ON A CLASS OF STOCHASTIC PROCESSES[1]

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1. Introduction. We shall introduce our problem by recalling (see, for instance, [1]) the example of a *branching process* $\{X_t\}$ with mean number of descendants per individual per unit time $= \mu > 1$. It is well known that

$$\Pr\,(\lim_{t\to\infty} X_t/\mu^t = Y \text{ exists}) = 1 \tag{1.1}$$

where Y is a random variable. This result implies that

$$\lim_{s\to\infty} \{X_{s+t}/\mu^s\} = \{Y_t\}, \qquad -\infty < t < \infty, \tag{1.2}$$

exists, where the brackets mean that we are considering the *processes* one of whose random variables is indicated, and the limit is in the sense of convergence of finite-dimensional distributions[3]. Of course, (1.1) implies also that

$$\{Y_t\} = \{\mu^t Y\}, \tag{1.3}$$

so that $\{Y_t\}$ is deterministic in the sense that if its state at some time t is given, the entire past and future are uniquely determined.

The problem studied in this paper is as follows: *Which processes can arise as limits in a manner similar to* (1.2)? That is, if for some stochastic process $\{X_t\}$ in Euclidean space there is a positive measurable function $f(s)$ such that

$$\lim_{s\to\infty} \{X_{s+t}/f(s)\} = \{Y_t\}, \qquad -\infty < t < \infty, \tag{1.4}$$

what can be inferred about the process $\{Y_t\}$? This question is very analogous to the one considered in [2]. The point there was to generalize the limiting operation by which the Wiener process derives from simple random walk under contraction of the space and time scales. The class of limiting processes which can be obtained in that way by varying the starting process (random walk) and the rates of contracting the axes was characterized by the "semi-stable" property. Some special classes of processes having this property were then described. The present investigation has a similar motivation; we are generalizing a well-known theorem (above) involving contraction of the space scale and *translation* of the time axis of one process to obtain another in the limit.

An outline of our results is as follows: in Theorem 1 a characterization is obtained for processes $\{Y_t\}$ which can arise in the manner (1.4). This does not, how-

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[3] All limiting statements for processes are to be interpreted in this way; in addition "equality" of two processes means that they have the same finite-dimensional distributions.

ever, make it clear which Markov processes with stationary transition probabilities belong to the class of possible limits. This question is perhaps more interesting and we shall solve it completely in Section 3 for one-dimensional, continuous parameter processes $\{Y_t\}$. The result is, roughly, that a process in the class is either a strictly stationary one or else it must be, apart from sign, deterministic in the manner of (1.3). In the discrete parameter case, however, or in several dimensions, there are many other possibilities, in contrast to the situation for branching. The general characterization in Theorem 1 is related to the "semi-stable" property of [2], but the Markov processes found are of a very different nature in the two cases.

2. Characterization of $\{Y_t\}$. We now give the general answer to the problem of which processes are possible limits in (1.4):

THEOREM 1. *Suppose that there exists a process $\{X_s\}$ and a positive measurable function $f(s)$ such that* (1.4) *holds; suppose also that for some t_1 the distribution of Y_{t_1} is not degenerate at* 0.[4] *Then*

$$f(s) = e^{\alpha s}M(s) \quad \text{for some real } \alpha, \tag{2.1}$$

where $M(s)$ is a function with the property that

$$\lim_{s\to\infty} M(s+c)/M(s) = 1 \quad \text{for any fixed } c. \tag{2.2}$$

The process $\{Y_t\}$ has the property that for each fixed u,

$$\{Y_{t+u}\} = \{e^{\alpha u}Y_t\}. \tag{2.3}$$

Conversely, each process $\{Y_t\}$ satisfying (2.3) *can be obtained as a limit of the form* (1.4).

REMARKS. The result holds in n-dimensions; the one-dimensional proof applies to each component separately as regards (2.1) and (2.2), and it will be clear that the argument from there to (2.3) is valid. We thus shall give the proof in the one-dimensional case only. The theorem also holds both for continuous and integral parameter, as we shall see. In the latter case, naturally u and t must both be integers in (2.3).

