STATISTICAL ANALYSIS BASED ON A CERTAIN MULTIVARIATE COMPLEX GAUSSIAN DISTRIBUTION (AN INTRODUCTION)[1]

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0. Summary. A complex Gaussian random variable is a complex random variable whose real and imaginary parts are bivariate Gaussian distributed. A p-variate complex Gaussian random variable is a p-tuple of complex Gaussian random variables such that the vector of real and imaginary parts is $2p$-variate Gaussian distributed. The present paper is an introduction to statistical analysis based on a certain multivariate complex Gaussian distribution which is the distribution of a p-tuple of complex Gaussian random variables whose real and imaginary parts are $2p$-variate Gaussian distributed with a $2p \times 2p$ real covariance matrix of special form. The special form of the $2p \times 2p$ real covariance matrix permits the distribution of the p-tuple of complex Gaussian random variables to be expressed in complex form, and to be (in the zero mean case) specified by a $p \times p$ Hermitian covariance matrix. (Certain simplifying conditions, e.g. zero mean random variables, non-singularity of matrices, etc. are retained throughout the paper. Such conditions may be removed at the expense of added complexity of exposition or results.) Statistical analysis based on the particular multivariate complex Gaussian distribution mentioned above possesses certain desirable properties:

(1) A theory that is a counterpart of (i.e. "parallels") classical multivariate real Gaussian statistical analysis may be developed.

(2) From the methods of proof and the distributional results stated in the paper there are indications that for every distributional result of classical multivariate real Gaussian statistical analysis obtainable in closed (explicit) form, the counterpart result in the multivariate complex Gaussian statistical analysis is also obtainable in closed (explicit) form. A comparison between certain counterpart distributional results of multivariate complex and real Gaussian statistical analysis indicates that the multivariate complex Gaussian distributional results often appear formally simpler and at times are simpler. Furthermore, not all distributional results of the multivariate complex Gaussian statistical analysis are counterpart results of multivariate real Gaussian statistical analysis so that,

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in some sense, the multivariate complex Gaussian statistical analysis is the richer of the two.

(3) The distributional results of the multivariate complex Gaussian statistical analysis are applicable in describing the statistical variability of estimators for the spectral density matrix of a multiple stationary Gaussian time series, and in describing the statistical variability of estimators for functions of the elements of a spectral density matrix of a multiple stationary Gaussian time series. Moreover, heuristic arguments suggest that the results of statistical analysis based on the multivariate complex Gaussian distribution may also be, under appropriate conditions, applicable in describing the sampling variability of empirical spectral analysis results for many *non-Gaussian* multiple stationary time series.

1. Introduction and technical summary. The paper constitutes an introduction to statistical analysis based on a certain multivariate complex Gaussian distribution. Preliminary algebraic results and the (zero mean) p-variate complex Gaussian distribution are initially derived. A complex Gaussian random variable $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$ is a complex random variable whose real and imaginary parts are bivariate Gaussian distributed. A p-variate complex Gaussian random variable $\boldsymbol{\xi}' = (\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_p)$ is a p-tuple of complex Gaussian random variables such that the vector of real and imaginary parts $\boldsymbol{\eta}' = (\mathbf{X}_1, \mathbf{Y}_1, \cdots, \mathbf{X}_p, \mathbf{Y}_p)$ is $2p$-variate Gaussian distributed. The $2p$-variate Gaussian random variables $\boldsymbol{\eta}'$ considered in the paper have zero mean and $2p \times 2p$ positive definite covariance matrices

$$(1.1)\qquad \Sigma_\eta = E\boldsymbol{\eta}\boldsymbol{\eta}' = \left\| \begin{bmatrix} E\mathbf{X}_j\mathbf{X}_k & E\mathbf{X}_j\mathbf{Y}_k \\ E\mathbf{Y}_j\mathbf{X}_k & E\mathbf{Y}_j\mathbf{Y}_k \end{bmatrix} \right\|$$

of the special form where

$$(1.2)\qquad \begin{bmatrix} E\mathbf{X}_j\mathbf{X}_k & E\mathbf{X}_j\mathbf{Y}_k \\ E\mathbf{Y}_j\mathbf{X}_k & E\mathbf{Y}_j\mathbf{Y}_k \end{bmatrix} = \begin{cases} \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sigma_k^2 & \text{if } j = k, \\ \frac{1}{2}\begin{bmatrix} \alpha_{jk} & -\beta_{jk} \\ \beta_{jk} & \alpha_{jk} \end{bmatrix} \sigma_j\sigma_k & \text{if } j \neq k. \end{cases}$$

(In (1.1) and (1.2) E denotes the expectation operator.) The corresponding (zero mean) p-variate complex Gaussian random variables $\boldsymbol{\xi}$ then have their distributions specified by $p \times p$ Hermitian positive definite complex covariance matrices

$$(1.3)\qquad \Sigma_\xi = E\boldsymbol{\xi}\bar{\boldsymbol{\xi}}' = \|E\mathbf{Z}_j\bar{\mathbf{Z}}_k\| = \|\sigma_{jk}\|$$

where

$$(1.4)\qquad \sigma_{jk} = \begin{cases} \sigma_k^2 & \text{if } j = k, \\ (\alpha_{jk} + i\beta_{jk})\sigma_j\sigma_k & \text{if } j \neq k. \end{cases}$$

In the present paper the phrase "multivariate complex Gaussian distribution" is restricted to that special case. The following results are established. The probability density function of the (zero mean) p-variate complex Gaussian distribution is given by

$$(1.5) \qquad p(\xi) = (1/\pi^p |\Sigma_\xi|) \exp(-\bar{\xi}'\Sigma_\xi^{-1}\xi).$$

If $\xi_1, \xi_2, \cdots, \xi_n$ is a sample of n complex valued vectors from such a distribution, then the sample Hermitian covariance matrix

$$\hat{\mathbf{\Sigma}}_\xi = (1/n)\sum_{j=1}^{n} \xi_j\bar{\xi}_j'$$

is the maximum likelihood estimator for Σ_ξ. The estimator $\hat{\mathbf{\Sigma}}_\xi$ is a sufficient statistic for the Hermitian covariance matrix Σ_ξ. Consider $\mathbf{A} = \|\mathbf{A}_{jkR} + i\mathbf{A}_{jkI}\| = n\hat{\mathbf{\Sigma}}_\xi$. The joint distribution of the distinct elements of the matrix $\mathbf{A}$ is called a *complex Wishart distribution.* The probability density function of the joint distribution of $\mathbf{A}_{11}, \cdots, \mathbf{A}_{pp}, \mathbf{A}_{12R}, \mathbf{A}_{12I}, \cdots, \mathbf{A}_{p-1,pR}, \mathbf{A}_{p-1,pI}$ is

$$(1.6) \qquad p_W(A) = [|A|^{n-p}/I(\Sigma_\xi)] \exp[-\mathrm{tr}(\Sigma_\xi^{-1}A)],$$

where

$$I(\Sigma_\xi) = \pi^{\frac{1}{2}p(p-1)}\Gamma(n) \cdots \Gamma(n-p+1)\,|\Sigma_\xi|^n.$$

The density $p_W(A)$ is defined over the domain D_A where A is Hermitian positive semi-definite. The characteristic function of the random variables $\mathbf{A}_{11}, \cdots, \mathbf{A}_{pp}, 2\mathbf{A}_{12R}, 2\mathbf{A}_{12I}, \cdots, 2\mathbf{A}_{p-1,pI}$ is

$$(1.7) \qquad C_W(\Theta) = |\Sigma_\xi|^{-n}\,|\Sigma_\xi^{-1} - i\Theta|^{-n},$$

where $\Theta = \|\theta_{jk}\|$ and $\theta_{kj} = \bar{\theta}_{jk}$ with $\theta_{jk} = \theta_{jkR} + i\theta_{jkI}$; $j, k = 1, 2, \cdots, p$. If $T = \|T_{jk}\| = \|T_{jkR} + iT_{jkI}\|$ denotes an upper triangular matrix of complex elements with the diagonal elements T_{jj}, $j = 1, \cdots, p$, real and positive, then there is a unique such $\mathbf{T}$ satisfying $\bar{\mathbf{T}}'\mathbf{T} = \mathbf{A}$. The probability density function of the distribution of the matrix $\mathbf{T}$ is

$$(1.8) \quad p(T) = 2^p[1/I(\Sigma_\xi)]T_{11}^{2n-1}T_{22}^{2n-3} \cdots T_{pp}^{2n-(2p-1)} \exp[-\mathrm{tr}(\Sigma_\xi^{-1}\bar{T}'T)].$$

The density $p(T)$ is defined over the domain D_T given by $T_{11} > 0, \cdots, T_{pp} > 0$, $-\infty < T_{jkR} < \infty$, $-\infty < T_{jkI} < \infty$; $j < k$, $j, k = 1, 2, \cdots, p$. The distribution of the matrix $\mathbf{T}$ is useful in deriving the distributions of functions of the elements of a complex Wishart distributed matrix. Two examples of such functions are now given. Let $\mathbf{A}^{-1} = \|\mathbf{A}^{jk}\|$. Consider the function

$$(1.9) \qquad \hat{\mathbf{R}}^2_{p\cdot p-1,p-2,\cdots,1} = 1 - (\mathbf{A}_{pp}\mathbf{A}^{pp})^{-1}.$$

In terms of the elements of the matrix $\mathbf{T}$ one has

$$(1.10) \qquad \hat{\mathbf{R}}^2_{p\cdot p-1,\cdots,1} = \left(\sum_{j=1}^{p-1} |\mathbf{T}_{jp}|^2\right)\left(\sum_{j=1}^{p} |\mathbf{T}_{jp}|^2\right)^{-1}.$$

Starting from (1.8) the probability density function of the distribution of

$\hat{\mathbf{R}}^2_{p\cdot p-1,p-2,\cdots,1}$ can be shown to be

$$(1.11)\quad p(\hat{R}^2) = \frac{\Gamma(n)}{\Gamma(p-1)\Gamma(n-p+1)} \cdot (1-R^2)(\hat{R}^2)^{p-2}(1-\hat{R}^2)^{n-p}F(n, n; p-1; R^2\hat{R}^2).$$

In (1.11) $R^2 \equiv 1 - (\sigma_{pp}\sigma^{pp})^{-1}$ where $\Sigma_\xi^{-1} \equiv \|\sigma^{jk}\|$, $\hat{R}^2$ denotes $\hat{R}^2_{p\cdot p-1,\cdots,1}$, and $F(,;;)$ denotes the hypergeometric function. Similarly, consider the function

$$(1.12)\quad \hat{\mathbf{R}}^2_{p,p-1|p-2,\cdots,1} = |\mathbf{A}^{p-1,p}|^2\,(\mathbf{A}^{p-1,p-1}\mathbf{A}^{pp})^{-1}.$$

In terms of the elements of the matrix $\mathbf{T}$ one has

$$(1.13)\quad \hat{\mathbf{R}}^2_{p,p-1|p-2,\cdots,1} = |\mathbf{T}_{p-1,p}|^2\,(|\mathbf{T}_{p-1,p}|^2 + \mathbf{T}^2_{pp})^{-1}.$$

Starting from (1.8) the probability density function of the distribution of $\hat{\mathbf{R}}^2_{p,p-1|p-2,\cdots,1}$ can be shown to be

$$(1.14)\quad p(\hat{R}^2) = (n-p+1)(1-R^2)^{n-p+2}(1-\hat{R}^2)^{n-p} \cdot F(n-p+2, n-p+2; 1; R^2\hat{R}^2).$$

In (1.14) now $R^2 \equiv |\sigma^{p-1,p}|^2(\sigma^{p-1,p-1}\sigma^{pp})^{-1}$ and $\hat{R}^2$ denotes $\hat{R}^2_{p,p-1|p-2,\cdots,1}$. The functions $\hat{\mathbf{R}}^2_{p\cdot p-1,\cdots,1}$ and $\hat{\mathbf{R}}^2_{p,p-1|p-2,\cdots,1}$ are respectively the sample *multiple coherence* between $\mathbf{Z}_p$ and $(\mathbf{Z}_{p-1}, \cdots, \mathbf{Z}_1)$ and the sample *conditional coherence* between $\mathbf{Z}_p$ and $\mathbf{Z}_{p-1}$ with respect to $(\mathbf{Z}_{p-2}, \cdots, \mathbf{Z}_1)$. The characteristic function of the complex Wishart distribution given essentially by (1.7) is also useful in deriving the distributions of functions of the elements of a complex Wishart distributed matrix. Consider for example the function $\mathrm{tr}(\mathbf{A})$. From (1.7) one obtains the characteristic function of $\mathrm{tr}(\mathbf{A})$

$$(1.15)\quad \psi_{\mathrm{tr}(\mathbf{A})}(\theta) = \frac{\lambda_1^{-n}\lambda_2^{-n}\cdots\lambda_p^{-n}}{(\lambda_1^{-1}-i\theta)^n(\lambda_2^{-1}-i\theta)^n\cdots(\lambda_p^{-1}-i\theta)^n}.$$

In (1.15) $\lambda_1, \cdots, \lambda_p$ denote the eigenvalues of Σ_ξ. Since $\psi_{\mathrm{tr}(\mathbf{A})}(\theta)$ is a rational function of θ, the probability density function of the distribution of $\mathrm{tr}(\mathbf{A})$ may readily be obtained by Fourier inversion.

