CENTRAL LIMIT THEOREMS FOR FAMILIES OF SEQUENCES OF RANDOM VARIABLES[1]

By F. Eicker

University of Freiburg im Breisgau

1. Introduction and summary. Let F be a (nonempty) set of distribution functions (d.f.'s) of random variables (r.v.'s) with zero means and positive, finite variances. Let $\mathfrak{F}(F)$ be the set of all sequences of independent r.v.'s (independent within each sequence) whose d.f.'s belong to F but are not necessarily the same from term to term of the sequence. No assumptions are made on the interrelations between the joint probability spaces of the r.v.'s of different sequences of $\mathfrak{F}(F)$. Accordingly, dependence or independence between r.v.'s of different sequences needs not be specified. A generic member of $\mathfrak{F}(F)$ will be denoted by $\epsilon = \{\epsilon_k ; k = 1, 2, \cdots\}$, or, when we discuss sequences of members of $\mathfrak{F}(F)$, by $\epsilon(n) = \{\epsilon_{nk} ; k = 1, 2, \cdots\}, n = 1, 2, \cdots$. In the following, $\mathfrak{F}(F)$ plays the role of a parameter space, the parameter points being $\epsilon, \epsilon(1), \epsilon(2), \cdots$. Since only the d.f.'s of a sequence of (independent) r.v.'s are relevant to the central limit theorem, instead of $\mathfrak{F}(F)$, the set of sequences of d.f.'s corresponding to the elements of $\mathfrak{F}(F)$ may also be regarded as the parameter space. It clearly is a map of $\mathfrak{F}(F)$. The inverse mapping subdivides $\mathfrak{F}(F)$ into certain equivalence classes. From elementary set theory it follows that $\mathfrak{F}(F)$ as well as its map has the cardinality of the continuum if F contains more than one element.

Further, let $\{a_{nk}; n = 1, 2, \cdots ; k = 1, 2, \cdots\}$ be a double sequence of real constants, and let $\{k_n\}$ be a sequence of positive integers such that $a_{nk_n} \neq 0$ and $a_{nk} = 0$ for $k > k_n$, $n = 1, 2, \cdots$. We denote the variance of $\sum_{k=1}^{k_n} a_{nk}\epsilon_{nk}$ by

$$B_n^2 = \sum_{k=1}^{k_n} a_{nk}^2 \sigma_{nk}^2, \qquad \sigma_{nk}^2 = \operatorname{var} \epsilon_{nk}, \tag{1}$$

and put

$$\zeta_n = B_n^{-1} \sum_{k=1}^{k_n} a_{nk}\epsilon_{nk}, \qquad n = 1, 2, \cdots, \tag{2}$$

where $B_n = +(B_n^2)^{\frac{1}{2}}$. If all sequences $\epsilon(n)$ are identical we denote them by ϵ and write instead of (2)

$$\zeta_n(\epsilon) = B_n^{-1}(\epsilon) \sum_{k=1}^{k_n} a_{nk}\epsilon_k, \qquad n = 1, 2, \cdots,$$

Received April 18, 1961; revised December 26, 1962.

[1] Part of this research was carried out at the Department of Statistics, University of North Carolina. It was supported by the Air Force Office of Scientific Research under Contract AF 49(638)–929. Some of the results have appeared in March, 1961, as Report No. 279 in the Mimeograph Series of the Institute of Statistics under the title: "Central Limit Theorem for Sums Over Sets of Random Variables". Reproduction is permitted for any purpose of the United States Government.

which shows the dependence on the parameter ϵ more clearly; $B_n^2(\epsilon)$ is defined analogously to (1).

This note deals with necessary and sufficient conditions on the set F and on the double sequence $\{a_{nk}\}$ in order that the d.f.'s of the ζ_n tend to the standard normal d.f. (denoted by $\Phi(x)$) and that the $B_n^{-1}a_{nk}\epsilon_{nk}$ are infinitesimal for every sequence of sequences $\epsilon(1), \epsilon(2), \cdots \varepsilon \mathfrak{F}(F)$ (Theorem 1). For the case of identical sequences $\epsilon(n)$ $(= \epsilon)$, which is of particular interest in applications, a normal convergence theorem (Theorem 3) holding uniformly for ϵ on $\mathfrak{F}(F)$ is obtained under the same necessary and sufficient conditions as in Theorem 1.

