A REPRESENTATION OF THE SYMMETRIC BIVARIATE CAUCHY DISTRIBUTION

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1. Introduction. There are very few distributions, which, like the normal distribution, have intrinsic extensions to the multivariate situation other than the trivial extension requiring the components to be independent. Of these distributions, the stable distributions occupy a unique position. The multivariate stable distributions may be characterized by the requirement that every one-dimensional marginal distribution (that is the distribution of *every linear* combination of the variables) is a stable distribution. A proof that such a requirement characterizes the multivariate normal distribution may be found in Anderson's book [1], pg. 37. This paper is concerned with an investigation of the symmetric multivariate stable distribution with characteristic exponent 1, namely, the symmetric multivariate Cauchy distribution.

DEFINITION. *A random vector* $\mathbf{X}' = (X_1, \cdots, X_r)$ *is said to have a multivariate Cauchy distribution if, and only if, for every real vector* $\mathbf{t}' = (t_1, \cdots, t_r)$, *the random variable* $\mathbf{t}'\mathbf{X} = \sum t_i X_i$ *has a Cauchy distribution. The distribution is said to be symmetric if the mass is distributed symmetrically with respect to some point in r-dimensional space.*

The following simple lemma is the basis for this study. A similar result for arbitrary stable distributions may be found in Lemma 2 of [2], but note that the word symmetric is used there in a different sense.

LEMMA 1. *The distribution of a random vector* $\mathbf{X}$ *is multivariate Cauchy if, and only if, the characteristic function of* $\mathbf{X}$ *has the form*

$$\phi_{\mathbf{X}}(\mathbf{t}) = e^{-g(\mathbf{t}) + i\gamma(\mathbf{t})}, \tag{1}$$

where $g(\mathbf{t}) \geqq 0$ *and* $\gamma(\mathbf{t})$ *are real functions satisfying the equations*

$$g(a\mathbf{t}) = |a| g(\mathbf{t}) \tag{2}$$

$$\gamma(a\mathbf{t}) = a\gamma(\mathbf{t}) \tag{3}$$

for every real number a. If the distribution is symmetric with respect to a point $\boldsymbol{\gamma}$ *in r-dimensional space, then*

$$\gamma(\mathbf{t}) = \boldsymbol{\gamma}'\mathbf{t}. \tag{4}$$

PROOF. If a characteristic function has the form (1) where g and γ satisfy (2) and (3), then

$$\begin{aligned} \phi_{\mathbf{t}'\mathbf{X}}(a) &= Ee^{ia\mathbf{t}'\mathbf{X}} = \phi_{\mathbf{X}}(a\mathbf{t}) \\ &= e^{-g(a\mathbf{t}) + i\gamma(a\mathbf{t})} \\ &= e^{-|a| g(\mathbf{t}) + ia\gamma(\mathbf{t})} \end{aligned} \tag{5}$$

Received November 7, 1961.

which is the characteristic function of a Cauchy distribution with median $\gamma(\mathbf{t})$ and semi-interquartile range $g(\mathbf{t})$.

Conversely, if for every $\mathbf{t}$,

$$\phi_{\mathbf{t}'\mathbf{X}}(a) = e^{-|a|g(\mathbf{t})+ia\gamma(\mathbf{t})}, \tag{6}$$

where $g(\mathbf{t}) \geqq 0$, we see by putting $a = 1$ that (1) is satisfied, which in turn implies that

$$\phi_{\mathbf{X}}(a\mathbf{t}) = e^{-g(a\mathbf{t})+i\gamma(a\mathbf{t})}. \tag{7}$$

Equating real and imaginary parts of (6) and (7) immediately yields equations (2) and (3).

If the distribution is symmetric about γ, then for every $\mathbf{t}$, $\mathbf{t}'(\mathbf{X} - \gamma)$ has a Cauchy distribution with median zero. This implies equation (4) and completes the proof.

It should be noted that not every function $\phi_{\mathbf{X}}(\mathbf{t})$ of the form found in Lemma 1 will be a characteristic function. It is the purpose of this paper to find necessary and sufficient conditions on a real function $g(\mathbf{t})$, when $\mathbf{t}$ is a vector in two dimensions, in order that $\exp\{-g(\mathbf{t})\}$ be a characteristic function of a bivariate Cauchy distribution symmetric with respect to the origin. It is proved in Theorem 1 that the only condition needed in addition to (2) and non-negativeness is that the contours of $g(\mathbf{t})$ be convex.

