A CHARACTERIZATION OF THE UNIFORM DISTRIBUTION ON A COMPACT TOPOLOGICAL GROUP

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1. Introduction. The random variables discussed in this paper take their values in a compact commutative topological group with a countable basis, e.g., the reals (mod 1). The major result is Theorem 4.1, a characterization of the uniform distribution on such a space. It is the similarity of this characterization to that of Skitovic [4] of the normal distribution on the real line which makes it of special interest. The uniform distribution on the spaces considered here seem to play the same central role that the normal distribution does on Euclidean spaces. For example, Theorem 2.3 is a central limit theorem for such space.

Many of the results stated without proof may be found in a 1940 paper by Kawada and Ito [1], which considers the case in which the group operation is non-commutative. The results presented in Sections 2 and 3 are easily obtained, some are well known, and proofs are sometimes omitted. A major purpose of these sections is to fully acquaint the reader with the background necessary to Section 4. Section 5 contains counterexamples which show the necessity of some of the hypotheses of the characterization.

2. Some preliminaries. Let Γ be a compact commutative topological group with a countable basis, the group operation being addition with the symbol $\oplus$. Let $\hat{\Gamma}$ be the character group of Γ; $\hat{\Gamma}$ is also a topological group. The compactness of Γ implies the discreteness of $\hat{\Gamma}$. Denote the value of a character $\hat{x}$ at a point $x \varepsilon \Gamma$ by $(x, \hat{x})$. Let the identities of Γ and $\hat{\Gamma}$ be e and $\hat{e}$ respectively. The character group of the cartesian product of groups such as Γ is the cartesian product of the corresponding character groups. The $\sigma-$ field for each space Γ will always be the class of Borel sets. Examples of Γ and the corresponding $\hat{\Gamma}$ are:

(1) I, the reals (mod 1), and $\hat{I}$, the multiplicative group of all functions $\exp(2\pi inx)$, $x \varepsilon I$, $n = 0, \pm 1, \cdots$.

(2) The product spaces $I^{(n)}$, $I^{(\infty)}$ and $\hat{I}^{(n)}$, $\hat{I}^{(\infty)}$.

(3) Λ_k, the integers (mod k) and $\hat{\Lambda}_k$, the functions $\exp(2\pi itx/k)$, $t = 0, \cdots, n-1$.

(4) Q, the n-adic integers and $\hat{Q}$ the functions $\exp(2\pi im \sum_{k=0}^{h-1} a_k n^k/m_k)$, m and h positive integers, $a_0, a_1, \cdots \varepsilon Q$.

Let (Ω, S, P) be a probability space, let ξ be a random variable defined on Ω and taking values in Γ and let $\mu = P\xi^{-1}$ be the distribution of ξ in Γ. If μ is a Haar measure on Γ, then μ is called the uniform distribution and ξ is said to be

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uniformly distributed (U.D.) in Γ. A random variable ξ with distribution μ on Γ is uniquely determined by its characteristic function

$$\phi_\xi(\hat{x}) = E(\xi, \hat{x}) = \int_\Gamma (x, \hat{x})\, d\mu,$$

the Fourier transform of μ, defined on Γ. A sequence of distributions $\mu_1, \mu_2, \cdots$ converges to a distribution μ on Γ if and only if the corresponding sequences of Fourier transforms converges to the Fourier transform of μ. ξ is U.D. on Γ if and only if $\phi(\hat{x}) = 0$ for all $\hat{x} \neq \hat{e}$. This fact makes it easy to find many interesting results. For example, from [1],

THEOREM 2.1. *Let ξ_1 and ξ_2 be independent random variables in Γ. If ξ_1 is* U.D. *in Γ then $\xi_1 + \xi_2$ is also. Moreover, ξ_2 and $\xi_1 + \xi_2$ are independent.*

A random variable ξ in Γ satisfying $P(\xi \varepsilon C) = 1$ for some coset C of a proper subgroup of Γ will be said to have property S. Theorems 2.2 and 2.3 are simple consequences of Lemmas 2.4 and 2.5. Theorem 2.3 might be called the central limit theorem for compact topological groups.

THEOREM 2.2. *Let ξ_1 and ξ_2 be independent random variables taking values in Γ and suppose ξ_2 does not have property S. Then if ξ_2 and $\xi_1 + \xi_2$ are independent, ξ_1 is* U.D. *in Γ.*

THEOREM 2.3. *Let $\xi_1, \xi_2, \cdots$ be identically and independently distributed, none having property S. Then $\eta_n = \sum_{n=1}^{n} \xi_j$, for $n \to \infty$, is asymptotically* U.D.