Theorem 3 yields, e.g., conditions for the asymptotic normality of the least squares estimators of the parameters in a linear regression with independent and not necessarily identically distributed errors whose d.f.'s are unknown but belong to a certain class F (Eicker (1963)). From the necessity of the conditions it follows that these conditions are the best possible ones under the limited information about the error terms provided by the model assumptions.

Frequently, when limit theorems for families of sequences of random variables are met in statistics and probability theory, the emphasis is on the uniformity of the convergence of the sequences with respect to the family parameter which assumes values in a given a priori set. (For an example, compare Parzen (1954), p. 38. That paper also cites some of the earlier publications on the subject.) The present note emphasizes not primarily the uniformity of the convergence, but the necessity of the conditions (including one on the parameter space $\mathfrak{F}(F)$) for the convergence on the parameter space. Accordingly, the set F cannot be prescribed arbitrarily. The uniformity of the convergence on $\mathfrak{F}(F)$ is shown without difficulty to be implied by the ordinary convergence on $\mathfrak{F}(F)$. Another difference from earlier work may be seen in the particular structure of $\mathfrak{F}(F)$, which is comparable to an infinite-dimensional vector space. Although many of the earlier convergence theorems are (or can be) formulated for abstract parameter spaces, in the applications such as estimation theory these spaces are usually specialized to intervals on the line or in a finite-dimensional vector space.

2. Central limit theorems for families of random sequences. In the following all limits are taken for $n \to \infty$, unless stated otherwise. With the notations of Section 1 we have

THEOREM 1. *In order that*

(A1) *the d.f.'s of the ζ_n converge uniformly in x to $\Phi(x)$, and*

(B1) *the random variables $B_n^{-1}a_{nk}\epsilon_{nk}$ are infinitesimal in the sense that for every $\delta > 0$*

$$\max_{k=1,\cdots,k_n} P(\,|B_n^{-1}a_{nk}\epsilon_{nk}| > \delta) \to 0,$$

both hold for every sequence of sequences $\{\epsilon(n)\}$ with $\epsilon(n) \varepsilon \mathfrak{F}(F)$, the following three conditions are jointly necessary and sufficient:

$$\text{(I)} \qquad \max_{k=1,\cdots,k_n} a_{nk}^2 \Big/ \sum_{k=1}^{k_n} a_{nk}^2 \to 0$$

(II) $$\sup_{G \varepsilon F} \int_{|x|>c} x^2\, dG(x) \to 0 \qquad \text{as} \quad c \to \infty,$$

(III) $$\inf_{G \varepsilon F} \int x^2\, dG(x) > 0.$$

Instead of proving the theorem directly we prove a slightly stronger theorem. For the case of identical sequences $\epsilon(n)(= \epsilon = \{\epsilon_k\})$ we define the statements

(A2) *the d.f.'s of the* $\zeta_n(\epsilon)$ *converge uniformly in* x *to* $\Phi(x)$,
(B2) $\max_{k=1,\cdots,k_n} P(\,|B_n^{-1}(\epsilon)a_{nk}\epsilon_k| > \delta) \to 0$ *for every* $\delta > 0$,

both for every sequence $\epsilon \,\varepsilon\, \mathfrak{F}(F)$.

Then the following chain of implications holds

Theorem 2. [(I) & (II) & (III)] ⇒ [(A1) & (B1)] ⇒ [(A2) & (B2)] ⇒ [(I) & (II) & (III)].

The proof will be given in the next section.

The following main theorem states necessary and sufficient conditions that the statements (A2) and (B2) hold uniformly in ϵ on $\mathfrak{F}(F)$.[2]

Theorem 3. *In order that*

(A3) $$\sup_{\substack{\epsilon \varepsilon \mathfrak{F}(F) \\ -\infty<x<\infty}} |P(\zeta_n(\epsilon) < x) - \Phi(x)| \to 0$$

and

(B3) $$\sup_{\epsilon \varepsilon \mathfrak{F}(F)} \max_{k=1,\cdots,k_n} P(\,|B_n^{-1}(\epsilon)a_{nk}\epsilon_k| > \delta) \to 0 \qquad \textit{for every } \delta > 0$$

hold, the Conditions (I), (II), *and* (III) *of Theorem 1 are jointly necessary and sufficient.*

The proof follows from Theorem 1 and the equivalence of (A1) with (A3), and of (B1) with (B3).