The distributions summarized above and other distributions of the multivariate complex Gaussian statistical analysis are applicable in describing the statistical variability of *empirical spectral analysis* results for *multiple stationary time series*. In that regard, for example, the *complex Wishart distribution* given by (1.6) describes the joint statistical variability of estimators for the elements of a spectral density matrix, and the sample coherence distributions given by (1.11) and (1.14) describe respectively the statistical variability of estimators for *multiple coherence* and *conditional coherence* between components of a multiple stationary time series. For a discussion of applications to empirical spectral analysis of time series the reader is referred to the "Applications" section of the present paper and to Goodman [7].

2. Algebraic preliminaries.

Notation. For a complex number z, $\bar{z}$ denotes the conjugate, $|z|$ the absolute value. A matrix M of elements m_{jk} is denoted by $\|m_{jk}\|$, the determinant of a square matrix by $|M|$ or $\det(M)$, the transpose by M', the trace of a square matrix by $\mathrm{tr}(M)$. The $p \times p$ identity matrix is denoted by I_p, a diagonal matrix by D_λ.

Lemma 2.1. *Multiplication of matrices of the form* $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *is isomorphic to multiplication of complex numbers* $c = a + ib$.

Proof. The lemma is a special case of Theorem 21, p. 240 of Birkhoff and MacLane [3].

Notation. Let $r_{jk} = \begin{bmatrix} a_{jk} & -b_{jk} \\ b_{jk} & a_{jk} \end{bmatrix}$ and $c_{jk} = a_{jk} + ib_{jk}$ for $j, k = 1, 2, \cdots, p$. Let $R = \|r_{jk}\|$ and $C = \|c_{jk}\|$.

Theorem 2.1. *Multiplication of the* $2p \times 2p$ *matrices* $R = \|r_{jk}\|$ *is isomorphic to multiplication of the* $p \times p$ *matrices* $C = \|c_{jk}\|$.

Proof. The theorem is an immediate consequence of Lemma 2.1 and the rule for multiplying compatibly partitioned matrices.

Notation. Isomorphic matrices R and C will be written $R \cong C$.

Theorem 2.2. *If R is symmetric, then C is Hermitian, and conversely.*

Proof. If R is symmetric, $R = R'$ where $'$ denotes transpose. Thus, $r_{jk} = r'_{kj}$, *i.e.*, $a_{jk} = a_{kj}$ and $b_{jk} = -b_{kj}$, consequently $c_{jk} = a_{jk} + ib_{jk} = \overline{a_{kj} + ib_{kj}} = \bar{c}_{kj}$, i.e., $C = \bar{C}'$. The converse is clear.

Notation. If C and R are nonsingular, let $C^{-1} = \|c^{jk}\| = \|a^{jk} + ib^{jk}\|$ and $R^{-1} = \|r^{jk}\|$.

Theorem 2.3. *If C is non singular then R is non singular and the inverse of R is*

$$R^{-1} = \|r^{jk}\| = \left\| \begin{bmatrix} a^{jk} & -b^{jk} \\ b^{jk} & a^{jk} \end{bmatrix} \right\|,$$

and conversely.

Proof. By Theorem 2.1, $\|r_{jk}\| \cdot \|r^{jk}\| \cong \|c_{jk}\| \cdot \|c^{jk}\| = I_p$. Thus, $\|r_{jk}\| \cdot \|r^{jk}\| = I_{2p}$ since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cong 1$. The converse is clear.

Theorem 2.4. *If R is orthogonal, then C is unitary, and conversely.*

Proof. If R is orthogonal, $R^{-1} = R' = \|r_{jk}\|' = \|r'_{kj}\|$. Thus, by Theorem 2.3, $C^{-1} = \|\bar{c}_{kj}\| = \|\overline{c_{jk}}\|' = \bar{C}'$, i.e., C is unitary. The converse is clear.

Theorem 2.5. *The* $\det(R) = |\det(C)|^2$.

Proof. Two matrices C and C^* are unitarily equivalent if there exists a unitary matrix $U = \|u_{jk}\|$ such that $C = \bar{U}'C^*U$. By Theorem 41.3, p. 75 of Mac Duffee [10] the matrix C is unitarily equivalent to a triangular matrix C^* where

$$C^* = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ c^*_{21} & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ c^*_{p1} & \cdots & \cdots & \lambda_p \end{bmatrix}$$

and $\lambda_1, \lambda_2, \cdots, \lambda_p$ are the characteristic roots of C. Thus, by Theorems 2.1 and 2.4, $R = M'R^*M$ where M denotes the orthogonal matrix isomorphic to U and R^* denotes the matrix isomorphic to C^*. The matrix R^* is

$$R^* = \left[\begin{array}{cc|cc|cc} \lambda_{1R} & -\lambda_{1I} & 0 & 0 & 0 & 0 \\ \lambda_{1I} & \lambda_{1R} & 0 & 0 & 0 & 0 \\ \hline & & \ddots & & & \\ \hline \begin{bmatrix} c^*_{jkR} & -c^*_{jkI} \\ c^*_{jkI} & c^*_{jkR} \end{bmatrix} & & & \ddots & 0 & 0 \\ & & & & 0 & 0 \\ \hline & & & & \lambda_{pR} & -\lambda_{pI} \\ & & & & \lambda_{pI} & \lambda_{pR} \end{array}\right]$$

where $\lambda_j = \lambda_{jR} + i\lambda_{jI}$ and $c^*_{jk} = c^*_{jkR} + ic^*_{jkI}$, $k \leqq j = 1, 2, \cdots, p$. By taking determinants in $R = M'R^*M$, $\det(R) = \det(M')\det(R^*)\det(M) = \det(R^*)$. Upon evaluating $\det(R^*)$ by Laplacian expansion (see p. 78 of Aitken [1])

$$\det(R^*) = \prod_{j=1}^{p} \begin{vmatrix} \lambda_{jR} & -\lambda_{jI} \\ \lambda_{jI} & \lambda_{jR} \end{vmatrix} = \prod_{j=1}^{p} (\lambda_{jR}^2 + \lambda_{jI}^2) = \prod_{j=1}^{p} |\lambda_j|^2.$$

Thus, $\det(R) = \det(R^*) = \prod_{j=1}^{p} \lambda_j\overline{\lambda_j} = (\prod_{j=1}^{p} \lambda_j)\overline{(\prod_{j=1}^{p} \lambda_j)} = \det(C^*)\,\overline{\det(C^*)} = \det(C)\,\overline{\det(C)} = |\det(C)|^2$.

COROLLARY 2.1. *If R is symmetric, then* $\det(R) = \det^2(C)$.

PROOF. By Theorem 2.2, if R is symmetric, C is Hermitian. The characteristic roots of an Hermitian matrix are real so that $\lambda_j = \lambda_{jR}$, $j = 1, 2, \cdots p$. Thus, $\det(C) = \prod_{j=1}^{p} \lambda_{jR}$ is real, consequently $\det(R) = |\det(C)|^2 = \det^2(C)$.

Notation. Let $\eta' = (x_1, y_1, x_2, y_2, \cdots, x_p, y_p)$ and $\xi' = (z_1, z_2, \cdots, z_p)$ where $z_j = x_j + iy_j$, $j = 1, \cdots, p$.

THEOREM 2.6. *If R is symmetric, the quadratic form* $\eta'R\eta = \bar{\xi}'C\xi$.

PROOF. The quadratic form

$$\eta'R\eta = \sum_{j,k=1}^{p} [x_j, y_j] r_{jk} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \sum_{j,k=1}^{p} [x_j, y_j] \begin{bmatrix} a_{jk} & -b_{jk} \\ b_{jk} & a_{jk} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$
(2.3)
$$= \sum_{j,k=1}^{p} [a_{jk}(x_j x_k + y_j y_k) - b_{jk}(x_j y_k - y_j x_k)].$$

The quadratic form

$$\bar{\xi}'C\xi = \sum_{j,k=1}^{p} \bar{z}_j c_{jk} z_k = \sum_{j,k=1}^{p} (x_j - iy_j)(a_{jk} + ib_{jk})(x_k + iy_k)$$
$$= \sum_{j,k=1}^{p} [a_{jk}(x_j x_k + y_j y_k) + b_{jk}(-x_j y_k + y_j x_k)]$$
(2.4)
$$+ i \sum_{j,k=1}^{p} [a_{jk}(x_j y_k - y_j x_k) + b_{jk}(x_j x_k + y_j y_k)]$$
$$= \sum_{j,k=1}^{p} [a_{jk}(x_j x_k + y_j y_k) + b_{jk}(-x_j y_k + y_j x_k)] = \eta'R\eta.$$

The imaginary part of the summation in (2.4) vanishes since $a_{jk} = a_{kj}$ and $b_{jk} = -b_{kj}$ if R is symmetric.

Corollary 2.2. *If R is symmetric positive definite then C is Hermitian positive definite, and conversely.*

Proof. If R is symmetric positive definite the quadratic form $\eta' R \eta$ assumes, for $\eta \neq 0$, exclusively positive values. Since $\eta' R \eta = \bar{\xi}' C \xi$, $\bar{\xi}' C \xi$ assumes, for $\xi \neq 0$, exclusively positive values. By Theorem 2.2, the matrix C is Hermitian. Thus, C is Hermitian positive definite. The converse is clear.

3. Complex Gaussian random variables and the multivariate complex Gaussian distribution.

Definition 3.1. A univariate *complex Gaussian random variable* $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$ is a complex random variable whose real and imaginary parts are bivariate Gaussian distributed.

Definition 3.2. A *p-variate complex Gaussian random variable* $\xi' = (\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_p)$ is a p-tuple of complex Gaussian random variables such that the vector of real and imaginary parts $\mathbf{n}' = (\mathbf{X}_1, \mathbf{Y}_1, \cdots, \mathbf{X}_p, \mathbf{Y}_p)$ is $2p$ variate Gaussian distributed.

Comment. The Gaussian random variables $\mathbf{X}_j$, $\mathbf{Y}_j$ considered in the paper are taken to have zero mean. The distribution of $\mathbf{n}$ is therefore specified by the covariance matrix

$$\Sigma_\eta = E\mathbf{n}\mathbf{n}' = \left\| \begin{bmatrix} E\mathbf{X}_j\mathbf{X}_k & E\mathbf{X}_j\mathbf{Y}_k \\ E\mathbf{Y}_j\mathbf{X}_k & E\mathbf{Y}_j\mathbf{Y}_k \end{bmatrix} \right\|,$$

where E denotes the expectation operator. Consider the special case where Σ_η is such that the 2×2 submatrices

$$\begin{bmatrix} E\mathbf{X}_j\mathbf{X}_k & E\mathbf{X}_j\mathbf{Y}_k \\ E\mathbf{Y}_j\mathbf{X}_k & E\mathbf{Y}_j\mathbf{Y}_k \end{bmatrix} \equiv \begin{cases} \dfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sigma_k^2 & \text{if } j = k, \\ \dfrac{1}{2}\begin{bmatrix} \alpha_{jk} & -\beta_{jk} \\ \beta_{jk} & \alpha_{jk} \end{bmatrix} \sigma_j \sigma_k & \text{if } j \neq k. \end{cases} \tag{3.1}$$

The paper is restricted to that case and henceforth, all covariance matrices Σ_η are taken to be of that form.