LEMMA 2.4 ([3], pp. 126, 137). *Let Γ_1 be a subgroup of Γ and let $\hat{\Gamma}_2$ be a subgroup on $\hat{\Gamma}$ consisting of all elements $\hat{x} \varepsilon \hat{\Gamma}$ with $(x, \hat{x}) = 1$ for all $x \varepsilon \Gamma_1$. Then the factor group $\hat{\Gamma}_1 = \hat{\Gamma}/\hat{\Gamma}_2$ is the character group of Γ_1 and $\hat{\Gamma}_2$ is the character group of the factor group $\Gamma_2 = \Gamma/\Gamma_1$. $\Gamma_1 = \Gamma$ if and only if $\hat{\Gamma}_2 = \{\hat{e}\}$.*

LEMMA 2.5. *$|\phi_\xi(\hat{x})| = 1$ for some $\hat{x} \varepsilon \hat{\Gamma}$, $\hat{x} \neq \hat{e}$, if and only if ξ has property S.*

PROOF: Suppose $\phi_\xi(\hat{x}_0) = e^{i\alpha}$ for $\hat{x}_0 \neq \hat{e}$ and some real number α. Then from $|(x, \hat{x}_0)| = 1$, the complex variable $(\xi, \hat{x}_0)$ must equal $e^{i\alpha}$ with probability one; thus for $C = \{x: (x, \hat{x}_0) = e^{i\alpha}\}$, we have $\mu(C) = 1$. Let $B = \{x: (x, \hat{x}_0) = 1\}$; since $\hat{x}_0 \neq \hat{e}$, B is a proper subgroup of Γ. Further, for $x_1, x_2 \varepsilon C$, we have

$$(x_1 - x_2, \hat{x}_0) = (x_1, \hat{x}_0)\overline{(x_2, \hat{x}_0)} = 1;$$

thus C is a coset of B.

Vice versa, let B be a proper subgroup of Γ and let $x_1 + B = C$ denote a coset of B such that $P\{\xi \varepsilon C\} = 1$. Now Lemma 2.4 implies that there exists an $\hat{x}_1$ such that $(x, \hat{x}_1) = 1$ for $x \varepsilon B$, thus $(x, \hat{x}_1) = (x_1, \hat{x}_1)$ for $x \varepsilon C$; hence $E\{\xi_1, \hat{x}_1\} = (x, \hat{x}_1) = e^{i\alpha}$ for some real α.

3. The uniform distribution under a homomorphism. Theorem 3.1 will prove valuable to the consideration of the problem of when certain functions of U.D. random variables are also U.D. Let Γ and G be two separable locally compact commutative topological groups and let Γ be compact. Then a homomorphic image K of Γ into G is necessarily a compact topological group (in its relative topology).

THEOREM 3.1. *Let T be a homomorphism of Γ into G and let μ be the uniform distribution on Γ. Then the distribution $\mu^* = \mu T^{-1}$ induced by T on $K = T\Gamma$ is also uniform (μ^* is a Haar measure on K).*

PROOF: For $y = T(x) \varepsilon K$ and a Borel set F of G,

$$\mu T^{-1}(F + y) = \mu(T^{-1}(F) + x) = \mu T^{-1}(F).$$

COROLLARY 3.2: *Suppose that $\xi_1, \cdots, \xi_n$ are independent* U.D. (I.U.D.) *random variables in Γ. Let $\eta_i = \sum_{j=1}^n a_{ij}\xi_j (i = 1, \cdots, m)$, where the a_{ij} are integers. Then the random variable $\eta = (\eta_1, \cdots, \eta_m)$ is* U.D. *in some compact subgroup of the m-fold product $\Gamma \times \cdots \times \Gamma = \Gamma^{(m)}$.*

In short, the homomorphic image of a U.D. random variable is also U.D. on some space K. The only problem left will be to find this image space K. The following is a simple consequence of Corollary 3.2 and Lemma 2.4 and uses the same notation as Corollary 3.2.

