# A. I. KHINCHIN'S WORK IN MATHEMATICAL PROBABILITY[1]

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Alexander Iacovlevich Khinchin was born in 1894 in the Russian village Kondrovo, of the Kaluga district, where his father was the chief engineer of a paper factory. In 1911 he entered the University of Moscow, with which he continued to be associated throughout his whole life, first as a student and later as an academic teacher and research worker. He died on November 18, 1959.

The mathematical work of Khinchin covers a broad field, including the theory of functions of a real variable, number theory, mathematical probability theory and its applications to statistical physics, queueing problems and information theory. In the present paper, it will only be possible to give a brief and incomplete review of his work on probability and its applications. However, it must be emphasized that this is an artificial limitation, which makes it impossible to give the reader an adequate idea of the remarkable internal unity of his work in the various fields where he was active as a creative mathematician.

Khinchin's earliest papers on probability appeared in 1924. During the 1920's and 1930's he published more than fifty works on probability and its applications, including the remarkable monographs [65] and [92]. In order to appreciate his work during this period at its full value, it is necessary to compare the scientific standing and general character of mathematical probability theory in 1920 and 1940.

About 1920, R. von Mises summed up a critical review of the situation in the statement that "to-day, probability theory is not a mathematical science." There was no satisfactory definition of mathematical probability, and the conceptual foundations of the subject were completely obscure. Moreover, with few exceptions, mainly belonging to the French and Russian schools, writers on probability did not seem aware of the standards of rigour which, in other mathematical fields, were regarded as obvious.

At the end of the 1930's, the picture has radically changed. Mathematical probability theory stands firmly established on an axiomatic foundation. It is a purely mathematical discipline, with problems and methods of its own, conforming to current standards of mathematical rigour, and entering into fruitful relations with other branches of mathematics. At the same time, the fields of applications of mathematical probability are steadily and rapidly growing in

Received February 5, 1962.

[1] *Editorial note:* This article was invited by the editor as an obituary of Khinchin. The Annals is indebted to Professor Jerzy Neyman and to the University of California Press for permission to reproduce here the bibliography of Khinchin's works that appeared in the *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* (1961) **2** 10–15. That *Proceedings* also contains articles on Khinchin by B. V. Gnedenko and by J. L. Doob. Another article by Gnedenko has appeared in *Uspehi Mat. Nauk* **15** 97–104, and one by Gnedenko and A. N. Kolmogorov in *Teor. Veroyatnost. i Primenen.* **5** 3–6.

number and importance. It is true that there are still (and this may apply even today, in 1962) some "pure" mathematicians who tend to look down on the "applied" science of probability. But this attitude will be bound to disappear within a generation.

The tremendous development which thus took place in the twenty years from 1920 to 1940 was, no doubt, a joint effect of the efforts of a large number of mathematicians and statisticians. However, it does not seem unlikely that future historians will ascribe this development, as far as the mathematical side of the subject is concerned, above all to the creative powers of three men: A. I. Khinchin, A. N. Kolmogorov, and P. Lévy. In fact, it may be said that the real turning point came with the publication in 1933 of the two famous monographs by Khinchin and Kolmogorov, and the appearance in 1934 of Lévy's paper on an important class of stochastic processes.

Khinchin's early work in probability was concerned with problems arising in the theory of summation of independent random variables, such as the law of the iterated logarithm, the law of large numbers, and the central limit theorem. Generalizing and modifying this type of problems in various directions, he was then led into other, more or less closely related fields. His works on topics such as the summation of inter-dependent variables, the theory of infinitely divisible distributions, stochastic processes with independent increments, and stationary processes, thus show an organic connection between themselves and may, to a great extent, be traced back to a common origin. The following brief analysis of some of his works should provide sufficient motivation for these remarks.