Remark. Since we are dealing with sums of random variables whose variances are finite, and since the variances of the sums (2) are normed to one, the infinitesimality conditions (B1), (B2), and (B3) in terms of probability statements can be seen to imply the corresponding stronger expressions (B1)′, (B2)′, and (B3)′ in terms of variances (compare, e.g., Loève (1960), pp. 295 and 316):

(B1)′ $$\max_{k=1,\cdots,k_n} |B_n^{-1}a_{nk}\sigma_{nk}| \to 0, \qquad \sigma_{nk} = (\operatorname{var}\epsilon_{nk})^{\frac{1}{2}}.$$

(B2)′ $$\max_{k=1,\cdots,k_n} |B_n^{-1}(\epsilon)a_{nk}\sigma_k| \to 0, \qquad \sigma_k = (\operatorname{var}\epsilon_k)^{\frac{1}{2}}.$$

(B3)′ $$\sup_{\epsilon \varepsilon \mathfrak{F}(F)} \max_{k=1,\cdots,k_n} |B_n^{-1}(\epsilon)a_{nk}(\operatorname{var}\epsilon_k)^{\frac{1}{2}}| \to 0.$$

3. Proof of Theorem 2.

1) From a well known theorem on the convergence of sequences of monotonic functions it follows that, if a sequence of d.f.'s converges to a continuous d.f., then pointwise convergence is equivalent to uniform convergence (Fréchet

[2] The observation that the statement (A2) and (B2) hold uniformly in ϵ is due to one of the referees.

(1950), p. 321). We therefore need only consider pointwise convergence to $\Phi(x)$ in the (A)-statements. It is easy to see, moreover, that in (A3) it does not matter whether $\sup_\epsilon$ or $\sup_x$ is taken first.

2) The remaining proof is based upon the general *Central Limit Criterion* (see, e.g., Gnedenko and Kolmogorov (1954), p. 103): In order that the d.f.'s of the sums

$$\zeta_n = \xi_{n1} + \xi_{n2} + \cdots + \xi_{nk_n} \tag{3}$$

of the independent random variables $\xi_{n1}, \xi_{n2}, \cdots, \xi_{nk_n}$ with

$$E\,\xi_{nk} = 0 \quad \text{for all } n \text{ and } k \tag{4}$$

and

$$\sum_{k=1}^{k_n} \operatorname{var} \xi_{nk} = 1 \tag{5}$$

converge to the standard normal d.f. $\Phi(x)$ and that the random variables ξ_{nk} be infinitesimal, it is necessary and sufficient that Lindeberg's condition

$$\sum_{k=1}^{k_n} \int_{|z|>\delta} z^2\, dF_{nk}(z) \to 0 \qquad \text{for } n \to \infty \tag{6}$$

be satisfied for every $\delta > 0$; F_{nk} is the d.f. of ξ_{nk}.

Putting $\xi_{nk} = B_n^{-1} a_{nk} \epsilon_{nk}$, where B_n and ζ_n are defined by (1) and (2), (3)-(5) are seen to be satisfied. With the notation $q_{nk} = | a_{nk} B_n^{-1} |$ the convergence criterion (6) becomes

$$\sum_{\substack{k=1 \\ q_{nk}\neq 0}}^{k_n} q_{nk}^2 \int_{|x|>\delta/q_{nk}} x^2\, dG_{nk}(x) \to 0 \qquad \text{as} \quad n \to \infty, \tag{7}$$

where G_{nk} is the d.f. of ϵ_{nk}.

For later use we introduce the further notations

$$a_n = \sum_{k=1}^{k_n} a_{nk}^2, \qquad n = 1\ 2, \cdots;$$

$$g(c) = \sup_{G \varepsilon F} \int_{|x| \geqq c} x^2\, dG(x), \qquad c > 0$$

$$m = \inf_{G\varepsilon F} \int x^2\, dG(x).$$

3) To prove the first implication (sufficiency part in Theorem 1) let $\{\epsilon(n)\}$ be a sequence of sequences of $\mathfrak{F}(F)$ and let F fulfill (II) and (III). By (I) we have $\max_k q_{nk}^2 \leqq m^{-1} \max_k a_{nk}^2 a_n^{-1} \to 0$. An upper bound for the left hand side in (7) is obtained by

$$a_n B_n^{-2} \max_k \int_{|x|>\delta\sigma/q_{nk}} x^2\, dG_{nk}(x) \leqq m^{-1} g(\delta/\max_k q_{nk})$$

which tends to zero as $n \to \infty$.