PROOF. First we shall show that the distribution of Y_t is not degenerate at 0 for any t. Suppose the contrary for t_0, while by assumption there is a t_1 where $\Pr(Y_{t_1} = 0) < 1$. Now from this and (1.4), we have $X_{s+t_1+t_0}/f(s+t_1)$ tending to 0 in law as $s \to \infty$, while the distribution of $X_{s+t_1+t_0}/f(s+t_0)$ has the law of Y_{t_1}, not concentrated at 0, for its limit. It follows that $f(s+t_1)/f(s+t_0) \to \infty$ as $s \to \infty$. But then we consider

$$\frac{X_{s+2t_1}}{f(s+t_1)} = \frac{X_{s+2t_1}}{f(s+t_0)} \cdot \frac{f(s+t_0)}{f(s+t_1)}.$$

The distribution of the left side converges to that of Y_{t_1} which has mass away

[4] When $Y_t = 0$ a.s. for each t (2.3) is trivially true but (2.1) fails, since if some function $f(s)$ will serve in (1.4), so will any larger one.

from 0; the distribution of the first term on the right tends to the law of $Y_{2t_1-t_0}$. Since the second factor on the right tends to 0, we have reached a contradiction which proves the assertion.

Next we state a lemma of a familiar kind; the proof is very easy and will be omitted.

LEMMA. *If constants* $a_n > 0$, $b_n > 0$ *and distribution functions* G_n *are such that* $\lim_{n\to\infty} G_n(a_n x) = F_1(x)$, $\lim_{n\to\infty} G_n(b_n x) = F_2(x)$ *exist where* F_1 *and* F_2 *are distribution functions not degenerate at* 0, *then* $\lim_{n\to\infty} a_n/b_n = \alpha$ *exists,* $0 < \alpha < \infty$.

Now from (1.4) we have

$$\lim_{s\to\infty} \Pr\,(X_s \leqq xf(s)) = \Pr\,(Y_0 \leqq x), \tag{2.4}$$

and also, for any fixed t,

$$\lim_{s\to\infty} \Pr\,(X_s \leqq xf(s-t)) = \Pr\,(Y_t \leqq x). \tag{2.5}$$

Since neither of the distributions on the right is degenerate at 0, we can conclude that

$$\lim_{s\to\infty} f(s+t)/f(s) = h(t) \tag{2.6}$$

exists for each t (integral or real, depending on the process $\{X_t\}$). If s and t are discrete, it is easy to see that $h(t) = h(1)^t$; upon defining $M(s)$ by (2.1) with $\alpha = \log h(1)$ and substituting in (2.6) we obtain (2.2). The continuous case is similar, except that the relation $h(t) = e^{\alpha t}$ is obtained by showing that $h(t)$ satisfies the functional equation $h(u+v) = h(u)h(v)$.[5]

To prove (2.3) is now very simple. Thus using (1.4), (2.1) and (2.2),

$$\begin{aligned} \Pr(Y_{t+u} \leqq x) &= \lim_{s\to\infty} \Pr\left(\frac{X_{s+t+u}}{f(s)} \leqq x\right) \\ &= \lim_{s\to\infty} \Pr\left(\frac{X_{s+t+u}}{f(s+u)} \leqq x\,\frac{f(s)}{f(s+u)}\right) = \Pr\,(Y_t \leqq xe^{-\alpha u}) \end{aligned} \tag{2.7}$$

for any x which is a continuity point of the distributions of Y_{t+u} and of $e^{\alpha u}Y_t$. The same argument works for the finite-dimensional joint distributions; hence (2.3) and the direct part of the theorem. The converse is trivial; if a process satisfies (2.3) we can simply let $f(s) = e^{\alpha s}$ and $\{X_t\} = \{Y_t\}$ and (1.4) will be satisfied.