In the papers [10], [14] and [28], published in 1924 and 1926, the celebrated law of the iterated logarithm was stated and proved for the first time. It is characteristic that the first-mentioned paper deals with a problem in number theory, namely the frequency of digits in binary fractions, while in the two other papers the same law is expressed as a statement on the frequency of successes in a series of independent random trials. In his monograph [65], Khinchin points out that this is only a change of terminology, since one is at liberty to express the same underlying theorem in arithmetic or in probabilistic language. The law as expressed in [10] and [14] was extended to more general cases by various other authors, a fairly general statement being the following:

Let $x_1, x_2, \cdots$ be a sequence of independent and uniformly bounded random variables, such that $Ex_n = 0$ and $Ex_n^2 = \sigma_n^2$. Write $y_n = \sum_1^n x_k$, $s_n^2 = \sum_1^n \sigma_n^2$, and suppose that $s_n \to \infty$ with $n$. Then the relation

$$\limsup_{n\to\infty} \frac{y_n}{\sqrt{2s_n^2 \log \log s_n^2}} \quad 1$$

is true with probability one. A more stringent theorem on the same lines has since been obtained by Feller.

The strong law of large numbers belongs to the same order of ideas as the law of the iterated logarithm. In [36] of 1928, Khinchin considers this law for the case of inter-dependent variables. Using the same notations as above, and writing

$$\tau_n^2 = \sup_{|i-k|=n} |r_{ik}|, \qquad t_n^2 = \sum_1^n \tau_n^2,$$

where $r_{ik}$ denotes the correlation coefficient of $x_i$ and $x_k$, the theorem given in [36] asserts that, if

$$s_n t_n = O(n^{1-\epsilon})$$

for some $\epsilon > 0$, then with probability one

$$\lim_{n\to\infty} (y_n/n) = 0.$$

It will be seen that this contains, in the particular case when the $x_n$ sequence is stationary, a proof of the strong ergodic theorem under fairly general conditions.

Among Khinchin's contributions to the central limit theorem, we shall only mention [79], where he gives a NS condition for the convergence to the normal law of the appropriately normed sum of $n$ independent and identically distributed random variables, as $n \to \infty$. If $F(x)$ denotes the common distribution function of the variables, the condition (found independently also by Feller and Lévy) is

$$\lim_{x\to\infty} x^2 \int_{|y|>x} dF(y) \Big/ \int_{|y|<x} y^2\, dF(y) = 0.$$

The continuous analogy to the cumulated sums of a sequence of independent random variables is a stochastic process with independent increments. About 1930, some important particular cases of this class of processes were known. Thus the Brownian movement process (with normally distributed increments) had been studied by Wiener, while the Poisson process and some of its generalizations had been considered by others, including the present writer. The general form of a process with independent increments was then established in 1932 by Kolmogorov, under some restrictive assumptions, and quite generally in 1934 by Lévy. In the monograph [65] of 1933, Khinchin gave an account of the properties of the particular processes mentioned above, and also considered certain important processes with non-independent increments, which play an important part in the mathematical treatment of diffusion problems. In this connection, he was able to make use of the now well-known Kolmogorov equations, which had then only recently been discovered.

An important part in the theory of stochastic processes with independent increments is played by the class of probability distributions known as infinitely divisible. A distribution is said to belong to this class if, for every $n = 1, 2, \cdots$, it may be represented as the convolution of $n$ identical distributions. Some of the most important discoveries of Khinchin were connected with this class of distributions. Thus in [91] he proved the following beautiful theorem:

Suppose that for every $n = 1, 2, \cdots$,

$$y_n = x_{n1} + x_{n2} + \cdots + x_{nk_n},$$

so that $y_n$ is the sum of the $k_n$ random variables $x_{nk}$, which are assumed to be independent, and such that

$$\sup_{k \leqq k_n} P\{|x_{nk}| > \epsilon\} \to 0$$

as $n \to \infty$, for any fixed $\epsilon > 0$. A NS condition that a given distribution function $F(x)$ may be represented as the limiting distribution function of a sequence of the form $y_1, y_2, \cdots$ is then that $F(x)$ should be infinitely divisible.