4) The second implication is clear.

5) The last implication is proved in steps 5) to 9). In order to obtain (I) from (A2) and (B2), let all the ϵ_k be identically distributed with d.f. $G \varepsilon F$. With $\int x^2 \, dG(x) = \sigma^2$, (7) is seen to be bounded from below by

$$\mu_n^2 \sigma^{-2} \int_{|x|>\delta\sigma/\mu_n} x^2 \, dG(x) \tag{8}$$

where $\mu_n = \max_{k=1,\cdots,k_n} | a_{nk} a_n^{-\frac{1}{2}} |$. Since (8) tends to zero for every $\delta > 0$ we must have $\mu_n \to 0$ as $n \to \infty$.

6) Suppose $\sup_{G\varepsilon F} \int x^2 \, dG(x) = \infty$. Then F has necessarily infinitely many elements. Because of (I), $k_n \to \infty$. Let $\{n_j, j = 1, 2, \cdots\}$ be a sequence of integers such that the numbers $k_{n_j} \equiv k(j)$ form a strictly monotonic increasing subsequence of $\{k_n\}$. Let γ be a constant with $1 > \gamma > 2^{-\frac{1}{2}}$ and let $\delta > 0$ be smaller than $2^{-\frac{1}{2}}$. Now select a sequence $\{\sigma_k, k = 1, 2, \cdots\}$ of variances associated with d.f.'s $G_k \varepsilon F$ by choosing all σ_k for $k \not\varepsilon \{k(j)\}$ equal to a constant and by determining the remaining σ_k inductively as follows but not smaller than that constant. Choose $\sigma_{k(1)}$. Then, for $\sigma_1, \sigma_2, \cdots, \sigma_{k(j)-1}, j \geqq 2$, already determined, choose $\sigma_{k(j)}$ so large that

$$\frac{1}{2}\left\{ \sum_{k=1}^{k(j)-1} (a_{n_j k}^2 / a_{n_j k(j)}^2) \sigma_k^2 + \sigma_{k(j)}^2 \right\} < \gamma^2 \sigma_{k(j)}^2 . \tag{9}$$

The left hand side equals $\frac{1}{2} q_{n_j k(j)}^{-2}$. Therefore $\delta^2 q_{n_j k(j)}^{-2} < \gamma^2 \sigma_{k(j)}$. Estimating the left hand side of (7) for $n = n_j$ from below by the $k(j)$th term we obtain

$$q_{n_j k(j)}^2 \int_{|x|>\gamma\sigma_{k(j)}} x^2 \, dG_{k(j)}(x) > q_{n_j k(j)}^2 \, \sigma_{k(j)}^2 (1 - \gamma^2) > \frac{1 - \gamma^2}{2\gamma^2} > 0,$$

where the second inequality follows from (9). This, however, contradicts to the statement in (7). Thus $\sup_{G \varepsilon F} \int x^2 \, dG(x) < \infty$ is necessary.

7) In order to prove the necessity of (II) consider the functions $g_G(c) = \int_{|x| \geqq c} x^2 \, dG(x)$ for $c > 0$ and $G \varepsilon F$. Then $g(c) = \sup_{G\varepsilon F} g_G(c)$. Suppose (II) is false. Then there exists a $C > 0$ such that $\lim_{c\to\infty} g(c) = 2C$ ($< \infty$ because of 6)), and there exists for every c at least one d.f. $G \varepsilon F$ with $g(c) - C < g_G(c) \leqq \int x^2 \, dG(x) = \sigma^2$. Let $\hat{F} \subset F$ be the (infinite) subset of F whose d.f.'s have second moments greater than C. We shall now consider sequences out of $\mathfrak{F}(\hat{F})$. For all $G \varepsilon \hat{F}$ we have $C < \int x^2 \, dG(x) < M$ where $M < \infty$ exists by 6). Hence

$$a_{nk}^2 / M \, a_n \leqq q_{nk}^2 \leqq a_{nk}^2 / C \, a_n, \qquad k = 1, \cdots, k_n, \quad n = 1, 2, \cdots. \tag{10}$$