Definition 3.3. The *complex covariance* between $\mathbf{Z}_j$ and $\mathbf{Z}_k$ is $E\mathbf{Z}_j\bar{\mathbf{Z}}_k$.

Definition 3.4. The *variance* of $\mathbf{Z}$ is $E|\mathbf{Z}|^2$.

Comment. From (3.1) one readily verifies that

$$\sigma_{jk} \equiv E\mathbf{Z}_j\bar{\mathbf{Z}}_k = \begin{cases} \sigma_k^2 & \text{if } j = k \\ (\alpha_{jk} + i\beta_{jk})\sigma_j\sigma_k & \text{if } j \neq k. \end{cases} \tag{3.2}$$

Thus the complex covariance matrix of the p-variate complex random variable ξ is

$$E\xi\bar{\xi}' = \|E\mathbf{Z}_j\bar{\mathbf{Z}}_k\| = \|\sigma_{jk}\| = \Sigma_\xi \tag{3.3}$$

where the Hermitian matrix $\Sigma_\xi \cong 2\Sigma_\eta$ under the isomorphism $\cong$ established in §2.

THEOREM 3.1. *For the special type of convariance matrices* Σ_η *given by* (3.1) *the probability density function* (p.d.f.) $p(\eta)$ *of the* $2p$*-variate Gaussian random variable* $\mathbf{n}'$ *is given by*

$$p(\eta) = p(\xi) \equiv (1/\pi^p|\Sigma_\xi|) \exp(-\bar{\xi}'\Sigma_\xi^{-1}\xi). \tag{3.4}$$

PROOF. The p.d.f. of $\mathbf{n}'$ is

$$p(\eta) = [1/(2\pi)^p|\Sigma_\eta|^{\frac{1}{2}}] \exp(-\tfrac{1}{2}\eta'\Sigma_\eta^{-1}\eta). \tag{3.5}$$

The matrix $\Sigma_\xi \cong 2\Sigma_\eta$. Since Σ_η is positive definite, Σ_ξ is positive definite by Corollary 2.2 and in particular Σ_ξ is non singular. Furthermore, by Theorem 2.5, $|2\Sigma_\eta| = |\Sigma_\xi|^2$ so that $2^{2p}|\Sigma_\eta| = |\Sigma_\xi|^2$ and hence $|\Sigma_\eta|^{\frac{1}{2}} = 2^{-p}|\Sigma_\xi|$. By Theorem 2.3, $\Sigma_\xi^{-1} \cong \frac{1}{2}\Sigma_\eta^{-1}$ so that by Theorem 2.6 $\eta'(\frac{1}{2}\Sigma_\eta^{-1})\eta = \bar{\xi}'\Sigma_\xi^{-1}\xi$. The p.d.f. $p(\eta)$ is therefore given by (3.4).

COMMENT. Henceforth the phrase "multivariate complex Gaussian distribution" is restricted to apply to a distribution with probability density function $p(\xi)$ given by (3.4).

EXAMPLE 3.1. The univariate complex Gaussian distribution. ($p = 1$). Here $\xi' = \mathbf{Z}_1 = \mathbf{X}_1 + i\mathbf{Y}_1$ and $\xi' = z_1 = x_1 + iy_1$. The covariance matrix $\Sigma_\xi = \sigma_1^2$, $|\Sigma_\xi| = \sigma_1^2$, and $\Sigma_\xi^{-1} = 1/\sigma_1^2$. Thus,

$$p(\eta) = \frac{1}{\pi\sigma_1^2} \exp\left(-\frac{x_1^2 + y_1^2}{\sigma_1^2}\right) = \frac{1}{\pi\sigma_1^2} \exp\left(-\frac{|z_1|^2}{\sigma_1^2}\right). \tag{3.6}$$

EXAMPLE 3.2. The bivariate complex Gaussian distribution. ($p = 2$). Here $\xi' = (\mathbf{Z}_1, \mathbf{Z}_2) = (\mathbf{X}_1 + i\mathbf{Y}_1, \mathbf{X}_2 + i\mathbf{Y}_2)$ and $\xi' = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$,

$$\Sigma_\xi = \begin{bmatrix} \sigma_1^2 & (\alpha_{12} + i\beta_{12})\sigma_1\sigma_2 \\ (\alpha_{12} - i\beta_{12})\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}. \tag{3.7}$$

From (3.7), $|\Sigma_\xi| = (1 - \alpha_{12}^2 - \beta_{12}^2)\sigma_1^2\sigma_2^2$, and

$$\Sigma_\xi^{-1} = \frac{1}{(1 - \alpha_{12}^2 - \beta_{12}^2)\sigma_1^2\sigma_2^2} \begin{bmatrix} \sigma_2^2 & -(\alpha_{12} + i\beta_{12})\sigma_1\sigma_2 \\ -(\alpha_{12} - i\beta_{12})\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}. \tag{3.8}$$

Thus,

$$\begin{aligned} p(\eta) = {} & \frac{1}{\pi^2(1 - \alpha_{12}^2 - \beta_{12}^2)\sigma_1^2\sigma_2^2} \\ & \cdot \exp\left(-\frac{\sigma_2^2|z_1|^2 + \sigma_1^2|z_2|^2 - 2\sigma_1\sigma_2 \text{ R. P. } (\alpha_{12} + i\beta_{12})\bar{z}_1 z_2}{(1 - \alpha_{12}^2 - \beta_{12}^2)\sigma_1^2\sigma_2^2}\right). \end{aligned} \tag{3.9}$$

Notation. When functional dependence on the Hermitian covariance matrix Σ_ξ is to be emphasized the p.d.f. $p(\xi)$ given by (3.4) will be written $p(\xi; \Sigma_\xi)$ and expectation with respect to $p(\xi; \Sigma_\xi)$ will be written $E_{p(\xi;\Sigma_\xi)}$.

4. The maximum likelihood estimator for the Hermitian covariance matrix of the multivariate complex Gaussian distribution.

LEMMA 4.1. *If H denotes an Hermitian positive definite matrix, then the charac-*

teristic function with respect to the density $p(\xi;\, \Sigma_\xi)$ *of the Hermitian form* $Q_H = \bar{\xi}'H\xi$ is

$$\Phi_{\Sigma_\xi, H}(\theta) = \det^{-1}(I - i\theta\Sigma_\xi H). \tag{4.0}$$

PROOF. From (3.4) one obtains the integral identity

$$\int_\xi \pi^{-p} \exp\,(-\bar{\xi}'\Sigma_\xi^{-1}\xi)\; d\xi = \det\,(\Sigma_\xi). \tag{4.1}$$

From (4.1) one has formally

$$\begin{aligned} \Phi_{\Sigma_\xi, H}(\theta) &= E_{p(\xi;\Sigma_\xi)} \exp\,(i\theta\bar{\xi}'H\xi) \\ &= \int_\xi \pi^{-p} \det^{-1}(\Sigma_\xi) \exp\,[-\bar{\xi}'(\Sigma_\xi^{-1} - i\theta H)\xi]\; d\xi \\ &= \det^{-1}(\Sigma_\xi) \det\,([\Sigma_\xi^{-1} - i\theta H]^{-1}) = \det^{-1}(I - i\theta\Sigma_\xi H). \end{aligned} \tag{4.2}$$

Comment. Lemma 4.1 is also a consequence of a result of Turin [12].

COROLLARY 4.1. *If H denotes an Hermitian positive definite matrix then*

$$E_{p(\xi;\Sigma_\xi)}\bar{\xi}'H\xi = \operatorname{tr}\,(\Sigma_\xi H). \tag{4.3}$$

PROOF. From (4.2) one has

$$\begin{aligned} E_{p(\xi;\Sigma_\xi)}\bar{\xi}'H\xi &= -i(d/d\theta)\Phi_{\Sigma_\xi, H}(\theta)]_{\theta=0} \\ &= -i(-)\det^{-2}(I - i\theta\Sigma_\xi H)(d/d\theta)\det\,(I - i\theta\Sigma_\xi H)]_{\theta=0}\,. \end{aligned} \tag{4.4}$$

Now,

$$(d/d\theta)\det\,(I - i\theta\Sigma_\xi H)]_{\theta=0} = -\operatorname{tr}\,(\Sigma_\xi H). \tag{4.5}$$

Equation (4.3) follows from (4.4) and (4.5).

THEOREM 4.1. *Consider n independent identically distributed p-variate complex Gaussian random variables* ξ_j, $j = 1, 2, \cdots, n$ *as a sample of size n from a population with* p.d.f. $p(\xi)$ *given by* (3.4). *Let* D_H *be the set of* $p \times p$ *Hermitian positive definite matrices. Over the domain* D_H *the maximum likelihood estimator* $\hat{\Sigma}_\xi$ *of the Hermitian covariance matrix* Σ_ξ *is*

$$\hat{\Sigma}_\xi = (1/n) \sum_{j=1}^{n} \xi_j\bar{\xi}_j'\,. \tag{4.6}$$

PROOF. From (3.4) the p.d.f. $p(\xi_1, \xi_2, \cdots, \xi_n)$ of $\xi_1, \xi_2, \cdots, \xi_n$ is

$$L = p(\xi_1, \xi_2, \cdots, \xi_n) = \pi^{-pn} \det^{-n}(\Sigma_\xi) \exp\,(-\sum_{j=1}^{n} \bar{\xi}_j'\Sigma_\xi^{-1}\xi_j). \tag{4.7}$$

Thus

$$\ln L = -pn \ln \pi - n \ln \det\,(\Sigma_\xi) - \sum_{j=1}^{n} \bar{\xi}_j'\Sigma_\xi^{-1}\xi_j\,. \tag{4.8}$$

Now,

$$
\begin{aligned}
\sum_{j=1}^{n} \bar{\xi}_j' \Sigma_\xi^{-1} \xi_j &= \sum_{j=1}^{n} \operatorname{tr}\,(\bar{\xi}_j' \Sigma_\xi^{-1} \xi_j) = \sum_{j=1}^{n} \operatorname{tr}\,(\Sigma_\xi^{-1} \xi_j \bar{\xi}_j') \\
&= \operatorname{tr}\left(\Sigma_\xi^{-1}\left(\sum_{j=1}^{n} \xi_j \bar{\xi}_j'\right)\right) = n \operatorname{tr}\,(\Sigma_\xi^{-1} B)
\end{aligned} \tag{4.9}
$$

where

$$
B = \frac{1}{n} \sum_{j=1}^{n} \xi_j \, \bar{\xi}_j' . \tag{4.10}
$$

Thus,

$$
\ln L = -pn \ln \pi - n \ln \det\,(\Sigma_\xi) - n \operatorname{tr}\,(\Sigma_\xi^{-1} B). \tag{4.11}
$$

Now, from (3.4) and (4.3) the integral

$$
\begin{aligned}
J &\equiv \int_\xi p(\xi; \Sigma_\xi^{-1}) \ln\,[p(\xi; B^{-1})/p(\xi; \Sigma_\xi^{-1})]\, d\xi \\
&= \int_\xi [p(\xi; \Sigma_\xi^{-1}) \ln p(\xi; B^{-1}) - p(\xi; \Sigma_\xi^{-1}) \ln p(\xi; \Sigma_\xi^{-1})]\, d\xi \\
&= \int_\xi \Big\{[\ln \det\,(B) - \bar{\xi}' B \xi] p(\xi; \Sigma_\xi^{-1}) \\
&\qquad - [\ln \det\,(\Sigma_\xi) - \bar{\xi}' \Sigma_\xi \, \xi] p(\xi; \Sigma_\xi^{-1})\Big\}\, d\xi \\
&= \ln \det\,(B) - \operatorname{tr}\,(\Sigma_\xi^{-1} B) - \ln \det\,(\Sigma_\xi) + \operatorname{tr}\,(I).
\end{aligned} \tag{4.12}
$$

On comparing the final result of (4.12) with (4.11) one observes that any Hermitian positive definite matrix Σ_ξ that maximizes $\ln L$ maximizes J and conversely. Now, $\ln u \leqq u - 1$ with equality holding if and only if $u = 1$. Thus,

$$
\begin{aligned}
J &= \int_\xi p(\xi; \Sigma_\xi^{-1}) \ln \left(\frac{p(\xi; B^{-1})}{p(\xi; \Sigma_\xi^{-1})}\right) d\xi \\
&\leqq \int_\xi p(\xi; \Sigma_\xi^{-1}) \left[\frac{p(\xi; B^{-1})}{p(\xi; \Sigma_\xi^{-1})} - 1\right] d\xi = 0.
\end{aligned} \tag{4.13}
$$

Equality will hold in (4.13) if and only if $p(\xi; B^{-1}) = p(\xi; \Sigma_\xi^{-1})$, i.e., if and only if $\Sigma_\xi^{-1} = B^{-1}$, i.e., if and only if $\Sigma_\xi = B$. Thus $\hat{\mathbf{\Sigma}}_\xi = \mathbf{B}$.