EXAMPLE 1. Let X_1 and X_2 be independent, each having a Cauchy distribution with median zero and semi-interquartile range (SIQR) one. The joint density of X_1 and X_2 is $f_{X_1,X_2}(x_1, x_2) = \pi^{-2}(1 + x_1^2)^{-1}(1 + x_2^2)^{-1}$. The joint characteristic function of X_1 and X_2 is $\phi_{X_1,X_2}(t_1, t_2) = \exp\{-|t_1| - |t_2|\}$. The contours of this characteristic function are squares with center at the origin and vertices on the axes.

EXAMPLE 2. Let X_1 be as in Example 1 and let $Y_1 = Y_2 = X_1$. The joint distribution of Y_1 and Y_2 is a singular bivariate Cauchy with characteristic function $\phi_{Y_1,Y_2}(t_1, t_2) = \exp\{-|t_1 + t_2|\}$. The contours of this characteristic function are lines equidistant and parallel to the line $t_1 + t_2 = 0$.

EXAMPLE 3. Let random variables X_1 and X_2 have a joint density $f_{X_1X_2}(x_1, x_2) = (2\pi)^{-1}(1 + x_1^2 + x_2^2)^{-\frac{3}{2}}$, the so-called circular bivariate Cauchy distribution. The characteristic function of X_1 and X_2 is $\phi_{X_1,X_2}(t_1, t_2) = \exp\{-(t_1^2 + t_2^2)^{\frac{1}{2}}\}$. The contours of this characteristic function are circles centered at the origin. Such distributions occur, for example, when $X_1 = U_1/V$ and $X_2 = U_2/V$, where U_1, U_2 and V are independent random variables having normal distributions with mean zero and variance one.

EXAMPLE 4. Let $X_1, X_2, \cdots, X_n$ be a sequence of independent random variables all having the same Cauchy distribution with median zero and SIQR one. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be real numbers, and let $Y_1 = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ and $Y_2 = b_1X_1 + b_2X_2 + \cdots + b_nX_n$. Examples 1 and 2 are special cases of this example. The joint distribution of Y_1 and Y_2 is easily seen to be bivariate Cauchy, and the characteristic function is

$$\phi_{Y_1,Y_2}(t_1, t_2) = e^{-\sum_1^n|a_it_1+b_it_2|}. \tag{8}$$

The contours of this characteristic function are convex polygonal paths symmetric with respect to the origin. This may be seen by noting that in a neighborhood of any point not lying on one of the lines $a_i t_1 + b_i t_2 = 0$ for $i = 1, \cdots, n$, the exponent in the right side of (8) is linear. In fact, between any two such neighboring lines, the function $g(t_1, t_2) = \sum |a_i t_1 + b_i t_2|$ represents a plane whose value at the origin is zero. Consequently, the contours $\sum |a_i t_1 + b_i t_2| = c, c > 0$, are convex polygons with vertices on the lines $a_i t_1 + b_i t_2 = 0$. If the lines are distinct, there will be exactly $2n$ vertices.

The distributions of Example 4 are very important. The corollary to Theorem 1 states that these distributions are dense in the set of all bivariate Cauchy distributions symmetric with respect to the origin. The proof of the sufficiency part of Theorem 1 (that every convex curve symmetric with respect to the origin can be attained as the contour of the characteristic function of a symmetric bivariate Cauchy distribution) consists merely of showing that every convex polygon symmetric with respect to the origin can be attained as a contour $\sum_1^n |a_i t_1 + b_i t_2| = 1$ for suitably chosen a_i and b_i.

In Section 3 we shall discuss the relation which exists between the representation of the symmetric bivariate Cauchy distribution found in Theorem 1 and a similar representation of Paul Lévy [4]. In the last section we show that the representation of Theorem 1 cannot be extended directly to three dimensions, in that there exist convex surfaces symmetric with respect to the origin which cannot be attained as a contour of the characteristic function of any symmetric three-dimensional Cauchy distribution.

