) \leqq x\} = \mathbf{P}\{\xi(u) \leqq u + x \quad \text{for} \quad 0 \leqq u \leqq t \quad \text{and} \quad \eta(0) + \xi(t) \leqq t + x\}.$$

REMARK 2. If $\eta(0) = 0$, then by (34) we can write that

$$(40)\qquad W(t + \Delta t, x) = W(t, x)(1 - \lambda\Delta t) + \lambda\Delta t \int_0^{t+x} H(t+x-y)\, d_y W(t, y) + o(\Delta t),$$

because $\{\xi(t), 0 \leqq t < \infty\}$ is a Markov process with independent increments. From (40) we get the integro-differential equation

$$(41)\quad \partial W(t, x)/\partial t = -\lambda \left[W(t, x) - \int_0^{t+x} H(t+x-y)\, d_y W(t, y) \right]$$

for the determination of $W(t, x)$.

Note added in proof. I have learned that Lemmas 1 and 2 of this paper have also been proved, in an unpublished report, by Professor Meyer Dwass.

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