Observe that from (2.3), a "construction" of the most general process arising as a limit (1.4) is possible. Let $\{Z_t\}$ be any strictly stationary process, and put $\{Y_t\} = \{e^{\alpha t}Z_t\}$; it is easy to see that this yields all processes satisfying (2.3). The unsatisfactory thing about it (as mentioned above) is that it does not help in identifying the Markov processes *with stationary transition probabilities* which arise from (1.4). A similar situation occurred in [2].

3. The one-dimensional Markov case. In this section we consider only con-

[5] Equation (2.6) with st in place of $s + t$ is the defining relation for what Karamata has called a function of *regular variation*; a change of variable would reduce our case to that one.

tinuous parameter processes $\{Y_t\}$ which are Markovian with a stationary transition-probability function

$$p_t(x, A) = \Pr (Y_{s+t} \varepsilon A \mid Y_s = x). \tag{3.1}$$

The class of strictly stationary Markov processes have this property and satisfy (2.3) with $\alpha = 0$; this class contains many members which are in no sense deterministic processes. By contrast, we have

THEOREM 2. *Suppose t is continuous, and that $\{Y_t\}$ is a Markov process on the positive real axis satisfying* (3.1) *and also* (2.3) *for some $\alpha \neq 0$. Then for each t,*

$$\Pr (Y_t = e^{\alpha t} Y_0) = 1. \tag{3.2}$$

PROOF. We first observe from (2.3) that the transition probabilities $p_t(x, A)$ must satisfy

$$p_t(x, A) = p_t(\xi x, \xi A) \qquad \text{for all } \xi > 0. \tag{3.3}$$

The reason is that transition probabilities are (by assumption) invariant under a translation of the time scale, and so by (2.3) they are the same for $\{e^{\alpha u} Y_t\}$ as for $\{Y_t\}$, for each u. Now consider the process $\{W_t\} = \{\log Y_t\}$. This is again a Markov process; from (3.3) we have

$$\begin{aligned} \Pr (W_{t+s} \leqq y \mid W_s = x) &= p_t(e^x, [0, e^y]) \\ &= p_t(e^u e^x, [0, e^u e^y]) = \Pr (W_{t+s} \leqq u + y \mid W_s = u + x) \end{aligned} \tag{3.4}$$

for all real u. It follows that $\{W_t\}$ has stationary independent increments. However, (2.3) implies that

$$\Pr (W_t \leqq y) = \Pr (W_0 \leqq y - \alpha t). \tag{3.5}$$

Since $W_t = W_0 + (W_t - W_0)$, and the two summands are independent, (3.5) implies that the distribution of $W_t - W_0$ assigns probability one to the point αt; this yields (3.2).

Next we remove the restriction that $Y_t > 0$, keeping all the other assumptions of Theorem 2. First notice that because of (2.3), $\Pr (Y_t = 0) = p$ is constant, while because of (3.3), $p_t(0, \{0\}) = 1$ for all t. (A distribution concentrated on 0 is the only one invariant under all expansions of the state space.) Thus the process $\{Y_t\}$ can (a.s.) neither enter nor leave state 0.[6] We accordingly can turn our attention to the process $\{Y_t'\}$, defined as $\{Y_t\}$ conditioned not to be in state 0. $\{Y_t'\}$ is again a Markov process satisfying (2.3) (and hence (3.3)). We will also assume that $\{Y_t'\}$ changes sign with positive probability, since otherwise Theorem 2 suffices to determine its structure.

THEOREM 3. *If, under the conditions above, for a constant $A > 0$ we define*

$$S_t = \begin{cases} -1 & \text{if } Y_t' < 0 \\ +A & \text{if } Y_t' > 0, \end{cases} \tag{3.6}$$

[6] This statement is true for all t if the process $\{Y_t\}$ is assumed separable; if not, it holds for any countable set of parameter values.

then $\{S_t\}$ *is a stationary Markov chain and there is a choice of* A *for which the representation*

$$Y'_t = S_t e^{\alpha t} Y \tag{3.7}$$

holds with Y *a positive random variable independent of the process* $\{S_t\}$. *Conversely, if* $\{S_t\}$ *is a stationary Markov process with states* $-1, A$ *and* Y *is a positive random variable independent of* $\{S_t\}$, *then the process defined by* (3.7) *is a Markov process, has stationary transition probabilities and obeys* (2.3).