Because of 5), the latter terms with fixed k form a sequence which tends to zero as $n \to \infty$. Thus, if we select a positive constant $\eta < C^{-1}$, we can uniquely determine a sequence of integers $\{\nu_n ; n = 1, 2, \cdots\}, 0 \leqq \nu_n \leqq k_n$, where ν_n is the largest integer, such that

$$(1/a_n) \sum_{k=1}^{\nu_n} a_{nk}^2 < \eta C \; (<1) \tag{11}$$

(for $\nu_n = 0$ we take the empty sum). Since (11) would tend to zero if the ν_n's were bounded it follows necessarily that $\nu_n \to \infty$; furthermore, $\nu_n < k_n$ holds because of $\sum_k a_{nk}^2 = a_n$ and $\eta C < 1$. Let now $\{n_\rho\}$ be an infinite sequence of integers such that

$$k_{n_\rho} \equiv k(\rho) < \nu_{n_{\rho+1}} \equiv \nu(\rho+1), \qquad \rho = 1, 2, \cdots .$$

Put

$$c_k = +(Ma_n/\min_j a_{n_\rho j}^2)^{\frac{1}{2}}, k \,\varepsilon\{\nu(\rho), \cdots, k(\rho)\}, \quad \rho = 1, 2, \cdots, \tag{12}$$

where the minimum is taken for all $j \,\varepsilon\{\nu(\rho), \cdots, k(\rho)\}$ except those j for which $a_{n_\rho j} = 0$ (because of (11), (12) is always defined). In order to obtain an infinite sequence $\{c_k, k = 1, 2, \cdots\}$ let all c_k's not determined by (12) be arbitrary positive numbers. Choose now a sequence $\{G_k\}$, $G_k \,\varepsilon\, \hat{F}$, so that

$$\int_{|x|>c_k} x^2\, dG_k(x) > C, \qquad k = 1, 2, \cdots . \tag{13}$$

With identical c_k's we associate the same d.f.'s G_k, and correspondingly choose a sequence $\{\epsilon_k\} \,\varepsilon\, \mathfrak{F}(\hat{F})$. We now obtain a contradiction with (7) since according to (10)-(13) for each n_ρ and $1 > \delta > 0$

$$\sum_{\substack{k=1\\ q_{n_\rho k}\neq 0}}^{k(\rho)} q_{n_\rho k}^2 \int_{|x|>\delta/q_{n_\rho k}} x^2\, dG_k(x) \geqq \left(\sum_{k=\nu(\rho)}^{k(\rho)} q_{n_\rho k}^2\right) \int_{|x|>c_{\nu(\rho)}} x^2\, dG_{\nu(\rho)}(x) >$$

$$\frac{C}{M}\left\{1 - \sum_{k=1}^{\nu(\rho)} a_{n_\rho k}^2/a_{n_\rho}\right\} > \frac{C}{M}(1 - \eta C) = \text{const} > 0.$$

8) In order to prove the necessity of (III), we suppose the set F does not have this property. It then contains sequences of d.f.'s whose variances tend to zero. We shall show that there exists a sequence of this kind for which the left hand side of (7) remains above a positive constant for a sequence of n-values $\{n_\rho\,; \rho = 1, 2, \cdots\}$ which tends to infinity. This, however, is contradictory.

We first give the proof for a special set F which we denote by S and which does not have property (III). It shall contain only step functions with two steps of $\frac{1}{2}$ at $+\sigma$ and $-\sigma$ with values of $\sigma > 0$. The jth term in (7) for a sequence $\{S_k(x)\}$ of d.f.'s $S_k(x) \varepsilon S$ with variances σ_k^2 then gives the contribution

$$q_{nj}^2 \int_{|x|>\delta/q_{nj}} x^2\, dS_j(x) = q_{nj}^2\, \delta_j^2$$

if

$$q_{nj}^2 \sigma_j^2 > \delta^2, \tag{14}$$

because then the integration interval contains the points $\pm\sigma_j$. Putting $\delta^2 = \eta$, (14) becomes

$$(1-\eta)\sigma_j^2 a_{nj}^2 - \eta \sum_{\substack{k=1\\ k\neq j}}^{k_n} \sigma_k^2 a_{nk}^2 > 0. \tag{15}$$

Consider now for every k the sequences

$$\alpha_k = \{a_{nk}^2 A_n^{-2}; n = 1, 2, \cdots\} \tag{16}$$

where $A_n^2 = \max_{k=1,\cdots,k_n} a_{nk}^2$, $n = 1, 2, \cdots$. We first consider the case that there exists a sequence α_κ which contains a subsequence with a positive limit value, $a_{n\kappa}A_n^{-2} > c > 0$ for $n = n_1, n_2, \cdots$. For these n_ρ, the left hand side of (15) for $j = \kappa$ is greater than

$$\{(1-\eta)\sigma_\kappa^2 c - \eta \sum_{k=1}^{\infty} \sigma_k^2\} A_{n_\rho}^2. \tag{17}$$

This can be made positive by choosing η small enough and $\{\sigma_k\}$ such that $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$. With (17) positive, (14) is true for $j = \kappa$, and thus η is a lower bound for (7) for $n = n_1, n_2, \cdots$. This, however, is a contradiction.

