zero entries in $m \varepsilon M$ determines whether or not $m \varepsilon I$. Now $A'(1), A'(2), \cdots$ is an increasing sequence of subsets of $M'$, which has less than $2^{n^2}$ elements, so there must be a smallest $r$, $1 \leqq r \leqq 2^{n^2}$, such that $A'(r) = A'(r + 1)$. To complete the proof we need only show that $A'(r) = A'$ since if $A(r) \subset A(2^{n^2}) \subset I$ then $A' = A'(r) \subset I'$ so $A \subset I$. Thus we need only prove that if $k \geqq 1$ and $A'(k) = A'(k + 1)$ then $A'(k + 1) = A'(k + 2)$. Now if $m \varepsilon A(k + 2)$ then $m = bc$, where $b \varepsilon A(k + 1)$ and $c \varepsilon A(1)$ so there exists a $d \varepsilon A(k)$ with $b' = d'$ so $m' = (dc)' \varepsilon A'(k + 1)$ and the proof is complete.

We conclude with three comments. Clearly $A'(1)$ determines whether or not $A \subset I$ so that if $A(1)$ is an infinite set, which is not the case for indecomposable channels, then $A(1)$ may, for the purpose of determining whether or not $A \subset I$, be replaced by any finite $B \subset M$ with $B' = A'(1)$. If $m \varepsilon A$ has a state which is periodic with period $d > 1$ then $m^d \not\varepsilon I$ and $m^d \varepsilon A$ so $A \not\subset I$. For any $A(1)$, $(A(2^{n^2}))' = A'$.

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# NOTE ON QUEUES IN TANDEM[1]

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**1. Introduction.** Assume that $Q_k$, $k = 1, 2, \cdots, m$, is a single server queue where customers are served with an exponential service time distribution of mean $1/\mu_k$. We shall assume that the $j$th customer, $C_j$, arrives at $Q_1$ at time $t_j$, where $\{t_j\}$ are the events of a Poisson process, and $\lambda$ the number of arrivals per unit time. The queues $Q_k$ are arranged in tandem; that is, after $C_j$'s service at $Q_k$ is completed he proceeds to $Q_{k+1}$ and joins the queue there. We shall extend a result of our previous paper [1] for the foregoing situation.

Let $T_{jk}$ denote $C_j$'s waiting time at $Q_k$, *including* the duration of $C_j$'s service at $Q_k$. The purpose of the present note is to show, using the results of [1], that under "equilibrium" conditions the probabilistic description of the random

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variables $T_{jk}$, $1 \leqq k \leqq m$, ($j$ fixed) is exceedingly simple. In order to automatically bring about the "equilibrium" conditions we shall assume that all the numbers $\rho_k = \lambda/\mu_k$ are strictly less than 1, and that the arrival process $\{t_j\}$ is defined from time $-\infty$ on. To achieve the latter we may take $t_0 = 0$, and then generate $0 < t_1 < t_2 < \cdots$, and $0 > t_{-1} > t_{-2} > \cdots$ as independent Poisson processes. Under the preceding assumptions we shall prove the following in the remainder of this note.

THEOREM 1. *For each fixed $j$, the random variables $T_{jk}$, $k = 1, 2, \cdots, m$, are mutually independent. The distribution of $T_{jk}$ is the convolution*

$$\Pr\{T_{jk} \leqq x\} = W(x)^* S(x), \qquad 1 \leqq k \leqq m, \tag{1.1}$$

*where*

$$W(x) = \begin{cases} 1 - \rho_k e^{(\lambda-\mu_k)x}, & x \geqq 0 \\ 0, & x < 0 \end{cases}$$

*and*

$$S(x) = \max [0, 1 - e^{-\mu_k x}].$$

As usual, let the non-negative integral valued stochastic processes $n_k(t)$, $1 \leqq k \leqq m$, be defined as the *state* of $Q_k$ at time $t$; that is, $n_k(t)$ is the total number of customers present at $Q_k$ at time $t$, the customer, if any, being served included. The process $n_1(t)$ is of course a birth-death process. The following result was stated as a part of Theorem 4 of [1], and the first proof goes back to Burke ([2], page 259, or [3], page 45).