Theorem 4.2. *The maximum likelihood estimator* $\hat{\mathbf{\Sigma}}_\xi$ *is a sufficient statistic for the Hermitian covariance matrix* Σ_ξ.

Proof. The p.d.f. $p(\xi_1, \xi_2, \cdots, \xi_n)$ of $\xi_1, \xi_2, \cdots, \xi_n$ is from (4.7)

$$
\begin{aligned}
p(\xi_1, \xi_2, \cdots, \xi_n) &= \pi^{-pn} \det{}^{-n}\,(\Sigma_\xi) \exp\left(-\sum_{j=1}^{n} \bar{\xi}_j' \Sigma_\xi^{-1} \xi_j\right) \\
&= \pi^{-pn} \det{}^{-n}\,(\Sigma_\xi) \exp\left(-\sum_{j=1}^{n} \operatorname{tr}\,(\Sigma_\xi^{-1} \xi_j \bar{\xi}_j')\right) \\
&= \pi^{-pn} \det{}^{-n}\,(\Sigma_\xi) \exp\,(-n \operatorname{tr}\,(\Sigma_\xi^{-1} \hat{\Sigma}_\xi)).
\end{aligned} \tag{4.14}
$$

Since $p(\xi_1, \xi_2, \cdots, \xi_n)$ can be expressed as a function of $\hat{\Sigma}_\xi$ it follows from the Neyman criterion for a sufficient statistic that $\hat{\mathbf{\Sigma}}_\xi$ is a sufficient statistic for Σ_ξ. (See Halmos and Savage [9].)

5. The complex Wishart distribution and related distributions.

THEOREM 5.1. *Consider* $\mathbf{A} = \|\mathbf{A}_{jk}\| = \|\mathbf{A}_{jkR} + i\mathbf{A}_{jkI}\| = n\hat{\mathbf{\Sigma}}_\xi$. *The probability density function of the joint distribution of* $\mathbf{A}_{11}, \cdots, \mathbf{A}_{pp}, \mathbf{A}_{12R}, \mathbf{A}_{12I}, \cdots, \mathbf{A}_{p-1,pR}, \mathbf{A}_{p-1,pI}$ *is*

$$p_W(A) = (|A|^{n-p}/I(\Sigma_\xi)) \exp(-\mathrm{tr}(\Sigma_\xi^{-1}A)), \tag{5.0}$$

where $I(\Sigma_\xi) = \pi^{\frac{1}{2}p(p-1)}\Gamma(n) \cdots \Gamma(n-p+1)|\Sigma_\xi|^n$. *The density is defined over the domain* D_A *where* A *is Hermitian positive semi-definite.*

PROOF. (The method of proof is as follows. The characteristic function of $\mathbf{A}_{11}, \cdots, \mathbf{A}_{pp}, \mathbf{A}_{12R}, \mathbf{A}_{12I}, \mathbf{A}_{13R}, \mathbf{A}_{13I}, \cdots, \mathbf{A}_{p-1,pR}, \mathbf{A}_{p-1,pI}$ is computed. The (multidimensional) Fourier transform of $p_W(A)$ is computed. The two are seen to be equal. The matrix methods employed in the details of proof introduce a triangular representation patterned after the Bartlett decomposition. (See Bartlett [2].))

The matrix $\mathbf{A}$ is (apart from a factor of $1/n$) the sample Hermitian covariance matrix. Let

$$\xi_s' = (\mathbf{Z}_{1s}, \mathbf{Z}_{2s}, \mathbf{Z}_{3s}, \cdots, \mathbf{Z}_{ps}), \qquad s = 1, 2, \cdots n \tag{5.1}$$

so that

$$\mathbf{A} = \|\mathbf{A}_{jk}\| = \left\|\sum_{s=1}^{n} \mathbf{Z}_{js}\bar{\mathbf{Z}}_{ks}\right\|, \tag{5.2}$$

and

$$\sum_{s=1}^{n} \mathbf{Z}_{js}\bar{\mathbf{Z}}_{ks} = \sum_{s=1}^{n} [(\mathbf{X}_{js}\mathbf{X}_{ks} + \mathbf{Y}_{js}\mathbf{Y}_{ks}) + i(-\mathbf{X}_{js}\mathbf{Y}_{ks} + \mathbf{Y}_{js}\mathbf{X}_{ks})] \tag{5.3}$$

where

$$\mathbf{Z}_{js} = \mathbf{X}_{js} + i\mathbf{Y}_{js}, \qquad j = 1, 2, \cdots p; s = 1, 2, \cdots n.$$

The joint distribution of $\mathbf{A}_{jk}$, $j \leqq k = 1, 2, \cdots p$ is called the *complex Wishart* distribution. The characteristic function of the complex Wishart distribution is now derived.

Let

$$\Theta = \|\theta_{jk}\| \tag{5.4}$$

where $\theta_{kj} = \bar{\theta}_{jk}$ and $\theta_{jk} = \theta_{jkR} + i\theta_{jkI}$; $j, k, = 1, 2, \cdots p$. Then,

$$\begin{aligned} \mathrm{tr}(\mathbf{A}\Theta) &= \sum_{j,k=1}^{p} \mathbf{A}_{jk}\bar{\theta}_{jk} = \sum_{j=1}^{p} \mathbf{A}_{jj}\theta_{jj} + \sum_{\substack{j<k \\ j,k=1}}^{p} (\mathbf{A}_{jk}\bar{\theta}_{jk} + \bar{\mathbf{A}}_{jk}\theta_{jk}) \\ &= \sum_{j=1}^{p} \mathbf{A}_{jj}\theta_{jj} + 2\sum_{\substack{j<k \\ j,k=1}}^{p} (\mathbf{A}_{jkR}\theta_{jkR} + \mathbf{A}_{jkI}\theta_{jkI}). \end{aligned} \tag{5.5}$$

By virtue of (5.5), the characteristic function of the joint distribution of $\mathbf{A}_{11}, \cdots, \mathbf{A}_{pp}, 2\mathbf{A}_{12R}, 2\mathbf{A}_{12I}, 2\mathbf{A}_{13R}, 2\mathbf{A}_{13I}, \cdots, 2\mathbf{A}_{p-1,pR}, 2\mathbf{A}_{p-1,pI}$ is

$$
\begin{aligned}
E \exp(i \operatorname{tr}(\mathbf{A}\Theta)) &= E \exp\left(i \operatorname{tr}\left(\sum_{s=1}^{n} \xi_s \bar{\xi}_s' \Theta\right)\right) \\
&= E \exp\left(i \operatorname{tr}\left(\sum_{s=1}^{n} \bar{\xi}_s' \Theta \xi_s\right)\right) \qquad (5.6)\\
&= E \exp\left(i \sum_{s=1}^{n} \bar{\xi}_s' \Theta \xi_s\right).
\end{aligned}
$$

Since the ξ_s, $s = 1, \cdots, n$ are independent and identically distributed

$$E \exp\left(i \sum_{s=1}^{n} \bar{\xi}_s' \Theta \xi_s\right) = [E \exp(i\bar{\xi}'\Theta\xi)]^n. \qquad (5.7)$$

For Θ Hermitian positive definite there exists a non singular matrix M such that

$$\bar{M}'\Sigma_\xi^{-1}M = I \quad \text{and} \quad \bar{M}'\Theta M = D, \qquad (5.8)$$

where D is a real diagonal matrix with $d_{jj} > 0$, $j = 1, \cdots, p$.
Let

$$\xi = M\zeta, \qquad \zeta' = (\zeta_1, \zeta_2, \cdots, \zeta_p); \qquad (5.9)$$

so that

$$
\begin{aligned}
E \exp(i\bar{\xi}'\Theta\xi) &= E \exp(i\bar{\zeta}'\bar{M}'\Theta M\zeta) = E \exp(i\bar{\zeta}'D\zeta) \\
&= \prod_{j=1}^{p} E \exp(i\, d_{jj}|\zeta_j|^2) = \prod_{j=1}^{p} (1 - i\, d_{jj})^{-1} = |I - i\, D|^{-1} \qquad (5.10)\\
&= |\bar{M}'\Sigma_\xi^{-1}M - i\bar{M}'\Theta M|^{-1} = |\bar{M}'|^{-1}|\Sigma_\xi^{-1} - i\Theta|^{-1}|M|^{-1}.
\end{aligned}
$$

From (5.8)

$$|\bar{M}'|\,|\Sigma_\xi^{-1}|\,|M| = 1. \qquad (5.11)$$

Thus, from (5.6), (5.7), and (5.10)

$$E \exp(i \operatorname{tr}(\mathbf{A}\Theta)) = |\Sigma_\xi|^{-n}|\Sigma_\xi^{-1} - i\Theta|^{-n}. \qquad (5.12)$$

Let Σ denote an Hermitian positive definite matrix (constant) and consider the problem of evaluating the following integral

$$I(\Sigma) = \int_{D_H} |H|^k \exp(-\operatorname{tr}(\Sigma^{-1}H))\, dH, \qquad (5.13)$$

where the domain D_H of integration is the space of Hermitian positive definite matrices $H = \|H_{jkR} + iH_{jkI}\|$ and the differential element dH is

$$dH = (dH_{11} \cdots dH_{pp}) \prod_{\substack{j<k \\ j,k=1}}^{p} dH_{jkR}\, dH_{jkI}. \qquad (5.14)$$

Let

$$T = \begin{bmatrix} T_{11} & \cdots & \cdots & T_{1p} \\ 0 & T_{22} & \cdots & T_{2p} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & T_{pp} \end{bmatrix}, \qquad T_{jk} = T_{jkR} + iT_{jkI},$$

and consider the equation

$$\bar{T}'T = H. \tag{5.16}$$

It is now stated that, with the additional restriction that the T_{jj} be real and positive($j = 1, \cdots, p$), there exists a unique T of the form given in (5.15) satisfying (5.16). To prove that consider the Hermitian positive definite form in $z_1, \cdots, z_p$ given by

$$H(z_1, \cdots, z_p) = \sum_{j,k=1}^{p} H_{jk}\bar{z}_j z_k . \tag{5.17}$$

Now,

$$\begin{aligned} H(z_1, \cdots, z_p) &= H_{11}|z_{11}|^2 + \sum_{j=2}^{p} (H_{1j}\bar{z}_1 z_j + \bar{H}_{1j} z_1 \bar{z}_j) + \sum_{j,k=2}^{p} H_{jk}\bar{z}_j z_k \\ &= H_{11}\left(z_1 + \sum_{j=2}^{p} \frac{H_{1j}}{H_{11}} z_j\right)\left(\bar{z}_1 + \sum_{j=2}^{p} \frac{\bar{H}_{1j}}{H_{11}} \bar{z}_j\right) + \sum_{j,k=2}^{p} \frac{(H_{11}H_{jk} - \bar{H}_{1j}H_{1k})}{H_{11}} \bar{z}_j z_k \\ &= H_{11}\left|z_1 + \sum_{j=2}^{p} \frac{H_{1j}}{H_{11}} z_j\right|^2 + H_{p-1}(z_2, \cdots, z_p). \end{aligned} \tag{5.18}$$