One is led to believe that the validity of Theorem 1 is an "accident", that there is some more intrinsic property defining the contours of the functions $g(\mathbf{t})$ when $\mathbf{t}$ is in n-dimensional space, which just happens to coincide with convexity when $n = 2$. If so, I believe that it is a fortunate accident, because it seems to me that a characterization in terms of the contours of the characteristic function is what one really wants. Given a symmetric bivariate Cauchy distribution and knowledge of its marginal distributions in several directions, one would like to know what possibilities are available for the marginal distributions in a few other directions. Information of this sort, which is difficult to obtain from Lévy's representation, can easily be obtained from Theorem 1.

Certain unsolved problems are suggested naturally by the results of this paper. Since Theorem 1 does not extend directly to higher dimensions, the problem arises to find the condition which characterizes the shape of these contours in three or more dimensions. This condition must be stronger than convexity. Another interesting problem is the extension of these results to symmetric bivariate stable laws with arbitrary characteristic exponent α, $0 < \alpha \leqq 2$. This paper treats the case $\alpha = 1$, and shows that the contours of the characteristic function are the convex curves symmetric with respect to a point in the plane. For $\alpha = 2$, the distributions are normal, and the contours of the characteristic function of bivariate normal laws are the ellipses. For each α between 1 and 2, there is a condition, yet to be determined, on the contours of the characteristic function, which characterizes the stable distributions with characteristic exponent α. This condition gets increasingly stronger as α increases from one to two, starting from the general convex symmetric curves, to the "most convex" curves of all—the ellipses. Equally interesting and unknown are the conditions on the contours for $\alpha < 1$. These contours do not have to be convex, as is exhibited by the case where the individual variables X and Y are independent.

2. The main theorem. A function satisfying (3) is said to be homogeneous of degree one. A function satisfying (2) is said to be positive homogeneous of degree one.

Suppose that $g(t_1, t_2)$ is positive homogeneous of degree one and that $g(t_1, t_2) \geqq 0$. Then on any straight line through the origin, either g is identically zero or g is zero at the origin and in either direction away from the origin increases linearly from zero to infinity. Thus, on any line through the origin, the contour $g(t_1, t_2) = 1$ will have either no points at all, or exactly two points symmetric with respect to the origin. Any contour, $g(t_1, t_2) = c$, $c > 0$, is a projection from the origin of any other contour. Thus $g(t_1, t_2)$ will be completely known when one contour is specified.

The distance from the origin to the contour $g(t_1, t_2) = 1$ in the direction θ (i.e., along the line $t_1 \sin\theta - t_2 \cos\theta = 0$) will be denoted by $\rho(\theta)$. The contour in polar coordinates (r, θ) may then be written as $r = \rho(\theta)$. Since $\rho(\theta)$ may be infinite but not zero we prefer to work with the reciprocal $h(\theta) = \rho(\theta)^{-1}$. The equation defining $h(\theta)$ may be written

$$h(\theta) = g(\cos\theta, \sin\theta). \tag{9}$$

We shall say that the contour $g(t_1, t_2) = 1$ is convex, if the set

$$\{(t_1, t_2): g(t_1, t_2) \leqq 1\}$$

is convex. This will apply equally well for unbounded contours. It takes an elementary calculation, which we omit, to show that the contour $g(t_1, t_2) = 1$ is convex if, and only if, for every $\theta_1 < \theta_2 < \theta_3$ for which $\theta_3 - \theta_1 < \pi$,

$$h(\theta_3)\sin(\theta_2 - \theta_1) + h(\theta_1)\sin(\theta_3 - \theta_1) \geqq h(\theta_2)\sin(\theta_3 - \theta_1). \tag{10}$$

The reader may refer to Hardy, Littlewood and Polya ([3], pp. 98, problem 123) for a discussion of circular convexity. If the function $h(\theta)$ admits two continuous derivatives, the above convexity inequality may be simplified to an equation involving only one arbitrary θ,

$$h(\theta) + h''(\theta) \geqq 0 \tag{11}$$

for all θ.

THEOREM 1. *In order that a function* $\phi(t_1, t_2) = \exp\{-g(t_1, t_2)\}$ *be the characteristic function of a bivariate Cauchy distribution symmetric with respect to the origin, it is necessary and sufficient that* $g(t_1, t_2)$ *be real, non-negative, and positive homogeneous of degree one, and have convex contours.*

PROOF.