THEOREM A. *The sequence of departure instants from $Q_1$ constitutes a Poisson process. If a departure occurs at $t = u$ then $n_1(u + 0)$ is independent of the sequence of past departure instants.*

As an immediate corollary of Theorem A we note that the arrivals at $Q_k$, $k = 1, 2, \cdots, m$ are Poisson processes, each with $\lambda$ happenings per unit time. The processes $n_k(t)$ are all birth-death processes with birth rate $\lambda$, and respective death rates $\mu_k$. This implies (1.1) by the classical result for queues with exponential service time and Poisson arrivals (see e.g. [3], page 41, Theorem 12). Furthermore, by Jackson's theorem ([2], page 262) $n_k(t)$, $1 \leqq k \leqq m$, are mutually independent for each fixed $t$, and

$$\Pr\{n_k(t) = r\} = (1 - \rho_k)\rho_k^r, \qquad r = 0, 1, 2, \cdots. \tag{1.2}$$

Hence, so far as waiting times of customers $C_j$, $j > 0$, are concerned there would be no change if we restricted arrivals to $t > 0$, with the initial condition that $\{n_k(0)\}$ are mutually independent, and distributed as in (1.2).

**2. Proof of Theorem 1.** For the special case $m = 2$, Theorem 1 becomes Theorem 5 of [1]. Thus, in fact, the pair of random variables $T_{jk}$, $T_{j,k+1}$ is independent for each $k$. We shall start from the beginning, however, proving the more general Theorem 1 directly by the method used in [1].

Let us introduce the following additional notation.
$s_{jk}$ = duration of $C_j$'s service period at $Q_k$
$t_j'$ = instant at which $C_j$ departs from $Q_1$ (upon completion of service)
$\tau_j' = t_j' - t_{j-1}'$

Consider the state $n_1(t_j' + 0)$ which $C_j$ leaves behind at $Q_1$ upon departure. Since this state is the number of customers (Poisson arrivals) who have appeared at $Q_1$ while $C_j$ was waiting to be served we may consider $n_1(t_j' + 0)$ to be determined by a random device a-posteriori to observing $T_{j1}, T_{j2}, \cdots, T_{jm}$, according to the law

$$\Pr\{n_1(t_j' + 0) = r \mid T_{j1} = \alpha_1, T_{j2} = \alpha_2, \cdots, T_{jm} = \alpha_m\} = e^{-\lambda\alpha_1}(\lambda\alpha_1)^r/r!, \qquad r = 0, 1, \cdots.$$

Hence, for any complex number $z$,

$$\begin{aligned} &E[z^{n_1(t_j'+0)} \mid T_{j2} = \alpha_2, T_{j3} = \alpha_3, \cdots T_{jm} = \alpha_m] \\ &\quad = \int_0^\infty e^{\lambda\beta(z-1)}\, d_\beta \Pr\{T_{j1} \leqq \beta \mid T_{j2} = \alpha_2, T_{j3} = \alpha_3, \cdots, T_{jm} = \alpha_m\}. \end{aligned} \tag{2.1}$$

Let us introduce the probability spaces $\Omega_j = \{\omega\}, j = 0, \pm1, \pm2, \cdots$, consisting of elementary events $\omega = (\tau_j', \tau_{j-1}', \tau_{j-2}', \cdots; s_{j2}, s_{j-1,2}, s_{j-2,2}, \cdots; s_{jm}, s_{j-1,m}, s_{j-2,m}, \cdots)$. The measure on the finite dimensional subsets of $\Omega_j$ is defined so as to be consistent with the fact that $\tau_i'$ has an exponential distribution with mean $1/\lambda$, and $s_{ik}$ has an exponential distribution with mean $1/\mu_k$, and consistent with the fact that the components of $\omega$ are mutually independent. This measure is extended to the Borel field $\mathfrak{F}_j$ generated by the finite dimensional subsets in the usual way.