PROOF. Form the auxiliary process

$$Z_t = (\log |Y'_t| - \alpha t, \text{ sign } Y'_t). \tag{3.8}$$

It is clear that this is a Markov process with state space consisting of the pair of lines $y = \pm 1$ in the plane. It follows from (2.3) that $\{Z_t\}$ is strictly stationary; we denote its transition probabilities by

$$q^t_{ij}(x, \tilde{x}) = \Pr\{Z_t \,\varepsilon\, ([-\infty, \tilde{x}], j) \mid Z_0 = (x, i)\}, \tag{3.9}$$

where i and j take the values ± 1. For each t, (3.3) implies that

$$q^t_{ij}(x + u, \tilde{x} + u) = q^t_{ij}(x, \tilde{x}) = Q^t_{ij}(\tilde{x} - x). \tag{3.10}$$

Thus Z_t is a stationary, translation-invariant Markov process on two parallel lines.

Let $F_i(x)$, $i = \pm 1$, be the (stationary) probability that Z_t is on the line $y = i$, to the left of x. For any t we have

$$F_j(\tilde{x}) = \int_{-\infty}^{\infty} q^t_{-1,j}(x, \tilde{x})\, dF_{-1}(x) + \int_{-\infty}^{\infty} q^t_{1,j}(x, \tilde{x})\, dF_1(x). \tag{3.11}$$

Letting $\varphi_j(\lambda)$ be the Fourier-Stieltjes transform of $F_j(\ )$, and $\psi_{ij}(\lambda)$ that of $Q^t_{ij}(\ )$, (3.11) with (3.10) yields

$$(\varphi_{-1}(\lambda), \varphi_1(\lambda)) \begin{bmatrix} \psi_{-1,-1}(\lambda) & \psi_{-1,1}(\lambda) \\ \psi_{1,-1}(\lambda) & \psi_{1,1}(\lambda) \end{bmatrix} = (\varphi_{-1}(\lambda), \varphi_1(\lambda)). \tag{3.12}$$

Thus for each λ in a neighborhood U of the origin (where $\varphi_{-1}(\lambda)$ and $\varphi_1(\lambda)$ don't vanish) the determinant of $[\delta_{ij} - \psi_{ij}(\lambda)]$ must be zero.

Notice next from their definition that the functions Q^t_{ij} are monotonic, and that if v_{ij} = variation of Q^t_{ij}, then $v_{i,-1} + v_{i,1} = 1$. In addition, we know that $|\psi_{ij}(\lambda)| \leqq v_{ij}$ for all λ. Now a complex determinant of the form

$$\begin{vmatrix} 1 - a & b \\ c & 1 - d \end{vmatrix}, \; |a| + |b| \leqq 1, \; |c| + |d| \leqq 1 \tag{3.13}$$

can vanish only if either $a = 1$ or $d = 1$, or else if a and d are real and non-negative, if bc is real, and if equality holds in both places in (3.13). Applied to $[\delta_{ij} - \psi_{ij}]$, the first possibility is eliminated by the condition that Y_t changes sign. This shows that $\psi_{ii}(\lambda) = v_{ii}$ for all $\lambda \,\varepsilon\, U$ so that Q^t_{ii} concentrates all its mass at 0 for both values of i. In addition, Q^t_{ij} is degenerate (not necessarily at 0) for $i \neq j$ since $|\psi_{ij}(\lambda)| = v_{ij}$ for all $\lambda \varepsilon U$. Thus from a point (x, i) in the state-space of $\{Z_t\}$, the only transitions possible in time t are to itself, or to some *one* point (x', j) on the other line; from that point the process, if it moves, can only go back to (x, i). The location of x' is independent of t, for if (x_1', j) and (x_2', j) were accessible from (x, i) in times t and s, then (x_2', j) would be accessible from (x_1', j) in time $t + s$ contrary to what we have shown.