Necessity. That $g(t_1, t_2)$ must be real non-negative and positive homogeneous of degree one is contained in Lemma 1. To show that the contours are convex, we must show that if (u_1, u_2) and (v_1, v_2) are two points for which $g(u_1, u_2) = g(v_1, v_2) = 1$, then for every α, $0 < \alpha < 1$, $g_\alpha = g(\alpha u_1 + (1-\alpha)v_1, \alpha u_2 + (1-\alpha)v_2) \leqq 1$. Let (X, Y) denote a random vector whose distribution has characteristic function $\exp\{-g(t_1, t_2)\}$. Then, for $0 < r < 1$,

$$g_\alpha^r E\{|(\alpha u_1 + (1-\alpha)v_1)X + (\alpha u_2 + (1-\alpha)v_2)Y|^r g_\alpha^{-r}\} \leqq \alpha^r E|u_1X + u_2Y|^r + (1-\alpha)^r E|v_1X + v_2Y|^r \tag{12}$$

from the c_r-inequality of Loève [5] pg. 155. The three expectations in the inequality (12) are all equal to $E|Z|^r$ where Z has a Cauchy distribution with median zero and SIQR one. We may cancel these expectations and write

$$g_\alpha^r \leqq \alpha^r + (1-\alpha)^r, \tag{13}$$

for all r, for which $0 < r < 1$. This inequality must also be true for $r = 1$, implying that $g_\alpha \leqq 1$ as was to be shown.

Sufficiency. Let $g(t_1, t_2)$ represent a non-negative function, positive homogeneous of degree one, having a contour, $g(t_1, t_2) = 1$, which is convex. If the function $h(\theta)$ defined by equation (9) is zero for all θ, then $g(t_1, t_2)$ is identically zero, and $\exp\{-g(t_1, t_2)\}$ is the characteristic function of the degenerate (bivariate Cauchy) distribution. If for some θ_0, $h(\theta_0) = 0$, but h is not identically zero, then the convexity implies that the contour consists of a pair of straight lines parallel to the line $t_1 \cos\theta_0 + t_2 \sin\theta_0 = 0$. In this case, $\exp\{-g(t_1, t_2)\} = \exp\{-c|t_1 \cos\theta_0 + t_2 \sin\theta_0|\}$, the characteristic function of a singular bivariate Cauchy distribution. Henceforth, we assume that $h(\theta) > 0$ for all θ.

Let $-\frac{1}{2}\pi = \theta_0 < \theta_1 < \cdots < \theta_{n-1} < \theta_n = \frac{1}{2}\pi$. We claim that there exist numbers $a_1, \cdots, a_n$, and $b_1, \cdots, b_n$ such that the polygon

$$\sum_{i=1}^n |a_i t_1 + b_i t_2| = 1 \tag{14}$$

have its vertices at the points where the contour $g(t_1, t_2) = 1$ intersects the lines $t_1 \sin\theta_j - t_2 \cos\theta_j = 0$ for $j = 1, \cdots, n$. If the vertices are to be on these lines, the discussion of Example 4 shows that we may as well choose $a_i = r_i \sin\theta_i$ and $b_i = -r_i \cos\theta_i$, for $i = 1, \cdots, n$. Thus we claim that there exist non-negative numbers $r_1, \cdots, r_n$ such that

$$\sum_{i=1}^n r_i \rho(\theta_j) |\sin\theta_i \cos\theta_j - \cos\theta_i \sin\theta_j| = 1, \tag{15}$$

for $j = 1, \cdots, n$. This equation may be written as

$$\sum_{i=1}^j r_i \sin(\theta_j - \theta_i) + \sum_{i=j+1}^n r_i \sin(\theta_i - \theta_j) = h(\theta_j), \tag{16}$$

for $j = 1, \cdots, n$. The solution to this set of n linear equations in n unknowns may be written as

$$(17)\quad r_i = \frac{\begin{array}{l} h(\theta_{i+1}) \sin (\theta_i - \theta_{i-1}) \\ \qquad + h(\theta_{i-1}) \sin (\theta_{i+1} - \theta_i) - h(\theta_i) \sin (\theta_{i+1} - \theta_{i-1}) \end{array}}{2 \sin (\theta_{i+1} - \theta_i) \sin (\theta_i - \theta_{i-1})},$$

for $i = 1, \cdots, n$, where θ_{n+1} is defined to be $\theta_1 + \pi$, and h is, of course, periodic of period π. This may be checked by substituting (17) into (16) and noting that the result is an identity in the $h(\theta_i)$. In addition, all the r_i will be nonnegative from the convexity inequality, (10). Our claim is thus verified.