By Theorem A, $n_1(t_j' + 0)$ is independent of $\mathfrak{F}_j$. On the other hand, $T_{jk}$, $2 \leqq k \leqq m$, is a random variable on $(\Omega_j, \mathfrak{F}_j)$, because $T_{jk}$, $2 \leqq k \leqq m$, is completely determined by the history of arrivals at $Q_2$ up to and including $t_j'$, and by specifying the numbers $s_{i\gamma}$, $-\infty < i \leqq j$, $2 \leqq \gamma \leqq k$. Hence $n_1(t_j' + 0)$ is independent of the vector $(T_{j2}, T_{j3}, \cdots, T_{jm})$. Therefore the left side of (2.1) is constant as a function of $\alpha_2, \alpha_3, \cdots, \alpha_m$. But the right side of (2.1) is a Laplace transform. By the uniqueness of the inverse Laplace transform it follows that

$\Pr\{T_{j1} \leqq \beta \mid T_{j2} = \alpha_2, T_{j3} = \alpha_3, T_{jm} = \alpha_m\}$ is constant as a function of $\alpha_2$, $\alpha_3, \cdots, \alpha_m$.

Thus,

$$\begin{aligned} &\Pr\{T_{j1} \leqq \beta_1, T_{j2} \leqq \beta_2, \cdots, T_{jm} \leqq \beta_m\} \\ &\quad = \Pr\{T_{j1} \leqq \beta_1\} \Pr\{T_{j2} \leqq \beta_2, \cdots, T_{jm} \leqq \beta_m\}. \end{aligned} \tag{2.2}$$

By reapplying the argument leading to (2.2), in turn, to the sets of queues

$(Q_2, Q_3, \cdots, Q_m)$, $(Q_3, Q_4, \cdots, Q_m)$, etc., we finally obtain

$$\Pr\{T_{j1} \leqq \beta_1, T_{j2} \leqq \beta_2, \cdots, T_{jm} \leqq \beta_m\} = \prod_{k=1}^{m} \Pr\{T_{jk} \leqq \beta_k\}$$

as was to be shown.

Finally, it should be mentioned that if the waiting times are defined so as *not* to include the service times, that is, as the quantities $T_{jk} - s_{jk}$, the question of mutual independence of these quantities for $k = 1, 2, \cdots, m$ is apparently an open problem.

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# A NOTE ON THE RE-USE OF SAMPLES[1]

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There are situations in statistical estimation in which the basic underlying distribution is invariant under some family of transformations. In this note a theorem similar to the Blackwell-Rao Theorem is proved demonstrating that this additional structure can sometimes be exploited to improve an estimator.

Theorem. *Consider a random variable $x$, sample space $X$, $\sigma$-algebra $\mathfrak{X}$, probability measure $P(\ )$. Suppose that $G$ is a set of measure-preserving transformations for the measure $P$,* i.e. *$P(gA) = P(A)$ for all $A$ in $\mathfrak{X}$, $g$ in $G$. Let $\mu(\ )$ be a measure of total mass* 1, *defined on a $\sigma$-algebra $\mathfrak{G}$ of subsets of $G$. Let $\phi(x)$ be an estimator such that $\phi(gx)$ is $\mathfrak{G} \times \mathfrak{X}$ measurable.*

(i) If $\phi(x)$ is an unbiased estimator of $\theta$ then,

$$\gamma(x) = \int_G \phi(gx)\, d\mu(g)$$

is also an unbiased estimator of $\theta$.

(ii) If $\phi(x)$ takes values in a $k$-dimensional space and has an associated real-valued, convex, bounded from below loss function $W[\phi(x)]$, such that $W[\phi(gx)]$ is $\mathfrak{G} \times \mathfrak{X}$ measurable then, $R_\phi \geqq R_\gamma$ where $R$ is the associated risk function, and in particular the ellipsoid of concentration of $\gamma$ is everywhere

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