The element H_{11} is real, >0. Let

$$H_{11}^{\frac{1}{2}} z_1 + \sum_{j=2}^{p} \frac{H_{1j}}{H_{11}^{\frac{1}{2}}} z_j = \sum_{j=1}^{p} T_{1j} z_j . \tag{5.19}$$

Since $H = H_p$ is Hermitian positive definite, H_{p-1} is Hermitian positive definite, so that by continuing the above procedure (completing the square)

$$\begin{aligned} H_p(z_1, \cdots, z_p) &= \left|\sum_{j=1}^{p} T_{1j} z_j\right|^2 \\ &\quad + \left|\sum_{j=2}^{p} T_{2j} z_j\right|^2 + \cdots + \left|\sum_{j=p-1}^{p} T_{p-1,j} z_j\right|^2 + |T_{pp}|^2 |z_p|^2. \end{aligned} \tag{5.20}$$

In (5.20) T_{jj} real, $>0, j = 1, 2, \cdots, p$, and it is clear that this condition makes the representation of $H_p(z_1, \cdots, z_p)$ in the triangular form (5.20) unique. It is also clear that the equation in matrices (5.16) is an equivalent way of expressing (5.20). Furthermore, from the equivalence of (5.20) and (5.16), any matrix of the form $\bar{T}'T$ with T_{jj} real, $>0, j = 1, \cdots, p$ is Hermitian positive definite. Thus, there is a one-to-one correspondence between triangular matrices of the form (5.15) with diagonal elements real and positive and Hermitian positive definite matrices. Consider now the real variables (p^2 in number) $H_{11}, \cdots, H_{pp}, H_{12R}, H_{12I}, \cdots, H_{p-1,pR}, H_{p-1,pI}$ satisfying the inequalities (leading minor determinants positive) that make H Hermitian positive definite and the variables $T_{11}, \cdots, T_{pp}, T_{12R}, T_{12I}, \cdots, T_{p-1,pR}, T_{p-1,pI}$ satisfying the inequalities

$$(5.21) \qquad \begin{aligned} &T_{11} > 0, \cdots, T_{pp} > 0 \\ D_T : -\infty < T_{jkR} < \infty \qquad &-\infty < T_{jkI} < \infty, \qquad j < k, j, k = 1, \cdots, p. \end{aligned}$$

From (5.16)

$$(5.22) \quad H_{jk} = H_{jkR} + iH_{jkI} = \sum_{s=1}^{\min(j,k)} \bar{T}_{sj}T_{sk}\ ; j < k; j, k = 1, \cdots, p.$$

Let J denote the Jacobian

$$(5.23) \quad J = \frac{\partial(H_{11}, H_{12R}, H_{12I}, \cdots, H_{1pR}, H_{1pI}, H_{22}, \cdots H_{2pR}, H_{2pI}, \cdots, H_{pp})}{\partial(T_{11}, T_{12R}, T_{12I}, \cdots, T_{1pR}, T_{1pI}, T_{22}, \cdots, T_{2pR}, T_{2pI}, \cdots, T_{pp})}.$$

The order of the variables in (5.23) is important. With the order indicated the Jacobian matrix is triangular since the summation in (5.22) extends only to $\min(j, k)$. Thus, to evaluate J it is only necessary to compute the diagonal elements of the Jacobian matrix. From (5.22) the diagonal elements are

$$(5.24) \quad \begin{aligned} &\partial H_{jkR}/\partial T_{jkR} = T_{jj}, \ \partial H_{jkI}/\partial T_{jkI} = T_{jj}, \\ &\qquad\qquad \partial H_{jj}/\partial T_{jj} = 2T_{jj}\ ; j < k, j, k = 1, 2, \cdots p. \end{aligned}$$

Thus,

$$(5.25) \quad \begin{aligned} J &= \prod_{j=1}^{p} (\partial H_{jj}/\partial T_{jj}) \prod_{\substack{j<k \\ j,k=1}}^{p} [(\partial H_{jkR}/\partial T_{jkR})(\partial H_{jkI}/\partial T_{jkI})] \\ J &= 2^p T_{11}^{2p-1} T_{22}^{2p-3} \cdots T_{pp}. \end{aligned}$$

Consider now the special case when $\Sigma = D_\lambda$, where

$$(5.26) \qquad D_\lambda = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_j & & \\ & & & \ddots & \\ 0 & & & & \lambda_p \end{bmatrix}, \qquad \lambda_j > 0, j = 1, \cdots, p.$$

From (5.13)

$$(5.27) \quad \begin{aligned} I(D_\lambda) &= \int_{D_H} |H|^k \exp(-\operatorname{tr}(D_{\lambda^{-1}} H))\, dH \\ &= \int_{D_T} |T|^{2k} \exp(-\operatorname{tr}(D_{\lambda^{-1}} \bar{T}'T)) 2^p T_{11}^{2p-1} \cdots T_{pp}^{1}\, dT \\ &= 2^p \int_{D_T} T_{11}^{2(p+k)-1} T_{22}^{2(p+k)-3} \cdots T_{p-1,p-1}^{2k+3} T_{pp}^{2k+1} \exp(-\operatorname{tr}(D_{\lambda^{-1}} \bar{T}'T))\, dT, \end{aligned}$$

where the domain D_T of integration is given by (5.21) and the differential element dT is

$$dT = dT_{11} \cdots dT_{pp} \prod_{\substack{j<k \\ j,k=1}}^{p} dT_{jkR}\, dT_{jkI} . \tag{5.28}$$

From (5.15)

$$\begin{aligned} &\operatorname{tr}(D_{\lambda^{-1}} \bar{T}' T) \\ &= \lambda_1^{-1} T_{11}^2 + \lambda_2^{-1}(|T_{12}|^2 + T_{22}^2) + \cdots + \lambda_p^{-1}(|T_{1p}|^2 + \cdots + T_{pp}^2). \end{aligned} \tag{5.29}$$

Thus, $I(D_\lambda)$ can be expressed as a product of integrals. In particular,

$$I(D_\lambda) = 2^p I_{11} \cdots I_{pp} \prod_{\substack{i<j \\ i,j=1}}^{p} I_{ij} , \tag{5.30}$$

where

$$\begin{aligned} I_{jj} &= \int_0^\infty T_{jj}^{2(p+k)-(2j-1)} \exp(-\lambda_j^{-1} T_{jj}^2)\, dT_{jj} \\ &= \frac{1}{2} \lambda_j^{p+k-j+1} \int_0^\infty u^{p+k-j} \exp(-u)\, du \\ &= \tfrac{1}{2} \lambda_j^{p+k-j+1} \Gamma(p+k-j+1), \quad \text{for} \quad j = 1, 2, \cdots p; \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} I_{ij} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\,[-\lambda_j^{-1}(T_{ijR}^2 + T_{ijI}^2)]\, dT_{ijR}\, dT_{ijI} \\ &= \int_{-\pi}^{\pi} \int_0^\infty \exp(-\lambda_j^{-1} R^2) R\, dR\, d\theta = 2\pi \int_0^\infty \exp(-\lambda_j^{-1} R^2) R\, dR \\ &= \pi \lambda_j , \text{ for } i < j, i, j = 1, 2, \cdots p. \end{aligned} \tag{5.32}$$

Thus, from (5.30), (5.31) and (5.32)

$$\begin{aligned} I(D_\lambda) &= 2^p \left[\prod_{j=1}^{p} \frac{1}{2} \lambda_j^{p+k-j+1} \Gamma(p+k-j+1) \right] \prod_{\substack{i<j \\ i,j=1}}^{p} (\pi \lambda_j) \\ &= \prod_{j=1}^{p} \pi^{j-1} \Gamma(p+k-j+1) \lambda_j^{p+k} \\ &= \pi^{\frac{1}{2}p(p-1)} \Gamma(p+k) \cdots \Gamma(1+k) |D_\lambda|^{p+k}. \end{aligned} \tag{5.33}$$

Any Hermitian positive definite Σ can be written in the form

$$\Sigma = \bar{U}' D_\lambda U, \tag{5.34}$$

where $\bar{U}'U = I$. Thus,

$$I(\Sigma) = \int_{D_H} |H|^k \exp(-\text{tr}(\bar{U}' D_{\lambda^{-1}} UH))\, dH \tag{5.35}$$
$$= \int_{D_H} |H|^k \exp(-\text{tr}(D_{\lambda^{-1}} UH\bar{U}'))\, dH.$$

Let

$$K = UH\bar{U}'. \tag{5.36}$$

The K corresponding to any Hermitian positive definite H is Hermitian positive definite, and conversely. Thus, (5.36) can be thought of as a one-to-one transformation of the space of Hermitian positive definite matrices onto itself. From (5.36) it is noted that any element of K is a linear function of the elements of H with coefficients quadratic functions of the elements of U. Thus, the Jacobian

$$\frac{\partial(K_{11}, \cdots, K_{pp}, K_{12R}, \cdots, K_{p-1,p,I})}{\partial(H_{11}, \cdots, H_{pp}, H_{12R}, \cdots, H_{p-1,p,I})} = J(U), \tag{5.37}$$

i.e. the Jacobian in(5.37) is a function of U alone. Explicitly, $J(U)$ is the $p^2 \times p^2$ determinant of the linear equations expressing the K_{jk} in terms of the H_{jk}. Thus, from (5.34), (5.35), (5.36) and (5.37) one has

$$I(\Sigma) = \int_{D_H} |H|^k \exp(-\text{tr}(D_{\lambda^{-1}} K))dH \tag{5.38}$$
$$= J^{-1}(U) \int_{D_K} |K|^k \exp(-\text{tr}(D_{\lambda^{-1}} K))\, dK.$$

In (5.38), let $D_\lambda = I_p$. Thus,

$$I(I_p) = J^{-1}(U) I(I_p). \tag{5.39}$$

The value of $I(I_p)$ is given by (5.33) and, in particular, $I(I_p) \neq 0$. Thus, from (5.39) $J(U) = 1$ for every unitary matrix U. From (5.38) and (5.33) one has

$$I(\Sigma) = \pi^{\frac{1}{2}p(p-1)} \Gamma(p + k) \cdots \Gamma(1 + k) |\Sigma|^{p+k} \tag{5.40}$$

From (5.13) and (5.40)

$$p(H) \equiv I^{-1}(\Sigma) |H|^k \exp(-\text{tr}(\Sigma^{-1} H)) \tag{5.41}$$

is a probability density on the space D_H of Hermitian positive definite matrices H. One has

$$\int_{D_H} p(H)\, dH = 1. \tag{5.42}$$

From (5.13), (5.40) and (5.41) one obtains

$$\int_{D_H} \exp(i \,\text{tr}(H\Theta)) p(H)\, dH$$
$$= \int_{D_H} I^{-1}(\Sigma) |H|^k \exp[-\text{tr}(\Sigma^{-1} H - i\Theta H)]\, dH \tag{5.43}$$
$$= I^{-1}(\Sigma) I((\Sigma^{-1} - i\Theta)^{-1}) = \frac{|(\Sigma^{-1} - i\Theta)^{-1}|^{p+k}}{|\Sigma|^{p+k}} = \frac{|\Sigma^{-1}|^{p+k}}{|\Sigma^{-1} - i\Theta|^{p+k}}.$$

Now, in(5.41) and(5.43) let $\Sigma = \Sigma_\xi$ and $k = n - p$. Thus,

$$\int_{D_H} \exp(i \operatorname{tr}(H\Theta))p(H)\, dH = |\Sigma_\xi|^{-n}|\Sigma_\xi^{-1} - i\Theta|^{-n}. \tag{5.44}$$

Upon comparing the final result in (5.44) with (5.12) one concludes (a characteristic function uniquely determines a distribution function) that the probability density function of the joint distribution of the random variables $\mathbf{A}_{11}, \cdots, \mathbf{A}_{pp}, \mathbf{A}_{12R}, \mathbf{A}_{12I}, \cdots, \mathbf{A}_{p-1,pR}, \mathbf{A}_{p-1,pI}$ is given by (5.0).