This theorem forms one of the main starting points of the modern theory of limit distributions for sums of independent random variables, as treated in the well-known book by Gnedenko and Kolmogorov.

A particularly interesting subclass of the class of all infinitely divisible distributions is formed by the so called stable distributions. A distribution function $F(x)$ is called stable if, to any positive $a_1$, $a_2$, and any real $b_1$, $b_2$, there correspond a positive $a$ and a real $b$ such that

$$F(a_1x + b_1) * F(a_2x + b_2) = F(ax + b),$$

where the sign $*$ denotes convolution. Khinchin wrote several papers on stable distributions and allied subjects. Particularly remarkable is the joint paper [86], where Khinchin and Lévy gave the general formula for the characteristic function of a stable distribution.

In the book [92], Khinchin developed the theory of infinitely divisible and stable distributions, and their applications to the theory of limit distributions for sums of independent random variables, as far as this was known in 1938, when the book appeared. In particular, one finds here the theorem that the class of all stable distributions is identical with the class of limiting distributions for normed sums of the form $(x_1 + x_2 + \cdots + x_n)/A_n - B_n$, where the $x_k$ are independent and identically distributed random variables, while $A_n > 0$ and $B_n$ are constants. An interesting detail, typical of Khinchin's interest in function theory and number theory, is the elegant example given on p. 35 of the book of an infinitely divisible distribution connected with the Riemann zeta function.

Perhaps the greatest of all Khinchin's achievements in probability theory was his work on stationary stochastic processes and their applications to statistical physics. During his investigations concerning the validity of various forms of the law of large numbers for sequences of inter-dependent variables he was led to consider, e.g., in [52] and [60], a certain type of dependence which he called stationary, where the probability laws are in some sense invariant under a translation in time. For sequences of stationary variables, the law of large numbers turned out to be closely related to the ergodic theorems of statistical mechanics, and Khinchin gave in [58] and [62] proofs of certain theorems generalizing the previously known ergodic theorems due to Birkhoff and others.

In the important paper [67], Khinchin then proceeded to consider a stochastic process with continuous time, of the type which is now known as second order stationary. In a slightly modernized form, he obtained in this paper the following results. Let $x(t)$, for every real $t$, be a complex-valued random variable such that

$$Ex(t) = 0, \qquad Ex(t)\overline{x(u)} = r(t - u),$$

and let it be assumed that the covariance function $r(t)$ is continuous at $t = 0$. Then $r(t)$ is a positive-definite function with the spectral representation

$$r(t) = \int_{-\infty}^{\infty} e^{it\lambda} \, dF(\lambda),$$

where $F(\lambda)$ is real, bounded and never decreasing. From this representation, an ergodic mean value theorem for $x(t)$ is deduced.

These basic results of Khinchin had a great influence on the subsequent powerful development of the theory and applications of stationary processes. By the work of Kolmogorov, Karhunen and others, it is now known that this theory may be expressed as a chapter in Hilbert space geometry. In particular, the above spectral representation of the covariance function, and the corresponding representation for the random variable $x(t)$ given by the present writer, are equivalent to certain Hilbert space theorems. The possibility of a probability approach to these problems has opened up interesting lines of research.

Khinchin's interest in the problems of statistical physics was by no means limited to ergodic theory and the applications of stationary processes. In a number of minor papers, and in the monographs [113] and [136], he developed his ideas on the mathematical foundations of classical statistical mechanics, and of quantum statistics. In both these fields, he emphasized the fundamental importance of the mathematical tools provided by probability theory, particularly the local limit theorems appearing in the theory of summation of independent random variables.

The works [146], [148] and [149] deal with the theory of queues, and contain among other things a very interesting analysis of certain generalizations of the simple Poisson process. Finally, during his last years, Khinchin also studied the theory of information and gave in [143] and [150] a clear exposition of the mathematical foundations of this theory.

The work of Khinchin is so many-sided, and so rich in new ideas, not yet fully explored, that it will certainly during many years continue to have a profound influence on the development of the mathematical theory of probability.

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