As n tends to infinity, and as the mesh of the set of points $(\theta_0, \theta_1, \cdots, \theta_n)$ tends to zero, the approximating functions will converge to g, pointwise.

$$(18)\qquad \sum_{i=1}^{n} r_i |t_1 \sin \theta_i - t_2 \cos \theta_i| \to g(t_1, t_2).$$

Since the negatives of the terms on the left side of (18) are all logarithms of characteristic functions from Example 4, and since the hypotheses on g imply that it is continuous at the origin, the two-dimensional continuity theory may be applied to show that the function $\exp\{-g(t_1, t_2)\}$ is a characteristic function of a distribution which Lemma 1 asserts to be bivariate Cauchy, symmetric with respect to the origin. This completes the proof.

In the proof of this theorem we have seen that the characteristic function of any bivariate Cauchy distribution, symmetric with respect to the origin, is the limit of characteristic functions of the type (8) where, as $n \to \infty$, the a_i and b_i may depend upon n. Conversely, any such limit, if it exists, must satisfy the conditions of Lemma 1, and, if it is a characteristic function, it must be the characteristic function of a bivariate Cauchy distribution. This proves the following corollary.

COROLLARY. *The set of distributions which are limits of distributions with characteristic function of the form* (8) *is the set of all bivariate Cauchy distributions symmetric with respect to the origin.*

3. Relation to Lévy's representation. Lévy, [4] Section 63, has derived a representation of the characteristic function of the general multivariate stable distributions. When specialized to the case of multivariate stable distributions of characteristic exponent one, Lévy's representation gives for the logarithm of the characteristic function

$$(19)\qquad \psi(\mathbf{t}) = i\gamma'\mathbf{t} - \int \left\{ |\mathbf{u}'\mathbf{t}| + i\frac{2}{\pi}\mathbf{u}'\mathbf{t} \log |\mathbf{u}'\mathbf{t}| \right\} d\Phi(\mathbf{u})$$

where $\Phi(\mathbf{u})$ is a finite measure giving all its mass to the unit circle $\mathbf{u}'\mathbf{u} = 1$. The integrand in formula (19) is the logarithm of the characteristic function of a random vector $\mathbf{X} = Y\mathbf{u}$, where Y is a scalar random variable having a stable distribution with characteristic exponent one and maximum positive asymmetry, and $\mathbf{u}$ is a fixed vector. If Φ is a discrete measure giving mass $\alpha_1, \alpha_2, \cdots, \alpha_n$ to vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$, then the integral in (19) becomes a sum and $\psi(\mathbf{t})$ is clearly the logarithm of the characteristic function of a random vector $\mathbf{X} = \alpha_1 Y_1 \mathbf{u}_1 + \alpha_2 Y_2 \mathbf{u}_2 + \cdots + \alpha_n Y_n \mathbf{u}_n$, analogous to Example 4, where $Y_1, Y_2, \cdots, Y_n$ are independent random variables having the same stable distribution with characteristic exponent one and maximum asymmetry. The general case, then, may be considered as a limit of sums like those defining $\mathbf{X}$, so that an analogue to the corollary of the preceeding section is valid in this more general case.

When the center of gravity of $\Phi(\mathbf{u})$ is at the origin, or equivalently, when $\int \mathbf{u}'\mathbf{t}\, d\Phi(\mathbf{u}) = 0$ for all vectors $\mathbf{t}$, it is easy to check that the distributions corresponding to the representation (19) become multivariate Cauchy according to the definition given in section one of this paper. Furthermore, when $\Phi(\mathbf{u})$ distributes mass symmetrically with respect to the origin, then $\psi(\mathbf{t})$ can obviously be written in the simpler form,

$$\psi(\mathbf{t}) = i\gamma'\mathbf{t} - \int |\mathbf{u}'\mathbf{t}|\, d\Phi(\mathbf{u}), \tag{20}$$

the logarithm of the characteristic function of a symmetric multivariate Cauchy distribution. When $\Phi(\mathbf{u})$ has center of gravity at the origin but is not symmetric, representation (19) provides an example of a multivariate Cauchy distribution which is not symmetric ((4) will not be satisfied). One sees that there is an error in the top paragraph of page 149 in my paper [2].