EXAMPLE 5.1. The univariate complex Wishart distribution. ($p = 1$). Here $\Sigma_\xi = \sigma_1^2$ and $A = A_{11}$. Thus,

$$p_W(A) = (A_{11}^{n-1}/\Gamma(n)\sigma_1^{2n}) \exp(-A_{11}/\sigma_1^2). \tag{5.45}$$

EXAMPLE 5.2. The bivariate complex Wishart distribution. ($p = 2$). Here

$$\begin{aligned} \Sigma_\xi &= \begin{bmatrix} \sigma_1^2 & (\alpha_{12} + i\beta_{12})\sigma_1\sigma_2 \\ (\alpha_{12} - i\beta_{12})\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \\ A &= \begin{bmatrix} A_{11} & A_{12R} + iA_{12I} \\ A_{12R} - iA_{12I} & A_{22} \end{bmatrix}. \end{aligned} \tag{5.46}$$

Thus, using the expression for Σ_ξ^{-1} given by (3.8)

$$\begin{aligned} p_W(A) = &\frac{(A_{11}A_{22} - A_{12R}^2 - A_{12I}^2)^{n-2}}{\pi\Gamma(n)\Gamma(n-1)(1 - \alpha_{12}^2 - \beta_{12}^2)^n\sigma_1^{2n}\sigma_2^{2n}} \\ &\cdot \exp\left(-\frac{\sigma_2^2 A_{11} - 2\alpha_{12}\sigma_1\sigma_2 A_{12R} - 2\beta_{12}\sigma_1\sigma_2 A_{12I} + \sigma_1^2 A_{22}}{(1 - \alpha_{12}^2 - \beta_{12}^2)\sigma_1^2\sigma_2^2}\right). \end{aligned} \tag{5.47}$$

Comment. Consider the matrix **A** of Theorem 5.1 and the matrix **T** defined by (5.15) with the diagonal elements $\mathbf{T}_{jj}$, $j = 1, \cdots, p$ real and positive. As proved earlier there is a unique such **T** satisfying

$$\bar{\mathbf{T}}'\mathbf{T} = \mathbf{A}. \tag{5.48}$$

From (5.25) the Jacobian

$$J = \partial(A)/\partial(T) = 2^p T_{11}^{2p-1} T_{22}^{2p-3} \cdots T_{pp}. \tag{5.49}$$

In (5.49) $\partial(A)/\partial(T)$ denotes as in (5.23) the Jacobian of the distinct elements of A with respect to the elements of T. From (5.15) and (5.48)

$$|A| = T_{11}^2 T_{22}^2 \cdots T_{pp}^2. \tag{5.50}$$

From (5.0) and (5.40) the probability density function of the matrix **T** is

$$p(T) = 2^p[1/I(\Sigma_\xi)]T_{11}^{2n-1}T_{22}^{2n-3} \cdots T_{pp}^{2n-(2p-1)} \exp(-\operatorname{tr}(\Sigma_\xi^{-1}\bar{T}'T)) \tag{5.51}$$

where $I(\Sigma_\xi) = \pi^{\frac{1}{2}p(p-1)}\Gamma(n) \cdots \Gamma(n - p + 1)|\Sigma_\xi|^n$. The distribution of the matrix **T** is useful in deriving the distributions of functions of the elements of a complex Wishart distributed matrix. Two examples of such functions are now given. Let $\mathbf{A}^{-1} = \|\mathbf{A}^{jk}\|$. Consider the function

$$\hat{\mathbf{R}}^2_{p\cdot p-1,p-2,\cdots,1} = 1 - (\mathbf{A}_{pp}\mathbf{A}^{pp})^{-1}. \tag{5.52}$$

In terms of the elements of the matrix $\mathbf{T}$ one has

$$\hat{\mathbf{R}}^2_{p\cdot p-1,\cdots,1} = \left(\sum_{j=1}^{p-1} |\mathbf{T}_{jp}|^2\right)\left(\sum_{j=1}^{p} |\mathbf{T}_{jp}|^2\right)^{-1}. \tag{5.53}$$

Starting from (5.51) the probability density function of the distribution of $\hat{\mathbf{R}}^2_{p\cdot p-1,p-2,\cdots,1}$ can be shown to be

$$\begin{aligned} p(\hat{R}^2) &= [\Gamma(n)/\Gamma(p-1)\Gamma(n-p+1)] \\ &\quad \cdot(1-R^2)^n(\hat{R}^2)^{p-2}(1-\hat{R}^2)^{n-p}F(n, n; p-1; R^2\hat{R}^2). \end{aligned} \tag{5.54}$$

In (5.54) $R^2 \equiv 1 - (\sigma_{pp}\sigma^{pp})^{-1}$ where $\Sigma_\xi^{-1} \equiv \|\sigma^{jk}\|$, $\hat{R}^2$ denotes $\hat{R}^2_{p\cdot p-1,\cdots,1}$, and $F(\ ,\ ;\ ;\ )$ denotes the hypergeometric function. Similarly, consider the function

$$\hat{\mathbf{R}}^2_{p,p-1|p-2,\cdots,1} = |\mathbf{A}^{p-1,p}|^2(\mathbf{A}^{p-1,p-1}\mathbf{A}^{pp})^{-1}. \tag{5.55}$$

In terms of the elements of the matrix $\mathbf{T}$ one has

$$\hat{\mathbf{R}}^2_{p,p-1|p-2,\cdots,1} = |\mathbf{T}_{p-1,p}|^2(|\mathbf{T}_{p-1,p}|^2 + \mathbf{T}^2_{pp})^{-1}. \tag{5.56}$$

Starting from (5.51) the probability density function of the distribution of $\hat{\mathbf{R}}^2_{p,p-1|p-2,\cdots,1}$ can be shown to be

$$\begin{aligned} p(\hat{R}^2) &= (n-p+1)(1-R^2)^{n-p+2}(1-\hat{R}^2)^{n-p} \\ &\quad \cdot F(n-p+2, n-p+2; 1; R^2\hat{R}^2). \end{aligned} \tag{5.57}$$

In (5.57) now $R^2 \equiv |\sigma^{p-1,p}|^2(\sigma^{p-1,p-1}\sigma^{pp})^{-1}$ and $\hat{R}^2$ denotes $\hat{R}^2_{p,p-1|p-2,\cdots,1}$. The functions $\hat{\mathbf{R}}^2_{p\cdot p-1,\cdots,1}$ and $\hat{\mathbf{R}}^2_{p,p-1|p-2,\cdots,1}$ are respectively the sample *multiple coherence* between $\mathbf{Z}_p$ and $(\mathbf{Z}_{p-1}, \cdots, \mathbf{Z}_1)$ and the sample *conditional coherence* between $\mathbf{Z}_p$ and $\mathbf{Z}_{p-1}$ with respect to $(\mathbf{Z}_{p-2}, \cdots, \mathbf{Z}_1)$. The distributions given by (5.54) and (5.57) are important in the theory of measuring *coherence* between the components of a multiple stationary Gaussian time series.

The characteristic function of the complex Wishart distribution given essentially by (5.12) is also useful in deriving the distributions of functions of the elements of a complex Wishart distributed matrix. Consider for example the function $\operatorname{tr}(\mathbf{A})$. The characteristic function of $\operatorname{tr}(\mathbf{A})$ is obtained from (5.12) by setting $\theta_{jk} = 0$ if $j \neq k$, $j, k = 1, \cdots, p$ and setting $\theta_{jj} = \theta$, $j = 1, \cdots, p$. Thus, from (5.12) the characteristic function of $\operatorname{tr}(\mathbf{A})$ is

$$\Psi_{\operatorname{tr}(\mathbf{A})}(\theta) = |\Sigma_\xi|^{-n}|\Sigma_\xi^{-1} - iD_\theta|^{-n} \tag{5.58}$$

where D_θ is the $p \times p$ diagonal matrix consisting of all θ's on the diagonal. Since Σ_ξ is Hermitian positive definite, there exists a unitary matrix U such that

$$\bar{U}'\Sigma_\xi^{-1}U = D_{\lambda^{-1}}. \tag{5.59}$$

In (5.59) $D_{\lambda^{-1}}$ denotes the $p \times p$ diagonal matrix consisting of $(\lambda_1^{-1}, \cdots, \lambda_p^{-1})$ down the diagonal and $\lambda_1, \cdots, \lambda_p$ denote the eigenvalues of Σ_ξ. From (5.58) and (5.59) one obtains

$$(5.60)\qquad \begin{aligned} \Psi_{\mathrm{tr}(\mathbf{A})}(\theta) &= |\bar{U}'|^n\, |\Sigma_\xi^{-1}|^n\, |U|^n\, |\bar{U}'|^{-n}\, |\Sigma_\xi^{-1} - iD_\theta|^{-n}\, |U|^{-n} \\ &= |D_{\lambda^{-1}}|^n\, |D_{\lambda^{-1}} - iD_\theta|^{-n} = \frac{\lambda_1^{-n}\lambda_2^{-n} \cdots \lambda_p^{-n}}{(\lambda_1^{-1} - i\theta)^n(\lambda_2^{-1} - i\theta)^n \cdots (\lambda_p^{-1} - i\theta)^n}. \end{aligned}$$

Since $\Psi_{\mathrm{tr}(\mathbf{A})}(\theta)$ is a rational function of θ, the probability density function of the distribution of tr $(\mathbf{A})$ may readily be obtained by Fourier inversion.

6. Applications. A continuous parameter real (zero mean) multiple stationary Gaussian time series $[\mathbf{X}_1(t), \mathbf{X}_2(t), \cdots, \mathbf{X}_j(t), \cdots, \mathbf{X}_p(t)]$ has the (real) spectral representation

$$(6.1)\qquad \mathbf{X}_j(t) = \int_0^\infty [\cos \omega t\, d\mathbf{U}_j(\omega) + \sin \omega t\, d\mathbf{V}_j(\omega)], j = 1, \cdots, p.$$

The integrals in (6.1) are stochastic integrals. Heuristically, one may regard the "differentials" $d\mathbf{U}_j(\omega), d\mathbf{V}_j(\omega)$ as infinitesimal Gaussian random variables. The means are

$$(6.2)\qquad E\, d\mathbf{U}_j(\omega) = 0 = E\, d\mathbf{V}_j(\omega), j = 1, \cdots, p.$$

The covariances are (in the case of absolutely continuous spectral distribution functions)

$$(6.3)\qquad \begin{aligned} &E\; d\mathbf{U}_j(\omega)\, d\mathbf{U}_j(\omega') = E\; d\mathbf{V}_j(\omega)\, d\mathbf{V}_j(\omega') = \begin{cases} 0 & \text{if } \omega \neq \omega' \\ s_{jj}(\omega)\, d\omega & \text{if } \omega = \omega', \end{cases} \\ &E\; d\mathbf{U}_j(\omega)\, d\mathbf{V}_j(\omega') = 0, j = 1, \cdots, p. \\ &E\; d\mathbf{U}_j(\omega)\, d\mathbf{U}_k(\omega') = E\; d\mathbf{V}_j(\omega)\, d\mathbf{V}_k(\omega') = \begin{cases} 0 & \text{if } \omega \neq \omega' \\ c_{jk}(\omega)\, d\omega & \text{if } \omega = \omega', \end{cases} \\ &E\; d\mathbf{U}_j(\omega)\, d\mathbf{V}_k(\omega') = -E\, d\mathbf{V}_j(\omega)\, d\mathbf{U}_k(\omega') = \begin{cases} 0 & \text{if } \omega \neq \omega' \\ q_{jk}(\omega)\, d\omega & \text{if } \omega = \omega', \end{cases} \\ &\qquad\qquad j \neq k; j, k = 1, \cdots, p. \end{aligned}$$