It is now not hard to see how the most general process $\{Z_t\}$ is formed. Associated with a point x on line $y = -1$ is a point x' on $y = +1$ such that from either point only transitions to itself or the other are possible; these transitions are those of a two-state Markov chain. Because of (3.10), the same transition probabilities hold for the points $(x + u, -1)$ and $(x' + u, +1)$. The stationary Markov process $\{Z_t\}$ must be a mixture of these elementary two-state ones. Such a mixture is uniquely determined by a probability measure on *one* of the lines. Indeed if μ is such a measure on line $y = -1$, and if p, q are the stationary probabilities of $(x, -1)$ and $(x', +1)$ for one of the elementary Markov chains, then the stationary measure for the whole process must be $p\mu$ on $y = -1$, $q\mu'$ on $y = +1$, where μ' is the measure μ translated by an amount $x' - x$. It follows that if the process $\{Z_t'\}$ were modified by translating the points on the line $y = +1$ horizontally by the amount $x - x'$, then the line occupied and the horizontal position would be independent.

It is now possible to "read off" the conclusions of the theorem. It is certainly clear that $\{\text{sign } Y_t'\}$, and hence $\{S_t\}$, is a stationary Markov chain for any A. Let $A = \exp(x' - x)$, and consider the further modified process

$$Z_t' = (\log |S_t^{-1}Y_t'| - \alpha t, \text{sign } Y_t');$$

this is nothing but the process $\{Z_t\}$ with the line $y = +1$ translated as described above. We then see from our earlier work that the random variables

$$\log |S_t^{-1}Y_t'| - \alpha t = \log Y$$

are constant in time and independent of the process $\{\text{sign } Y_t'\}$. It is now simple to write $Y_t' = |Y_t'| \text{ sign } Y_t'$ in terms of Y and S_t, and (3.7) is the result. The converse part of the theorem is easily verified.

4. Other cases. Let us first consider the situation of Theorem 2 with only one factor changed: $\{Y_n\}$ will have integral parameter. The "invariance" relation (3.3) then holds only if $\xi = e^{\alpha m}$ for some integer m, and consequently we can not conclude as before that $\{W_n\}$ has independent increments. The theorem is, in fact, false, and it is easy to construct examples. For instance, let $\{Z_n\}$ be a strictly stationary Markov process with state-space interior to the interval $(0, \alpha)$, $\alpha > 0$. Define

$$Y_n = e^{\alpha n} e^{Z_n}; \tag{4.1}$$

it is clear that $\{Y_n\}$ is again a Markov process, $Y_n > 0$, and that (2.3) holds. It is therefore only necessary to verify that the transition probabilities are independent of n. However this too is trivial in the following sense: with probability one $\alpha n < Y_n < \alpha(n+1)$. Thus the sets of possible states which can be attained by Y_n are disjoint as n varies, which automatically yields the desired stationarity.

The possibilities seem to be still more varied in several dimensions, even in the continuous-parameter case. For instance, consider a process $\{(U_t, V_t)\}$ satisfying (2.3) and with positive components. Then (3.3) holds, but only for multiplication by positive scalars, and so the process

$$Z_t = (\log U_t, \log V_t) \tag{4.2}$$

has transition probabilities which are invariant under translations in a direction parallel to the line $y = x$. The process

$$Z_t^* = (\log U_t - \alpha t, \log V_t - \alpha t) \tag{4.3}$$

has the same property and, by (2.3), is strictly stationary. Rotating coordinates through 45 degrees, we are led to consider the possibilities for finding strictly stationary plane Markov processes $\{(U_t^*, V_t^*)\}$ whose transition probabilities are invariant under translations in the u-direction. Every such process would lead, reversing the transformations above, to a solution of (2.3) of the type we are considering. They can be construed as follows: Let w_t be a strictly stationary 1-dimensional Markov process, and f a real measurable function. Then

$$(U_t^*, V_t^*) = (f(w_t), w_t) \tag{4.4}$$

has the desired properties, as does any mixture of horizontal translates of this "elementary" process. The situation is somewhat analogous to that occurring in the proof of Theorem 3; one is tempted to conjecture that we have described the most general case, but this has not yet been proved.

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