THEOREM 3.3. *$\eta_1, \cdots, \eta_n$ are* I.U.D. *if and only if for $\hat{x}_i \varepsilon \hat{\Gamma}$ $(i = 1, \cdots, m)$,*

$$\prod_{i=1}^m \hat{x}_i^{a_{ij}} = \hat{e} \text{ for } j = 1, \cdots, n \text{ implies that } \hat{x}_i = \hat{e},\ i = 1, \cdots, m. (*)$$

PROOF: By Corollary 3.2, $\eta = (\eta_1, \cdots, \eta_m)$ is U.D. on a subgroup Γ_1 of $\Gamma^{(m)}$. Hence we need only prove that $\Gamma_1 = \Gamma^{(m)}$ if and only if (*) holds. Let $\hat{\Gamma}_2$ be a subgroup of $\hat{\Gamma}^{(m)}$ consisting of all $\hat{x}$ such that $(x, \hat{x}) = 1$ for all $x \varepsilon \Gamma_1$. By Lemma 2.4 $\hat{\Gamma}_1 = \hat{\Gamma}^{(m)}$ if and only if $\hat{\Gamma}_2$ contains only the element $\hat{e}^{(m)} = (\hat{e}, \cdots, \hat{e})$. For $x = (y_1, \cdots, y_m) \varepsilon \Gamma_2$ we have $y_i = \sum_{j=1}^n a_{ij}x_j$ for $i = 1, \cdots, m$ with $x_j \varepsilon \Gamma$, $(j = 1, \cdots, m)$. Then for $(\hat{x}_1, \cdots, \hat{x}_m) \varepsilon \hat{\Gamma}_2$ the following statements are equivalent:

(1) $\prod_{i=1}^m (\sum_{j=1}^n a_{ij}x_j, \hat{x}_i) = \prod_{j=1}^n (x_j, \prod_{i=1}^m \hat{x}_i^{a_{ij}}) = 1$

(2) $\prod_{i=1}^m \hat{x}_i^{a_{ij}} = \hat{e}$ for $j = 1, \cdots, n$.

This proves the theorem.

A compact topological group is connected if and only if its character group has no elements of finite order. The spaces I, $I^{(n)}$ and $I^{(\infty)}$ are each connected. From Theorem 3.3, using the same notation, we have

COROLLARY 3.4. *Suppose that Γ is connected. Then $\eta_1, \cdots, \eta_m$ are* I.U.D. *if and only if $A = (a_{ij})$ has rank m.*

PROOF. Suppose A has rank $r = m$ and that for certain $\hat{x}_i \varepsilon \hat{\Gamma}, \prod_{i=1}^m \hat{x}_i^{a_{ij}} = \hat{e}$ for $j = 1, \cdots, n$. Now, for fixed $i = i_0$ there exist integers $h_1, \cdots, h_n$ and a constant $C_{i_0} \neq 0$ such that $\sum_{j=1}^n a_{ij}h_j = C_{i_0}$ for $i = i_0$ and is zero otherwise. Then

$$\hat{e} = \prod_{j=1}^n \left(\prod_{i=1}^m \hat{x}_i^{a_{ij}}\right)^{h_j} = \hat{x}_{i_0}^{C_{i_0}}.$$

Hence, since Γ has no elements of finite order, $\hat{x}_{i_0} = \hat{e}$ for $i_0 = 1, \cdots, m$. It follows from Theorem 3.3 that $\eta_1, \cdots, \eta_m$ are I.U.D. in Γ.

Now suppose A has rank $r < m$. Then there exist integers $k_1, \cdots, k_m$, not all zero with $\sum_{i=1}^m k_i a_{ij} = 0$ for $j = 1, \cdots, n$. Then for arbitrary fixed $\hat{x} \neq \hat{e}$, let $\hat{x}_i = \hat{x}^{k_i}$, $i = 1, \cdots, n$. Then $\hat{x}_i \neq \hat{e}$ for at least one i. Further,

$$\prod_{i=1}^{n} \hat{x}_i^{a_{ij}} = \hat{x}^{\sum_{i=1}^{n} k_i a_{ij}} = \hat{e} \text{ for all } j.$$

Hence, from Theorem 3.3, $\eta_1, \cdots, \eta_m$ are not I.U.D.

The following theorem follows by similar arguments.

THEOREM 3.5. *Suppose that* Γ *is a cyclic group* (e.g. Γ *is either* I *or* Λ_k). *Then* $\xi_1, \cdots, \xi_n$ *are* I.U.D. *in* Γ *if and only if* $\sum_{j=1}^{n} t_j \xi_j$ *is* U.D. *for every set of integers* $t_1, \cdots, t_n$ *such that* $\hat{x} \in \hat{\Gamma}$, $\hat{x}^{t_j} = \hat{e}$ $(j = 1, \cdots, n)$ *implies* $\hat{x} = \hat{e}$.