In (6.3) $s_{jj}(\omega)$ is the *spectral dentity* of the $\mathbf{X}_j(t)$ component of the multiple time series, $c_{jk}(\omega)$ the *cospectral density* between the $\mathbf{X}_j(t)$ and $\mathbf{X}_k(t)$ components, and $q_{jk}(\omega)$ the *quadrature spectral density* between the $\mathbf{X}_j(t)$ and $\mathbf{X}_k(t)$ components. Let

$$(6.4)\qquad \begin{aligned} d\mathbf{S}_j(\omega) &= \tfrac{1}{2}[d\mathbf{U}_j(\omega) - i\, d\mathbf{V}_j(\omega)] \quad \text{for} \quad \omega \geqq 0, \\ d\mathbf{S}_j(-\omega) &= \overline{d\mathbf{S}_j(\omega)}; j = 1, \cdots, p. \end{aligned}$$

One may then write

$$(6.5)\qquad \mathbf{X}_j(t) = \int_{-\infty}^\infty e^{i\omega t}\, d\mathbf{S}_j(\omega), j = 1, \cdots, p.$$

Furthermore, by virtue of (6.2) and (6.3)

$$E\, d\mathbf{S}_j(\omega) = 0, j = 1, \cdots, p \tag{6.6}$$

and

$$E\ d\mathbf{S}_j(\omega)\ \overline{d\mathbf{S}_j(\omega')} = \begin{cases} 0 \text{ if } \omega \neq \omega' \\ \frac{1}{2} s_{jj}(\omega)\ d\omega \text{ if } \omega = \omega', \end{cases} \qquad j = 1, \cdots, p;$$

(6.7)

$$E\ d\mathbf{S}_j(\omega)\ \overline{d\mathbf{S}_k(\omega')} = \begin{cases} 0 \text{ if } \omega \neq \omega' \\ \frac{1}{2} s_{jk}(\omega)\ d\omega \text{ if } \omega = \omega', \end{cases} \qquad j \neq k; j, k = 1, \cdots, p.$$

In (6.7) $s_{jk}(\omega) = c_{jk}(\omega) + iq_{jk}(\omega)$. The *spectral density matrix* $S(\omega)$ of the multiple stationary Gaussian time series $[\mathbf{X}_1(t), \cdots, \mathbf{X}_p(t)]$ is given by

$$S(\omega) \equiv \|s_{jk}(\omega)\|. \tag{6.8}$$

By virtue of (6.3) the infinitesimal Gaussian random variables $d\mathbf{U}_j(\omega)$, $d\mathbf{V}_j(\omega)$' $j = 1, \cdots, p$ at a given frequency ω possess the special covariance structure described by (3.1). Furthermore, by virtue of (6.3) the infinitesimal Gaussian random variables $d\mathbf{U}_j(\omega)$, $d\mathbf{V}_j(\omega)$, $j = 1, \cdots, p$ and $d\mathbf{U}_j(\omega')$, $d\mathbf{V}_j(\omega')$, $j = 1, \cdots, p$ at two distinct frequencies ω, ω' $(\omega \neq \omega')$ are uncorrelated and therefore independent. Therefore, from (6.6) and (6.7) the complex "differentials" $[d\mathbf{S}_1(\omega), \cdots, d\mathbf{S}_p(\omega)]$ at a given frequency ω have a (zero mean) p-variate complex Gaussian distribution with Hermitian covariance matrix given essentially by (6.8). Furthermore, at two distinct frequencies ω, ω' $(\omega \neq \omega')$ the complex "differentials" $[d\mathbf{S}_1(\omega), \cdots, d\mathbf{S}_p(\omega)]$ and $[d\mathbf{S}_1(\omega'), \cdots, d\mathbf{S}_p(\omega')]$ are independently distributed.

Consider now a finite $-T \leqq t \leqq T$ realization (sample) of the (zero mean) multiple stationary Gaussian time series $[\mathbf{X}_1(t), \cdots, \mathbf{X}_p(t)]$. Let $\omega_k \geqq 0$ denote a fixed frequency, and

$$\begin{aligned} \Delta\mathbf{U}_j(\omega_k) &\equiv \int_0^T \frac{1}{2} K_{c\omega_k}(t)[\mathbf{X}_j(t) + \mathbf{X}_j(-t)]\ dt, \\ \Delta\mathbf{V}_j(\omega_k) &\equiv \int_0^T \frac{1}{2} K_{s\omega_k}(t)[\mathbf{X}_j(t) - \mathbf{X}_j(-t)]\ dt, \qquad j = 1, \cdots, p. \end{aligned} \tag{6.9}$$

It is intended that $K_{c\omega_k}(t)$, $K_{s\omega_k}(t)$ be chosen so that $\Delta\mathbf{U}_j(\omega_k)$, $\Delta\mathbf{V}_j(\omega_k)$ approximate $d\mathbf{U}_j(\omega_k)$, $d\mathbf{V}_j(\omega_k)$ respectively. From (6.1)

$$\begin{aligned} \frac{1}{2}[\mathbf{X}_j(t) + \mathbf{X}_j(-t)] &= \int_0^\infty \cos \omega t\ d\mathbf{U}_j(\omega), \\ \frac{1}{2}[\mathbf{X}_j(t) - \mathbf{X}_j(-t)] &= \int_0^\infty \sin \omega t\ d\mathbf{V}_j(\omega), \qquad j = 1, \cdots, p. \end{aligned} \tag{6.10}$$

From (6.9) and (6.10) one has

$$\Delta \mathbf{U}_j(\omega_k) = \int_0^\infty F_{c\omega_k}(\omega)\, d\mathbf{U}_j(\omega) \tag{6.11}$$

where

$$F_{c\omega_k}(\omega) = \int_0^T K_{c\omega_k}(t) \cos \omega t\, dt. \tag{6.12}$$

Similarly, one has

$$\Delta \mathbf{V}_j(\omega_k) = \int_0^\infty F_{s\omega_k}(\omega)\, d\mathbf{V}_j(\omega) \tag{6.13}$$

where

$$F_{s\omega_k}(\omega) = \int_0^T K_{s\omega_k}(t) \sin \omega t\, dt. \tag{6.14}$$

The functions $F_{c\omega_k}(\omega)$ and $F_{s\omega_k}(\omega)$ are called *filters*. Notice that if the filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ were Dirac delta functions centered at $\omega = \omega_k$, one would have from (6.11) and (6.13) $\Delta \mathbf{U}_j(\omega_k) = d\mathbf{U}_j(\omega_k)$, and $\Delta \mathbf{V}_j(\omega_k) = d\mathbf{V}_j(\omega_k)$. The filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ approximating Dirac delta functions centered at $\omega = \omega_k$ that are attainable with a finite T are such that in the frequency range $\omega \geqq 0$ $F_{c\omega_k}(\omega)$ and $F_{s\omega_k}(\omega)$ are very nearly equal and the smallest attainable bandwidth B of each is of the order of $4\pi/T$. (Attainable filters are discussed in Goodman [8].) Let

$$\Delta \mathbf{S}_j(\omega_k) = \tfrac{1}{2}[\Delta \mathbf{U}_j(\omega_k) - i\Delta \mathbf{V}_j(\omega_k)] \text{ for } \omega_k \geqq 0, j = 1, \cdots, p; \tag{6.15}$$

and

$$\xi'_{\omega_k} = [\Delta \mathbf{S}_1(\omega_k), \cdots, \Delta \mathbf{S}_p(\omega_k)]. \tag{6.16}$$

If the filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ were exactly equal, say $F_{c\omega_k}(\omega) = F_{s\omega_k}(\omega) = F_{\omega_k}(\omega)$, and $F_{\omega_k}(\omega)$ vanished outside the frequency band $\omega_k - \frac{1}{2}B \leqq \omega \leqq \omega_k + \frac{1}{2}B$, and the spectral density matrix $S(\omega)$ were constant over the frequency band $\omega_k - \frac{1}{2}B \leqq \omega \leqq \omega_k + \frac{1}{2}B$, then ξ'_{ω_k} would be a p-variate (zero mean) complex Gaussian random variable with Hermitian covariance matrix

$$\Sigma_{\xi_{\omega_k}} = C_F S(\omega_k) \tag{6.17}$$

where the constant C_F is given by

$$C_F = \int_{\omega_k - \frac{1}{2}B}^{\omega_k + \frac{1}{2}B} F^2_{\omega_k}(\omega)\, d\omega. \tag{6.18}$$

If the spectral density matrix $S(\omega)$ were constant over the frequency band $\omega_k - (m + \frac{1}{2})B \leqq \omega \leqq \omega_k + (m + \frac{1}{2})B$ then (if the frequencies ω_{k+r} are equally spaced a frequency interval B apart) $\xi'_{\omega_{k+r}}$, $r = -m, \cdots, m$ would be $(2m + 1)$ independent and identically distributed p-variate (zero mean) complex Gaussian random variables with Hermitian covariance matrix $\Sigma_{\xi_{\omega_k}}$. By the preceding theory of statistical analysis based on the multivariate complex Gaussian distribution one would have

$$\hat{\Sigma}_{\xi_{\omega_k}} = [1/(2m + 1)] \sum_{r=-m}^{m} \xi_{\omega_{k+r}} \bar{\xi}'_{\omega_{k+r}} \tag{6.19}$$

(apart from a constant factor) a complex Wishart distributed estimator for the spectral density matrix $S(\omega_k)$. Functions of the elements of the spectral density matrix $S(\omega_k)$ are statistically estimated by taking for estimators the corresponding functions of the elements of the random matrix $\hat{\Sigma}_{\xi_{\omega_k}}$.

Attainable filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ do not possess exactly the ideal properties described above. Certain conditions on the spectral density matrix $S(\omega)$ must therefore be imposed to make the preceding theory practically applicable. These conditions are: (1) $S(\omega)$ effectively vanish beyond a suitable cutoff frequency ω_c, (2) $S(\omega)$ be sensibly constant within the frequency band $\omega_k - (m + \frac{1}{2})B \leqq \omega \leqq \omega_k + (m + \frac{1}{2})B$, and (3) each of the elements of $S(\omega)$ be such that the integral of its absolute value over the frequency range exterior to the band $\omega_k - (m + \frac{1}{2})B \leqq \omega \leqq \omega_k + (m + \frac{1}{2})B$ be bounded by a not too large constant.

In various fields such as geophysics and electrical engineering multiple records of fluctuations considered stationary Gaussian are quite common. For example, in geophysics such records may be simultaneous measurements at several positions in the ocean of the height of gravity waves generated by the wind. The preceding theory is useful in measuring (estimating) the statistical structure of such fluctuations from finite lengths of record. The theory is also useful in measuring (estimating) deterministic physical constants. For applications in that direction the reader is referred to Goodman [7].

Comment. Consider a (zero mean) multiple stationary time series $[\mathbf{X}_1(t), \cdots, \mathbf{X}_p(t)]$ that is *not* Gaussian. Equations (6.1) through (6.8) still hold except that one now may no longer regard the "differentials" $d\mathbf{U}_j(\omega)$, $d\mathbf{V}_j(\omega)$ and $d\mathbf{S}_j(\omega)$ as Gaussian random variables. One desires that $\hat{\Sigma}_{\xi_{\omega_k}}$ given by (6.19) be (apart from perhaps a constant factor) a complex Wishart distributed estimator for the spectral density matrix $S(\omega_k)$. Clearly, that will be so if $\xi'_{\omega_{k+r}}$, $r = -m, \cdots, m$ are independent and identically distributed p-variate (zero mean) complex Gaussian random variables with Hermitian covariance matrix $\Sigma_{\xi_{\omega_k}}$ given by (6.17). If the filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ nearly possess the ideal properties described above and the spectral density matrix $S(\omega)$ satisfies the three conditions enumerated above, $\xi'_{\omega_{k+r}}$, $r = -m, \cdots, m$ will very nearly have these properties *if* $\Delta\mathbf{U}_j(\omega_{k+r})$, $\Delta\mathbf{V}_j(\omega_{k+r})$, $r = -m, \cdots, m; j = 1, \cdots, p$ given by (6.9) are *Gaussian.* Now, from (6.9), (6.11), and (6.13) one notes that the $\Delta\mathbf{U}_j(\omega_{k+r})$, $\Delta\mathbf{V}_j(\omega_{k+r})$ are obtained by "*filtering*" the $\mathbf{X}_j(t)$. If the filters $F_{c\omega_{k+r}}(\omega)$, $F_{s\omega_{k+r}}(\omega)$ "performing" the filtering approximate the ideal filters described above, they are *narrow-band* filters. Many stationary *non-Gaussian* processes become nearly Gaussian when "passed through" sufficiently narrow-band filters. The band-width B of the filters $F_{c\omega_{k+r}}(\omega)$, $F_{s\omega_{k+r}}(\omega)$ may be made small for moderately large sample lengths T. These heuristic remarks suggest that the $\Delta\mathbf{U}_j(\omega_{k+r})$, $\Delta\mathbf{V}_j(\omega_{k+r})$ may be nearly Gaussian for many stationary *non-Gaussian* processes $\mathbf{X}_j(t)$. Consequently, for many *non-Gaussian* (zero mean) multiple stationary time series $[\mathbf{X}_1(t), \cdots, \mathbf{X}_p(t)]$, $\hat{\Sigma}_{\xi_{\omega_k}}$ given by (6.19) is (apart from perhaps a constant factor) an estimator for the spectral density matrix $S(\omega_k)$ that is nearly complex Wishart distributed. More generally, these heuristic remarks suggest that the results of statistical analysis based on the multivariate complex Gaussian distribution may be practically applicable to empirical spectral analysis of many *non-Gaussian* multiple stationary time series when sufficiently long finite realizations are available.