From the representation (20), the corollary of the preceeding section would follow immediately. Indeed, Lévy's representation in the general form shows that similar corollaries are valid for all multivariate stable distributions. Moreover, a proof of Theorem 1 could undoubtedly be based on the representation (20). We have preferred, however, to prove Theorem 1 directly without using Lévy's representation, since it can be derived by elementary methods. Here we shall merely note the correspondence between the function $h(\theta)$, representing the semi-interquartile range in the direction θ, and the measure Φ, now a symmetrical measure on the unit circle in the plane. We introduce a cumulative distribution function (with total mass not necessarily equal to one) Ψ on the half open interval $(0, 2\pi]$ by the formula

$$\Psi(\phi) = \int_{S_\phi} d\Phi \tag{21}$$

where S_ϕ is the arc on the unit circle where the angle θ is in the half open interval $(-\frac{1}{2}\pi, \phi - \frac{1}{2}\pi]$. From symmetry, $d\Psi(\phi)$ is periodic of period π. We may write

$$g(u_1, u_2) = \int_0^{2\pi} |u_1 \sin \phi - u_2 \cos \phi|\, d\Psi(\phi). \tag{22}$$

Then, from $\Psi(\phi)$ one may immediately find $h(\theta)$ by the formula

$$h(\theta) = \int_0^{2\pi} |\sin(\theta - \phi)|\, d\Psi(\phi). \tag{23}$$

Conversely, in order to obtain the distribution function $\Psi(\phi)$ from the function $h(\theta)$, one may use the following *inversion formula*, of the integral transform (23). *The derivative of* $h(\theta)$ *exists at a point* ϕ_0 *if and only if* $\Psi(\phi)$ *is continuous at* ϕ_0. *If* ϕ_0 *and* ϕ_1 *are any two such points, and* $0 < \phi_0 < \phi_1 < 2\pi$, *then*

$$4[\Psi(\phi_1) - \Psi(\phi_0)] = \int_{\phi_0}^{\phi_1} h(\theta)\, d\theta + h'(\phi_1) - h'(\phi_0) \tag{24}$$

This may be checked directly, or it may be derived from formulas (17) and (18). The proof is omitted.

When the second derivative of $h(\theta)$ exists everywhere and is continuous, the distribution Ψ will have a continuous density, say $f(\phi) = \Psi'(\phi)$, and the inversion formula (24) takes on the simpler form,

$$4f(\phi) = h(\phi) + h''(\phi) \tag{25}$$

for all values of ϕ. The convexity inequality (11) will imply that $f(\phi) \geqq 0$.

4. Non-extendability to three dimensions. In this section it will be shown that Theorem 1 is not true in three dimensional space. Theorem 1 does imply that all contours of the function $g(t_1, t_2, t_3)$ will be convex. However, there do exist convex surfaces symmetric with respect to a point which cannot be achieved as contours of a characteristic function of a symmetric trivariate Cauchy distribution. The cube is one such surface. It is shown in Theorem 2 that if the contour of the characteristic function of a symmetric trivariate Cauchy distribution agrees with two pairs of opposite faces of a cube, then the whole contour must be a cylinder with a square base unbounded in two directions.

In order to prove Theorem 2 we will need a lemma giving upper and lower bounds for $E|X + Y|^r$ when $0 < r < 1$. The upper bound replaces the c_r-inequality used in the corresponding part of Theorem 1. If X and Y are random variables, $I(XY < 0)$ is used to denote the random variable equal to one if $XY < 0$ and zero otherwise. More generally, $I(A)$, where A is a measurable set in a probability space, denotes the random variable equal to one if A occurs and zero otherwise.

LEMMA 2. *Let* $0 < r < 1$ *and let* X *and* Y *be random variables for which* $E|X|^r$ *and* $E|Y|^r$ *are finite. Then*

$$\begin{aligned}(2^r - 1)\{E|X|^r + E|Y|^r\} &\leqq E|X + Y|^r \\ &+ 2E\{\min(|X|^r, |Y|^r) I(XY < 0)\} \leqq 2^{1-r}\{E|X|^r + E|Y|^r\}.\end{aligned} \tag{26}$$

PROOF.