COROLLARY 3.6. $\xi_1, \cdots, \xi_n$ *are* I.U.D. *in* $\Gamma = I$ *if and only if* $\sum_{j=1}^{n} t_j \xi_j$ *is* U.D. *in* I *for each set of integers* $t_1, \cdots, t_n$ *not all zero.*

COROLLARY 3.7. $\xi_1, \cdots, \xi_n$ *are* I.U.D. *in* $\Gamma = \Lambda_k$ *if and only if* $\sum_{j=1}^{n} t_j \xi_j$ *is* U.D. *in* Λ_k *for each set of integers* $t_1, \cdots, t_n$ *with* g.c.d. $(t_1, \cdots, t_n, k) = 1$.

PROOF: Let $\hat{x}_0$ be a generator of $\hat{\Lambda}_k$, let d be an arbitrary integer, $0 \leq d < k$, let $t_1, \cdots, t_n$ be integers, and let $\lambda = \text{g.c.d.}\ (t_1, \cdots, t_n, k)$. By Theorem 3.5 we must show that the following are equivalent:

(1) $(\hat{x}_0^d)^{t_j} = \hat{e}, j = 1, \cdots, n$ implies $d = 0$.

(2) $\lambda = 1$.

We may also write

(1) $dt_j = 0 \pmod k$ for all j implies $d = 0$. If $\lambda = 1$ then $d\lambda = d = 0 \pmod k$ implies $d = 0$, a conclusion stronger than (1). If $\lambda \neq 1$, let $d = k/\lambda$ and (1) does not hold.

4. A characterization of the uniform distribution. Let $\xi_1, \cdots, \xi_n$ be I.U.D. random variables in the connected compact group Γ. Let $A = (a_{ij})$ be an $n \times n$ non-singular matrix of integers and define $\eta_i = \sum_{j=1}^{n} a_{ij}\xi_j$ for $i = 1, \cdots, n$. Then, from Corollary 3.4, $\eta_1, \cdots, \eta_n$ are I.U.D. if and only if A is non-singular. This property of U.D. random variables is an interesting one, and it would be of further interest to discover the degree to which it characterizes the uniform distribution. Theorem 4.1 partially answers this question, and is the major result of this section. In this connection it may be interesting to mention the following related result due to Skitovic [4]: if $\xi_1, \cdots, \xi_n$ are independent real random variables and a_j, b_j are real constants such that $\eta_1 = \sum_{j=1}^{n} a_j\xi_j$ and $\eta_2 = \sum_{j=1}^{n} b_j\xi_j$ are independent, then each ξ_j with $a_j b_j \neq 0$ is normally distributed. It was this result which suggested the results of this section and the next.

THEOREM 4.1. *Let* Γ *be connected and let* $\xi_1, \cdots, \xi_n$ *be* n *independent random variables in* Γ, *such that for no* j *does* ξ_j *take* (*with a probability one*) *all its values in a fixed coset of a proper* (*compact*) *subgroup of* Γ. *Let*

$$\eta_i = \sum_{j=1}^{n} a_{ij}\xi_j, \; i = 1, \cdots, n,$$

where $A = (a_{ij})$ *is an* $n \times n$ *matrix of integers such that for each* i *at least two* a_{ij} *are non-zero, and* $\Delta = \det A = \pm 1$. *Then, if* $\eta_1, \cdots, \eta_n$ *are independent, we* *have for each j that ξ_j is* U.D. *in* Γ *as soon as its distribution* μ_j *has an absolutely continuous component with respect to the Haar measure* ν *on* Γ.

In proving Theorem 4.1 we need the following theorem, which is a consequence of a general theorem of homomorphism of a normed algebra given by Loomis [2] and is a generalization of the ordinary Riemann-Lebesgue theorem for the real line. We need this theorem only for the case that G is compact. The theorem can be proved for this case by showing first that each complex-valued function on Γ integrable with respect to μ can be ϵ approximated in the mean by a linear combination of characters. This approximation can then be applied to the Radon-Nikodyn derivative of the absolutely continuous part of μ with respect to the Haar measure.

