# BOUNDS FOR THE FREQUENCY OF MISLEADING BAYES INFERENCES

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A misleading Bayes inference is one which either assigns a low probability to a true hypothesis, or a high probability to a false hypothesis. If a Bayesian assumes that each of a finite number of simple hypotheses has equal prior probability, then an upper bound exists for the frequency with which his inferences will be misleading.

The result to be discussed may be regarded as a generalisation of Wald's [2], p. 41, bounds for the frequency of errors of each kind in a sequential probability ratio test. Here, however, in addition to allowing any finite number of hypotheses, the stopping rule is completely general: it need not even have probability one of terminating.

Suppose that there are $k$ simple hypotheses, a true hypothesis $T$ and false hypotheses $F_i (i = 1, 2, \cdots, k - 1)$. Let $X_n$ denote the complete set of observations available at the $n$th stage of sampling: if the sampling is terminated at the $r$th stage, $X_n \equiv X_r$ for all $n \geqq r$. $\sum^*$ denotes summation only over those $X_n$ for which both (i) the sampling has terminated, and (ii) the posterior probability of $T$ is $p$ or less. Then if $R_n = \sum_i P(X_n \mid F_i)/P(X_n \mid T)$, applying Bayes Theorem, $R_n = \sum_i P(F_i \mid X_n)/P(T \mid X_n)$, since the prior probabilities are assumed equal. Hence $\sum^* R_n P(X_n \mid T) \geqq [(1 - p)/p] \sum^* P(X_n \mid T)$. But $\sum^* R_n P(X_n \mid T) = \sum_i \{\sum^* P(X_n \mid F_i)\} \leqq k - 1$. Hence $\sum^* P(X_n \mid T) \leqq (k - 1)p/(1 - p)$: but $\sum^* P(X_n \mid T)$ is a monotone increasing function of $n$. Hence $\lim_{n\to\infty} \sum^* P(X_n \mid T)$ exists and $\leqq (k - 1)p/(1 - p)$, that is: *The frequency with which, at the termination of sampling the posterior probability of the true hypothesis is $p$ or less cannot exceed* $(k - 1)p/(1 - p)$.

Since a probability of $p$ or more for a false hypothesis implies $(1 - p)$ or less for the true hypothesis, an upper bound of $(k - 1)(1 - p)/p$ for the frequency of misleading inferences of this kind is an obvious corollary: this may be a more useful form in practice, although the inequality is weaker.

For $k = 2$, the strictness of the inequality is quite remarkable, especially in view of the completely general stopping rule which is allowed. For example, the frequency with which a true hypothesis is found to have posterior probability $\frac{1}{100}$ or less cannot be more than $\frac{1}{99}$. This remains true even if the stopping rule were biassed against the true hypothesis by requiring, to take an extreme case, that the sampling should continue until its posterior probability becomes $\frac{1}{100}$ or less if it ever does: in such a case the probability of terminating is not greater than $\frac{1}{99}$. A particular example of this type has been considered by Savage [1], p. 72.

For larger $k$ the inequality is less sharp, but it still remains true, at any rate

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in the somewhat restricted class of problems considered, that a sufficiently high posterior probability is unlikely to be achieved by a false hypothesis and must be considered as convincing evidence even from a strict frequency point of view. This is not, of course, a reason for using Bayesian rather than non-Bayesian methods, but it does show that even the unintelligent, routine application of Bayes Postulate to this class of problems does not have disastrous practical results. It is reasonable to assume that an intelligent choice of prior probabilities would lead to something better. The fact that any kind of general frequency statement is possible in the case of biassed stopping rules should be of direct practical as well as theoretical interest, since biassed stopping rules occur in statistical applications more than we usually like to admit.

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## REFERENCES

[1] Savage, L. J. (1962). *The Foundations of Statistical Inference*. Methuen, London.
[2] Wald, A. (1947). *Sequential Analysis*. Wiley, New York.