Case 1. $XY \geqq 0$. We use the elementary inequality $2^r - 1 \leqq (1 + z)^r - z^r \leqq 1$ when $0 \leqq z \leqq 1$. Suppose, first, that $|X| \geqq |Y|$; then $|X + Y|^r = |X|^r(1 + z^r)$ where $z = |Y|/|X|$ and $0 \leqq z \leqq 1$. Thus, $(2^r - 1)\{|X|^r + |Y|^r\} \leqq (2^r - 1)|X|^r + |Y|^r \leqq |X + Y|^r \leqq |X|^r + |Y|^r \leqq 2^{1-r}\{|X|^r + |Y|^r\}$. By symmetry, the same inequality must be true when $|Y| \geqq |X|$.

Case 2. $XY < 0$. We use the elementary inequality $1 \leqq (1 - z)^r + z^r \leqq 2^{1-r}$

when $0 \leqq z \leqq 1$. Suppose, first that $|X| \geqq |Y|$; then $|X + Y|^r = |X|^r(1 - z^r)$ where $z = |Y|/|X|$ and $0 \leqq z \leqq 1$. Thus, $(2^r - 1)\{|X|^r + |Y|^r\} - 2|Y|^r \leqq |X|^r - |Y|^r \leqq |X + Y|^r \leqq 2^{1-r}|X|^r - |Y|^r \leqq 2^{1-r}\{|X|^r + |Y|^r\} - 2|Y|^r$. By interchanging X and Y and combining the resulting inequality with the one just derived, we find $(2^r - 1)\{|X|^r + |Y|^r\} \leqq |X + Y|^r + 2 \min (|X|^r, |Y|^r) \leqq 2^{1-r}\{|X|^r + |Y|^r\}$.

Combining cases 1 and 2 and taking expectations will yield the lemma.

THEOREM 2. *Suppose that (X, Y, Z) has a trivariate Cauchy distribution symmetric with respect to the origin. If the marginal distributions of X, Y, $X + Y + Z$, $X + Y - Z$, $X - Y + Z$, and $X - Y - Z$ are identical, then Z is degenerate at zero.*

REMARK. The hypotheses of this theorem imply that the points $(1, 0, 0)$ $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(1, 1, 1)$ $(-1, -1, -1)$, $(1, 1, -1)$, $(-1, -1, 1)$ $(1, -1, 1)$, $(-1, 1, -1)$ $(1, -1, -1)$, $(-1, 1, 1)$ all lie on the same contour of the characteristic function of (X, Y, Z). The points without zeros form the eight corners of a cube. The four points with zeros lie at the center of four of the six faces of this cube. Since the contours must be convex, these four faces are entirely contained in this contour. If the cube itself was the contour, then the points $(0, 0, 1)$ and $(0, 0, -1)$ would have to lie on the contour, which is equivalent to saying that Z has the same marginal distribution as X, Y, etc. The conclusion of the theorem contradicts this, so that the cube cannot be a contour of the characteristic function of a symmetric trivariate Cauchy distribution. In fact, since Z must be degenerate at zero, the contour must consist of a cylinder with square base containing in its center the entire z axis.

PROOF. Let Q_r denote the expectation of $|X|^r$, and hence of $|Y|^r$, $|X + Y + Z|^r$, etc. and suppose that Z is not degenerate at zero. Then there exists a positive number g such that Z/g has the same distribution as X, so that $E|Z|^r = Q_r g^r$. We will show that $g = 0$ contradicting the non-degeneracy of Z.

We apply the left inequality of Lemma 2 in the forms

$$(27) \quad (2^r - 1)\{E|X + Y|^r + E|Z|^r\} \leqq E|X + Y + Z|^r + 2E\{m_1 I(A_1)\}$$

$$(28) \quad (2^r - 1)\{E|X + Y - Z|^r + E|Z|^r\} \leqq E|X + Y|^r + 2E\{m_2 I(A_2)\}$$

where $m_1 = \min (|X + Y|^r, |Z|^r)$, $m_2 = \min (|X + Y - Z|^r, |Z|^r)$,

$$A_1 = \{(X + Y)Z < 0\}, \quad \text{and} \quad A_2 = \{(X + Y - Z)Z < 0\}.$$

Eliminating $E|X + Y|^r$ from these two inequalities one may deduce that

$$(29) \quad 2^r g^r Q_r \leqq \frac{2^{r+1} - 2^{2r}}{2^r - 1} Q_r + \frac{2}{2^r - 1} E\{m_1 I(A_1)\} + 2E\{m_2 I(A_2)\}.$$