THEOREM 4.2. *Let μ be a distribution on a separable commutative locally compact topological group having an absolutely continuous component with respect to the Haar measure ν on G. Let $\hat{x}_1, \hat{x}_2, \cdots$ be a sequence of characters of $\hat{G}$ such that, for $n \to \infty$, $\hat{x}_n$ is eventually outside every compact set in $\hat{G}$ (every finite set if G is compact, thus $\hat{G}$ is discrete). Then*

$$\limsup_{n\to\infty} \left| \int_G (x, \hat{x}_n)\, d\mu \right| = b < 1.$$

Moreover, if $\mu \ll \nu$, then $b = 0$.

PROOF OF THEOREM 4.1: Let $\hat{x}_0$ be an arbitrary fixed non-trivial character of Γ. Let $\phi_j(t) = E\{(t\xi_j, \hat{x}_0)\}$ for integer t. Let $(t_1, \cdots, t_n)$ be a set of n integers. Then by the independence of $\eta_1, \cdots, \eta_n$ and $\xi_1, \cdots, \xi_n$ we have

(1) $\prod_{j=1}^n \phi_j(\sum_{i=1}^n a_{ij}t_i) = \prod_{j=1}^n \prod_{i=1}^n \phi_j(a_{ij}t_i)$.

We may assume without loss of generality that $\Delta = 1$, for otherwise we may replace η_n by $-\eta_n$, thus changing the sign of Δ. Let Δ_{ij} denote the cofactor of the element a_{ij} in the matrix (a_{ij}) and let $t_i^{(k)} = \Delta_{ik}t$ $(k, i = 1, \cdots, n)$, where t is an integer. Then

$$\sum_{i=1}^n a_{ij}t_i^{(k)} = \sum_{i=1}^n a_{ij}\Delta_{ik}t = \delta_j^k t.$$

Applying (1) for $t_i = t_i^{(k)}$, we obtain

(2) $\phi_k(t) = \prod_{j=1}^n \prod_{i=1}^n \phi_j(a_{ij}\Delta_{ik}t)$, (since $\phi_j(0) = 1$ for $j = 1, \cdots, n$).

For all j and integers t, $|\phi_j(t)| \leqq 1$; so we have

(3) $|\phi_k(t)| \leqq \prod_{i=1}^n |\phi_k(a_{ik}\Delta_{ik}t)|$ for all integers t and $k = 1, \cdots, n$. Let k be a fixed integer, $1 \leqq k \leqq n$ and assume that ξ_k is not U.D., i.e., there exists a character $\hat{x}_1 \neq \hat{e}$ such that $E\{(\xi_k, \hat{x}_1)\} \neq 0$. Since $\hat{x}_0$ was an arbitrary non-trivial character, we may choose $\hat{x}_0 = \hat{x}_1$, hence $\phi_k(1) = E\{(\xi_k, \hat{x}_1)\} \neq 0$.

We assert that at least two among the integers $a_{ik}\Delta_{ik}$ $(i = 1, \cdots, n)$ are non-zero. Suppose this is not so. We have $\sum_{i=1}^n a_{ik}\Delta_{ik} = \Delta = 1$, so exactly one $a_{ik}\Delta_{ik}$ is non-zero, say $a_{i_0k}\Delta_{i_0k}$: then $a_{i_0k}\Delta_{i_0k} = 1$ and $a_{ik}\Delta_{ik} = 0$ for $i \neq i_0$. Hence, from (2), applied for $t = 1$,

(4) $\prod_{j\neq k} \prod_{i=1}^n |\phi_j(a_{ij}\Delta_{ik})| = 1$.

From Lemma 2.5 and the fact that no ξ_j is entirely restricted to a coset of a proper subgroup of Γ, $|\phi_j(t)| < 1$ for all j and $t \neq 0$, so that (4) holds only when $a_{ij}\Delta_{ik} = 0$ ($j \neq k$, $i = 1, \cdots, n$). From $\Delta_{i_0k} \neq 0$ it follows that $a_{i_0j} = 0$ for $j \neq k$, which violates the assumption that $a_{i_0j} \neq 0$ for at least two j.

Hence we may assume that $a_{ik}\Delta_{ik} \neq 0$ for $i = i_1$ and $i = i_2$ ($i_1 \neq i_2$). We have from (3)

(5) $|\phi_k(t)| \leqq |\phi_k(a_{i_1k}\Delta_{i_1k}t)\phi_k(a_{i_2k}\Delta_{i_2k}t)|$.