If there does not exist a κ as defined above then the sequences α_k for every $k = 1, 2, \cdots$ have the limit zero. Hence, in many different ways, non-decreasing sequences of integers $m(n) \to \infty$, $m(n) < k_n$, can be found such that

$$\sum_{k=1}^{m(n)} a_{nk}^2 A_n^{-2} < \mu \tag{18}$$

where $\mu < 1$ is a positive constant (here and later on small values of n are possibly excluded). Now determine inductively a pair of sequences $\{m(n)\}$ and $\{n_\rho\}$ such that for $\rho = 1, 2, \cdots$

$$m(n_{\rho+1}) = k_{n_\rho} \equiv k(\rho). \tag{19}$$

(The sequences $\{n_\rho\}$, $\{k(\rho)\}$ are of course not those which occur in 7).) Let $K(n)$ for $n = 1, 2, \cdots$ be one of the k-values for which $a_{nk}^2 = A_n^2$. Because $\mu < 1$, certainly $m(n) < K(n)$. For $j = K(n)$, the left hand side of (15) is greater than

$$A_n^2 \left\{ (1-\eta)\, \sigma_{K(n)}^2 - \eta\mu \max_{k=1,\cdots,m(n)} \sigma_k^2 - \eta \sum_{\substack{k=m(n)+1 \\ k\neq K(n)}}^{k_n} \sigma_k^2 \right\},$$

where (18) has been used. This is positive for every n_ρ, $\rho = 1, 2, \cdots$, if we choose $\{\sigma_k\}$ as follows:

(i) $\sigma_{K(n_\rho)} = \text{const} > 0$, all ρ,

(ii) $\sum_{\substack{k=m(n_\rho)+1 \\ k\neq K(n_\rho)}}^{k(\rho)} \sigma_k^2 < \text{const}$, all ρ,

(iii) η sufficiently small.

These conditions are consistent because of (19), and thus the contradiction is established.

9) We now turn back to an arbitrary set F not having property (III). There are sequences of d.f.'s G_k in F such that for their second moments the relation $\liminf_{k\to\infty} \sigma_k = 0$ holds. Associate with every G_k a step function S_k as defined in 8), which has the same variance σ_k. We have for every k, every $\delta > 0$ and every positive number q the inequality

$$\int_{|x|\geq\delta/q} x^2\, dG_k(x) \geqq \frac{1}{2} \int_{|x|\geq\delta'/q} x^2\, dS_k(x)$$

with $\delta' = 2^{\frac{1}{2}}\delta$. It now follows from 8) that we can find a sequence of errors with d.f.'s G_k and a sequence $\{n_\rho\}$ such that for $\rho = 1, 2, \cdots$

$$\sum_{k=1}^{k(\rho)} q_{n_\rho k}^2 \int_{|x|>\delta/q_{n_\rho k}} x^2\, dG(x) \geqq \frac{1}{2} \sum_{k=1}^{k(\rho)} q_{n_\rho k}^2 \int_{|x|>2^{\frac{1}{2}}\delta/q_{n_\rho k}} x^2\, dS_k(x) > \delta^2 > 0,$$

utilizing (14). This completes the proof of Theorem 2.

REFERENCES

EICKER, F. (1963). Asymptotic normality and consistency of the least squares estimators for families of linear regressions. *Ann. Math. Statist.* **34** 447-456.

FRÉCHET, MAURICE (1950). *Généralités sur les Probabilités. Élements Aléatoires* (2nd ed.). Gauthier-Villars, Paris.

GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables.* (Translated by K. L. Chung.) Addison-Wesley, Cambridge.

LOÈVE, MICHEL (1960). *Probability Theory.* (2nd ed.). Van Nostrand, Princeton.

PARZEN, E. (1954). On uniform convergence of families of sequences of random variables. *Univ. California Publ. Statist.* **2** 23-54.