ON THE DUALS OF SYMMETRIC PARTIALLY-BALANCED INCOMPLETE BLOCK DESIGNS[1]

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1. Summary. It is shown that, under certain hypotheses, the dual of a symmetric partially balanced incomplete block design with m associate classes is also a PBIB design with m associate classes and all parameters the same as before. In the case $m = 1$, and in the case that the design is group divisible, these hypotheses coincide with assumptions previously known to be sufficient to ensure duality. To show that some hypotheses are needed for duality, an example is given of a group divisible design whose dual is not a group divisible design.

2. Introduction. Let v, r, $n_1, \cdots, n_m, p_{ij}^k(i, j, k = 1, \cdots, m)$, $\lambda_1, \cdots, \lambda_m$ be given non-negative integers. Let J be the square matrix of order v, each of whose entries is 1, and let I be the identity matrix of order v. A symmetric PBIB design with m associate classes [1], and parameters $v, \cdots, \lambda_m$ is a square matrix N of order v, each of whose entries is 0 or 1, such that there exist m symmetric matrices $K_1, \cdots, K_m$, each entry of which is 0 or 1, with

$$NN^T = rI + \textstyle\sum_{s=1}^m \lambda_s K_s, \tag{2.1}$$

$$J = I + \textstyle\sum_{s=1}^m K_s, \tag{2.2}$$

$$K_i^2 = n_i I + \textstyle\sum_{s=1}^m p_{ii}^s K_s, \qquad i = 1, \cdots, m, \tag{2.3}$$

$$K_i K_j = K_j K_i = \textstyle\sum_{s=1}^m p_{ij}^s K_s, \quad i, j = 1, \cdots, m, i \neq j. \tag{2.4}$$

Our object is to obtain sufficient conditions under which N^T is a PBIB design with m associate classes and the same parameters $v, \cdots, \lambda_m$ as N. We suspect these conditions can be substantially weakened, but the problem of weakening the conditions is open.

Let A be the square matrix of order m given by

$$\begin{aligned} a_{ij} &= -\lambda_i n_i + \sum_{j=1}^m \lambda_s p_{is}^j \qquad & i \neq j \\ a_{ii} &= r - \lambda_i n_i + \sum_{j=1}^m \lambda_s p_{is}^j & \end{aligned} \tag{2.5}$$

We will also consider a matrix B with $m + 2$ rows $(-1, 0, 1, \cdots, m)$ and m columns. We define

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\begin{aligned}
b_{-1,j} &= 1 \\
b_{0,j} &= \lambda_j \\
b_{ij} &= \sum_{s=1}^{m} \lambda_s p^j_{is}, \qquad i, j = 1, \cdots, m, i \neq j \\
b_{ii} &= r + \sum_{s=1}^{m} \lambda_s p^i_{is}, \qquad i = 1, \cdots, m.
\end{aligned} \tag{2.6}
$$

THEOREM. *If*

(2.7) $|A| \neq 0$, *and*

(2.8) *the greatest common divisor of the* $\binom{m+2}{m}$ *determinants of order* m *contained in* B *is* 1,

then N^T *is a* PBIB *design with* m *associate classes whose parameters are identical with the corresponding parameters of* N.

3. Proof of the theorem. The proof will be developed in a sequence of lemmas.

LEMMA 1. $|N| \neq 0$.

This has been shown in [3] to be a consequence of (2.7).

LEMMA 2. $NJ = JN = rJ$.

PROOF. We know $NJ = rJ$; so $N^{-1}J = (1/r)$J. We also known from (2.3) that $K_iJ = n_iJ$. If we multiply (2.1) on the left by N^{-1} and on the right by J, we obtain $N^TJ = cJ$, for some c. This says that all columns of N have the same sum; this common sum must be r, since the sum of all column sums of N = the sum of all row sums of $N = rv$. Hence $JN = rJ$.

Next, define the matrices $L_s = N^{-1}K_sN$, $s = 1, \cdots, m$. Then Lemma 2 immediately implies

LEMMA 3. *If* $K_s (s = 1, \cdots, m)$ *is replaced by* L_s, *and* N *and* N^T *are interchanged, then* (2.1)–(2.4) *are satisfied.*

It follows that all we need prove is that the matrices L_s are symmetric, with entries 0 or 1.

LEMMA 4. *The matrices* $L_s (s = 1, \cdots, m)$ *are symmetric.*

PROOF. From (2.1) and (2.4), $NN^TK_s = K_sNN^T$. Thus, $L_s^T = N^TK_s(N^T)^{-1} = N^{-1}K_sN = L_s$.

LEMMA 5. *The matrices* L_s *are integral.*

PROOF. From (2.2), we have

$$J - I = \sum_{j=1}^{m} L_j. \tag{3.1}$$

Multiply (2.1) on the left by N^{-1} and on the right by N to obtain

$$N^TN - rI = \sum_{j=1}^{m} \lambda_j L_j. \tag{3.2}$$

Next, multiply (2.1) on the left by N^{-1} and on the right by $K_i N$. The result is

$$(3.3) \qquad N^T K_i N - n_i \lambda_i I = \sum_{j=1}^{m} \left(r\delta_{ij} + \sum_{s=1}^{m} \lambda_s p_{is}^j \right) L_j , \qquad i = 1, \cdots , m.$$

Notice that the coefficients of L_j on the right hand sides of (3.1)–(3.3) are precisely the coefficients of (2.6).

Let λ_{tu} be the vector of $m + 2$ components which are the entries in position (t, u) of the left sides of (3.1)–(3.3). Let y_{tu} be the vector with m components whose jth component is the entry in position (t, u) of L_j. Then $\lambda_{tu} = By_{tu}$. It is well known that (2.8) implies that y_{tu} has integral components.

LEMMA 6. *The entries in* $L_i (i = 1, \cdots , m)$ *are* 0 *or* 1.

PROOF. Denote the entry in position (t, u) of L_i by $L_i(t, u)$. It follows from Lemma 3 and (2.3) that

$$(3.4) \qquad \sum_{t,u} L_i(t, u) = n_i v.$$

By Lemma 3, Lemma 4, and (2.3), we have

$$(3.5) \qquad \sum_{t,u} (L_i(t, u))^2 = \mathrm{Tr}(L_i^2) = n_i v,$$

since $\mathrm{Tr}\, L_i = \mathrm{Tr}\, K_i = 0$. But the only way (3.4) and (3.5) can be reconciled with Lemma 5 is for each $L_i(t, u)$ to be 0 or 1.

Clearly, Lemmas 3, 4, and 6 prove the theorem.

4. Remarks. If $m = 1$, (2.7) reduces to $r > \lambda$, and (2.8) is automatically satisfied, since one of the 3 determinants of order one is 1. This is the result of [4]. If we have a group divisible design which is nonsingular, then [2] the condition $(r^2 - \lambda_2 v, \lambda_1 - \lambda_2) = 1$ implies (2.8).

As was shown by Connor at the end of [2], (2.7) and (2.8) are undoubtedly too strong, and they are certainly not necessary. On the other hand, the theorem needs *some* hypotheses. Here is a group divisible design for which neither of the hypotheses nor the conclusion of our theorem hold:

$$N = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Another interesting question is (assuming our hypotheses): is the assocation scheme corresponding to N^T permutationally equivalent to the association scheme corresponding to N? In other words, does there exist a permutation matrix P such that $P^{-1}K_iP = L_i$, $(i = 1, \cdots, m)$? All our proof gives is the result $N^{-1}K_iN = L_i$.

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