Putting $a = \sup_{t=1,2,\cdots} |\phi_k(t)|$, we have $a \leqq 1$. By $\phi_k(1) \neq 0$ we have $a > 0$. For every $\epsilon > 0$ there exists a non-zero integer t' such that $|\phi_k(t')| \geqq a(1 - \epsilon)$. Then from (5), $a(1 - \epsilon) \leqq a^2$ for each $\epsilon > 0$, implying that $a = 1$. Because $|\phi_k(n)| < 1$ for all $n \neq 0$, it follows that

$$1 = a = \limsup_{n\to\infty} |\phi_k(n)|.$$

Now, since Γ is connected, $\hat{\Gamma}$ has no elements of finite order. Thus, since $\hat{\Gamma}$ is discrete, $\hat{x}_0^n$ is, for $n = 1, 2, \cdots$, eventually outside each compact set of $\hat{\Gamma}$.

Now, assume (besides the condition that ξ_k is not U.D.) that the distribution of ξ_k has an absolutely continuous component with respect to the Haar measure on Γ. Then, by Theorem 4.2, $\limsup_{n\to\infty} |\phi_k(n)| < 1$, a contradiction. This proves the theorem.

5. Counterexamples. In this section we will justify some of the many conditions of Theorem 4.1 with some counterexamples.

EXAMPLE 5.1: The necessity of the condition of Theorem 4.1 that $\Delta = \pm 1$ is demonstrated by the following counterexample: to any integer matrix (a_{ij}) with $|\Delta| = |\det(a_{ij})| \geqq 2$ one can associate independent random variables $\xi_1, \cdots, \xi_n$ in $\Gamma = I$, each having an absolutely continuous distribution (with respect to the Haar measure), such that the $\eta_i = \sum_{j=1}^n a_{ij}\xi_j$ ($i = 1, \cdots, n$) are independent while nevertheless not each ξ_j is U.D. Let m ($1 \leqq m \leqq n$) be such that Δ does not divide g.c.d. $(\Delta_{1m}, \cdots, \Delta_{nm}) = d$. In order to show that such an m exists suppose Δ divides each Δ_{ij}. Then Δ^n divides

$$\det(\Delta_{ij}) = [\det(a_{ij})]^{n-1} = \Delta^{n-1},$$

which is impossible for $|\Delta| \geqq 2$. It follows from $\Delta = \sum_{i=1}^n a_{im}\Delta_{im}$, that $\Delta = pd$ for some integer p, $|p| \geqq 2$.

Let $\xi_1, \cdots, \xi_n$ be independent random variables in I. Assume that ξ_j, $j \neq m$, is U.D. in I and that $p\xi_m$ is U.D. in such a way that ξ_m is not U.D. (e.g., ξ_m is U.D. in $[0, 1/p]$). It will be shown that $\eta_1, \cdots, \eta_n$ are independent, in fact I.U.D. (while ξ_m is not U.D., though its distribution has an absolutely continuous component).

Let $t_1, \cdots, t_n$ be integers, not all zero. In view of Corollary 3.6 it suffices to show that $\sum_{i=1}^n t_i\eta_i$ is U.D. in I. Now

$$\sum_{i=1}^n t_i\eta_i = \sum_{j=1}^n q_j\xi_j \quad \text{with} \quad q_j = \sum_{i=1}^n a_{ij}t_i.$$

$\sum_{i=1}^n t_i\eta_i$ is, by Corollary 3.6, certainly U.D. when some q_j ($j \neq m$) differs from zero. Otherwise, from Cramér's rule, $t_i = (1/\Delta)(\Delta_{im}q_m)$, $i = 1, \cdots, m$. Therefore,

$\Delta = pd$ is a divisor of $\Delta_{im}q_m$ $(i = 1, \cdots, n)$ and hence of dq_m, showing that p divides q_m. Thus, also in this case, $\sum_{i=1}^{n} t_i\eta_i = q_m\xi_m$ is U.D. in I.

EXAMPLE 5.2: The following result clearly demonstrates the necessity of the hypothesis of Theorem 4.1 that for fixed i at least two a_{ij} are non-zero.

Assume $\Delta = I$, let $\xi_1, \cdots, \xi_n$ be independent in I, and let $\xi_2, \cdots, \xi_n$ be U.D. in I. The distribution of ξ_1 is arbitrary. Let $\eta_1 = \xi_1$, $\eta_i = \sum_{j=1}^{n} a_{ij}\xi_j$ $(i = 1, \cdots, n)$, where (a_{ij}) is any non-singular integer matrix, $a_{12} = \cdots = a_{1n} = 0$, $a_{11} = 1$. Assertion: $\eta_1, \cdots, \eta_n$ are independent.