7. A comparison between certain distributions of complex and real multivariate Gaussian statistical analysis.

Notation. In order to compare the results of complex and real multivariate Gaussian statistical analysis the following notation is introduced. A subscript C on a symbol indicates that the symbol pertains to multivariate complex Gaussian statistical analysis and a subscript R on a symbol indicates that the symbol pertains to multivariate real Gaussian statistical analysis. Thus, a (zero mean) p-variate complex Gaussian random variable is now denoted by ξ_C. A (zero mean) p-variate real Gaussian random variable is denoted by ξ_R. The $p \times p$ Hermitian positive definite complex covariance matrix of ξ_C is denoted by $\Sigma_{C\xi}$ and $\Sigma_{C\xi} = \|\sigma_{Cjk}\|$. The $p \times p$ symmetric positive definite real covariance matrix of ξ_R is denoted by $\Sigma_{R\xi}$ and $\Sigma_{R\xi} = \|\sigma_{Rjk}\|$. The inverse matrices $\Sigma_{C\xi}^{-1} = \|\sigma_C^{jk}\|$ and $\Sigma_{R\xi}^{-1} = \|\sigma_R^{jk}\|$.

Comparison 7.1. The probability density functions of the distributions of a complex and real (zero mean) p-variate Gaussian random variable are respectively

$$(7.1_C) \qquad p(\xi_C) = (1/\pi^p\,|\Sigma_{C\xi}|)\exp\left(-\bar{\xi}_C'\Sigma_{C\xi}^{-1}\xi_C\right),$$

$$(7.1_R) \qquad p(\xi_R) = (1/(2\pi)^{p/2}\,|\Sigma_{R\xi}|^{\frac{1}{2}})\exp\left(-\tfrac{1}{2}\xi_R'\Sigma_{R\xi}^{-1}\xi_R\right).$$

Notation. Let $\mathbf{A}_C \equiv \sum_{j=1}^{n}\xi_{jC}\bar{\xi}_{jC}'$ where ξ_{jC}, $j = 1, \cdots, n$ are independent identically distributed p-variate complex Gaussian random variables constituting a sample of size n from a population with probability density function $p(\xi_C)$ given by (7.1_C). Let $\mathbf{A}_R \equiv \sum_{j=1}^{n}\xi_{jR}\xi_{jR}'$ where ξ_{jR}, $j = 1, \cdots, n$ are independent identically distributed p-variate real Gaussian random variables constituting a sample of size n from a population with probability density function $p(\xi_R)$ given by (7.1_R).

Comparison 7.2. The random $p \times p$ Hermitian matrix $\mathbf{A}_C$ is complex Wishart distributed. The random $p \times p$ symmetric matrix $\mathbf{A}_R$ is Wishart distributed. The probability density functions of the distributions of the matrices $\mathbf{A}_C$ and $\mathbf{A}_R$ are respectively

$$(7.2_C) \qquad p_W(A_C) = [|A_C|^{n-p}/I_C(\Sigma_{C\xi})]\exp\left(-\operatorname{tr}(\Sigma_{C\xi}^{-1}A_C)\right)$$

where

$$I_C(\Sigma_{C\xi}) = \pi^{\frac{1}{2}p(p-1)}\Gamma(n) \cdots \Gamma(n-p+1)\, |\Sigma_{C\xi}|^n,$$

and

$$(7.2_R) \qquad p_W(A_R) = \frac{|A_R|^{\frac{1}{2}(n-p-1)}}{I_R(\Sigma_{R\xi})} \exp\left(-\tfrac{1}{2} \operatorname{tr}(\Sigma_{R\xi}^{-1} A_R)\right)$$

where $I_R(\Sigma_{R\xi}) = 2^{\frac{1}{2}pn}\pi^{\frac{1}{4}p(p-1)}\Gamma(n/2) \cdots \Gamma[(n-p+1)/2]\, |\Sigma_{R\xi}|^{n/2}$. The density $p_W(A_C)$ is defined over the domain D_{A_C} where A_C is Hermitian positive semi-definite. The density $p_W(A_R)$ is defined over the domain D_{A_R} where A_R is symmetric positive semi-definite.

Notation. Let $\mathbf{A}_C = \|\mathbf{A}_{Cjk}\|$ and $\mathbf{A}_C^{-1} = \|\mathbf{A}_C^{jk}\|$. Let $\mathbf{A}_R = \|\mathbf{A}_{Rjk}\|$ and $\mathbf{A}_R^{-1} = \|\mathbf{A}_R^{jk}\|$. The multiple coherence between the pth component of ξ_C and the first $(p-1)$ components of ξ_C is $R^2_{Cp\cdot p-1,\cdots,1} \equiv 1 - (\sigma_{Cpp}\sigma_C^{pp})^{-1}$. The sample multiple coherence between the pth component of ξ_C and the first $(p-1)$ components of ξ_C is $\hat{\mathbf{R}}^2_{Cp\cdot p-1,\cdots,1} = 1 - (\mathbf{A}_{Cpp}\mathbf{A}_C^{pp})^{-1}$. The square of the multiple correlation coefficient between the pth component of ξ_R and the first $(p-1)$ components of ξ_R is $R^2_{Rp\cdot p-1,\cdots,1} \equiv 1 - (\sigma_{Rpp}\sigma_R^{pp})^{-1}$. The square of the sample multiple correlation coefficient between the pth component of ξ_R and the first $(p-1)$ components of ξ_R is $\hat{\mathbf{R}}^2_{Rp\cdot p-1,\cdots,1} = 1 - (\mathbf{A}_{Rpp}\mathbf{A}_R^{pp})^{-1}$.

COMPARISON 7.3. The probability density function of the distribution of $\hat{\mathbf{R}}^2_{Cp\cdot p-1,\cdots,1}$ is

$$(7.3_C) \quad p(\hat{R}_C^2) = [\Gamma(n)/\Gamma(p-1)\Gamma(n-p+1)] \cdot (1-R_C^2)^n(\hat{R}_C^2)^{p-2}(1-\hat{R}_C^2)^{n-p} F(n, n; p-1; R_C^2\hat{R}_C^2).$$

In (7.3_C) for brevity $R^2_{Cp\cdot p-1,\cdots,1}$ and $\hat{R}^2_{Cp\cdot p-1,\cdots,1}$ are denoted by R_C^2 and $\hat{R}_C^2$ respectively. The probability density function of the distribution of $\hat{\mathbf{R}}^2_{Rp\cdot p-1,\cdots,1}$ is

$$(7.3_R) \quad \begin{aligned} p(\hat{R}_R^2) &= [\Gamma(\tfrac{1}{2}n)/\Gamma(\tfrac{1}{2}(p-1))\Gamma(\tfrac{1}{2}(n-p+1))] \\ &\cdot (1-R_R^2)^{\frac{1}{2}n}(\hat{R}_R^2)^{\frac{1}{2}(p-3)}(1-\hat{R}_R^2)^{\frac{1}{2}(n-p-1)}F(\tfrac{1}{2}n, \tfrac{1}{2}n;\ \tfrac{1}{2}(p-1); R_R^2\hat{R}_R^2). \end{aligned}$$

In (7.3_R) for brevity $R^2_{Rp\cdot p-1,\cdots,1}$ and $\hat{R}^2_{Rp\cdot p-1,\cdots,1}$ are denoted by R_R^2 and $\hat{R}_R^2$ respectively.

Notation. The conditional coherence between the pth and $(p-1)$st components of ξ_C with respect to the first $(p-2)$ components of ξ_C is $R^2_{Cp,p-1|p-2,\cdots,1} \equiv |\sigma_C^{p-1,p}|^2(\sigma_C^{p-1,p-1}\sigma_C^{pp})^{-1}$. The sample conditional coherence between the pth and $(p-1)$st components of ξ_C with respect to the first $(p-2)$ components of ξ_C is $\hat{\mathbf{R}}^2_{Cp,p-1|p-2,\cdots,1} = |\mathbf{A}_C^{p-1,p}|^2(\mathbf{A}_C^{p-1,p-1}\mathbf{A}_C^{pp})^{-1}$. The square of the partial correlation coefficient between the pth and$(p-1)$st components of ξ_R with respect to the first $(p-2)$ components of ξ_R is $R^2_{Rp,p-1\cdot p-2,\cdots,1} \equiv (\sigma_R^{p-1,p})^2(\sigma_R^{p-1,p-1}\sigma_R^{pp})^{-1}$. The square of the sample partial correlation coefficient between the pth and $(p-1)$st components of ξ_R with respect to the first $(p-2)$ components of ξ_R is $\hat{\mathbf{R}}^2_{Rp,p-1\cdot p-2,\cdots,1} = (\mathbf{A}_R^{p-1,p})^2(\mathbf{A}_R^{p-1,p-1}\mathbf{A}_R^{pp})^{-1}$.

COMPARISON 7.4. The probability density function of the distribution of

$\hat{\mathbf{R}}^2_{Cp,p-1|p-2,\cdots,1}$ is

$$p(\hat{R}^2_C) = (n-p+1) \cdot (1-R^2_C)^{n-p+2}(1-\hat{R}^2_C)^{n-p}F(n-p+2, n-p+2; 1; R^2_C\hat{R}^2_C). \quad (7.4_C)$$

In (7.4_C) R^2_C and $\hat{R}^2_C$ now denote $R^2_{Cp,p-1|p-2,\cdots,1}$ and $\hat{R}^2_{Cp,p-1|p-2,\cdots,1}$ respectively. The probability density function of the distribution of $\hat{\mathbf{R}}^2_{Rp,p-1\cdot p-2,\cdots,1}$ is

$$p(\hat{R}^2_R) = \frac{\Gamma(\frac{1}{2}(n-p+2))}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(n-p+1))}(1-R^2_R)^{\frac{1}{2}(n-p+2)}(\hat{R}^2_R)^{-\frac{1}{2}} \cdot (1-\hat{R}^2_R)^{\frac{1}{2}(n-p-1)}F(\tfrac{1}{2}(n-p+2), \tfrac{1}{2}(n-p+2); \tfrac{1}{2}; R^2_R\,\hat{R}^2_R). \quad (7.4_R)$$

In(7.4_R) R^2_R and $\hat{R}^2_R$ now denote $R^2_{Rp,p-1\cdot p-2,\cdots,1}$ and $\hat{R}^2_{Rp,p-1\cdot p-2,\cdots,1}$ respectively.

Comment. In the present introduction to statistical analysis based on the multivariate complex Gaussian distribution attention is directed to distributions that have real counterparts. Not all distributions of multivariate complex Gaussian statistical analysis are of that type. For example, if one represents in polar form the complex random variables that occur in multivariate complex Gaussian statistical analysis, then the marginal distributions of phase angles have no real counterparts. The reader is referred to Goodman [7] for examples of noncounterpart distributions.

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