We shall proceed to derive upper bounds for the expectations on the right side of equation (29). To this end we apply the right hand inequality of Lemma 2 in the form

$$(30) \quad \begin{aligned} &2^r E|X|^r + 2E\{\min (|X + Y - Z|^r, |X - Y + Z|^r) I(B_1)\} \\ &\qquad \leqq 2^{1-r}\{E|X + Y - Z|^r + E|X - Y + Z|^r\}, \end{aligned}$$

where $B_1 = \{X^2 < (Y - Z)^2\}$. But when $X^2 < (Y - Z)^2$, $\min(|X + Y - Z|, |X - Y + Z|) = |Y - Z| - |X|$, so that inequality (30) may be written

$$E\{(|Y - Z| - |X|)^r I(B_1)\} \leqq c_1, \tag{31}$$

where $c_1 = (2^{1-r} - 2^{r-1})Q_r$. Similarly, since the problem is symmetric in X and Y, we may deduce the analogous inequality

$$E\{(|X - Z| - |Y|)^2 I(B_2)\} \leqq c_2, \tag{32}$$

where $B_2 = \{Y^2 < (X - Z)^2\}$, and $c_2 = (2^{1-r} - 2^{r-1})Q_r = c_1$.

Now we shall find a bound for the first expectation on the right side of (29). Suppose, then, that A_1 holds and $Z > 0$, so that $X + Y < 0$. If $X \leqq Y$, implying that $X < 0$, we will have

$$|Z| = Z \leqq \min(Z - X - Y, Z - X + Y) = |Z - X| - |Y|.$$

If $Y \leqq X$, we will have $|Z| \leqq |Z - Y| - |X|$. Together, this implies that

$$|Z|^r \leqq \max((|Z - X| - |Y|)^r, \quad (|Z - Y| - |X|)^r). \tag{33}$$

Similarly, if A_1 holds and $Z < 0$, we will again be able to arrive at equation (33). Thus, (33) will hold whenever A_1 holds. If we now write the first expectation on the right side of (29) as

$$\begin{aligned} Em_1 I(A_1) &= Em_1 I(A_1 B_1 B_2) + Em_1 I(A_1 B_1 B_2^c) \\ &\quad + Em_1 I(A_1 B_1^c B_2) + Em_1 I(A_1 B_1^c B_2^c), \end{aligned} \tag{34}$$

the first term will be bounded by $c_1 + c_2$, the second term by c_1, the third term by c_2, and the last term will vanish. Thus we find that

$$Em_1 I(A_1) \leqq 2(c_1 + c_2). \tag{35}$$

Next we shall bound the second expectation on the right side of (29). Suppose that A_2 holds and $Z > 0$, so that $X + Y - Z < 0$. If $Y > 0$, then

$$|X + Y - Z| = Z - X - Y = |Z - X| - |Y|.$$

If $Y < 0$ and $X > 0$, then $|X + Y - Z| = |Z - Y| - |X|$. But if $Y < 0$ and $X < 0$, then $|Z| = Z \leqq \max(Z - X + Y, Z + X - Y) = \max(|Z - X| - |Y|, |Z - Y| - |X|)$. Combining these cases we see that

$$m_2 \leqq \max((|Z - X| - |Y|)^r, (|Z - Y| - |X|)^r). \tag{36}$$

If A_2 holds and $Z < 0$, equation (36) may be proved in a similar manner. Thus, (36) will hold whenever A_2 holds. Proceeding as in the previous paragraph, we may prove that

$$Em_2 I(A_2) \leqq 2(c_1 + c_2). \tag{37}$$

Equations (29), (35), and (37) together imply that

$$2^r g^r Q_r \leqq \frac{2^{r+1} - 2^{2r}}{2^r - 1} Q_r + (c_1 + c_2) 4\left(\frac{2^r}{2^r - 1}\right), \tag{38}$$

which, after replacing c_1 and c_2 by their values and canceling Q_r, reduces to

$$2^r g^r \leqq \frac{2^{r+1} - 2^{2r}}{2^r - 1} + 8\left(\frac{2^r}{2^r - 1}\right)(2^{1-r} - 2^{r-1}) \tag{39}$$

which is valid for all r between zero and one. It must also be valid in the limit as $r \rightarrow 1$, which implies $g = 0$, finishing the proof.

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