For, let $(h_1, \cdots, h_n)$ be a set of n integers. Then

$$\sum_{i=1}^{n} h_i\,\eta_i = \left(h_1 + \sum_{i=2}^{n} a_{i1}\,h_i\right)\xi_1 + \sum_{j=2}^{n}\left(\sum_{i=2}^{n} a_{ij}\,h_i\right)\xi_j\,.$$

From Theorem 4.1 and Corollary 3.4, this sum is U.D. on I unless $h_i = 0$, $i \geqq 2$, (for the matrix $(a_{ij}\,;\, i, j = 2, \cdots, n)$ is non-singular). Hence, in all cases,

$$E\left\{\exp\left(2\pi\,(-1)^{\frac{1}{2}}\sum_{i=1}^{n} h_i\,\eta_i\right)\right\} = \prod_{i=1}^{n} E\{\exp\,(2\pi\sqrt{-1}\,h_i\,\eta_i)\}$$

$$= E\{\exp\,(2\pi(-1)^{\frac{1}{2}}h_1\xi_1)\} \text{ when } h_2 = h_3 = \cdots = h_n = 0, \; = 0 \text{ otherwise.}$$

Thus, since their joint characteristic function factors, $\eta_1, \cdots, \eta_n$ are independent.

EXAMPLE 5.3: If $|\Delta| \geqq 2$, it may happen that there exist positive numbers $\sigma_1^2, \cdots, \sigma_n^2$ such that

(6) $\sum_{j=1}^{n} a_{i_1j}a_{i_2j}\sigma_j^2 = 0$ whenever $i_1 \neq i_2$.

Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent real normally distributed random variables such that ξ_j has a variance σ_j^2 and mean zero. Then $\eta_i = \sum_{j=1}^{n} a_{ij}\xi_j$ $(i = 1, \cdots n)$ are jointly normally distributed. Further (6) implies that $E(\eta_i\eta_j) = 0$ for $i \neq j$; hence, the η_j are independent. On the other hand, a normally distributed random variable ξ_j is absolutely continuous (mod 1), but never uniformly distributed (mod 1), in view of

$$E\{\exp\,(2\pi(-1)^{\frac{1}{2}}m\xi_j)\} = e^{-\frac{1}{2}(2\pi m\sigma_j)^2} \neq 0.$$

It may be interesting to prove that (6) implies $|\Delta| \geqq 2$. More precisely, let $A = (a_{ij})$ be an $n \times n$ real matrix with $\Delta = \det A \neq 0$. Let k, m, r be fixed, $a_{km} \neq 0$, $a_{kr} \neq 0$, $\sigma_m \neq 0$, $\sigma_r \neq 0$. Then (6) implies $0 < |a_{km}\Delta_{km}| < |\Delta|$, hence $|\Delta| \geqq 2$ if the a_{ij} are all integers. Moreover, if all the a_{ij} are integers, for each i we have $a_{ij} \neq 0$ for at least two j, and all $\sigma_j \neq 0$, then (6) implies $|\Delta| \geqq 1 + \mathrm{Max}_{k,m}\,|a_{km}|$. To prove this note that

$$\begin{aligned} a_{km}\sigma_m^2\Delta &= \sum_{j=1}^{n} a_{kj}\sigma_j^2 \sum_{i=1}^{n} \Delta_{im}a_{ij} \\ &= \sum_{j} \Delta_{im} \sum_{j} a_{ij}a_{kj}\sigma_j^2 \\ &= \Delta_{km} \sum_{j} a_{kj}^2\sigma_j^2\,. \end{aligned}$$

Now, if $a_{km} \neq 0$, $a_{kr} \neq 0$, $\sigma_m \neq 0$, $\sigma_r \neq 0$, $\Delta \neq 0$, then $\Delta_{km} \neq 0, \sum_j a_{kj}^2 \sigma_j^2 > a_{km} \sigma_m^2$; hence

$$| a_{km} | \sigma_m^2 | \Delta | > | \Delta_{km} | a_{km}^2 \sigma_m^2 ,$$

and $| \Delta | > | \Delta_{km} a_{km} | > 0$, proving the assertion.

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