ON ESTIMATES FOR FRACTIONS OF A COMPLETE FACTORIAL EXPERIMENT AS ORTHOGONAL LINEAR COMBINATIONS OF THE OBSERVATIONS[1]

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0. Summary. A procedure for adjusting the treatment design matrix to furnish estimates as orthogonal linear functions of stochastic variates has been given for fractional replicates of factorial experiments composed of $ks^{-m}s^n$ treatments for $m < n$, for s a prime number, and k not a multiple of s. These are irregular fractional replicates. Ordinarily irregular fractional replicates do not lead to estimates of effects which are orthogonal linear combinations of the observations. In this paper a new relationship and some generalizations on the theory of irregular fractional replicates of complete factorials have been developed. En route to these developments two theorems in matrix transformations were proved. In addition, the relationship between the method utilized here and ordinary missing plot techniques is pointed out.

1. Introduction. Regular fractional replicates are those obtained by *completely* confounding one or more effects with the mean resulting in an s^{-m} fraction of an s^n complete factorial for s a prime number, and for $m < n$. These designs and their properties have been discussed at length by Finney [12], Kempthorne [15], Daniel [6], [7], Rao [18], and others (for e.g. see references in [7]).

Irregular fractional replicates, then, are those which are not regular as defined above, and are of various types. For example, one type of an irregular fractional replicate is the fraction of ks^{-m} of an s^n complete factorial where k is not a multiple of s. The method of construction for this type of irregular fractional replicate was presented by Banerjee [3] together with a worked numerical example of a $\frac{3}{4}$ replicate of a 2^6 factorial. The possibility of using a $\frac{3}{4}$ replicate of a complete factorial was suggested also by Kempthorne [16], and Davies and Hay [8]. After a decade, additional papers have appeared on $\frac{3}{4}$ replicates [14] and on ks^{-m} fractions of an s^n factorial [1]. Another type of irregular fractional replicate of a complete factorial is obtained by taking an s^{-m} fraction of an s^n factorial such that some effects are partially confounded with the mean while others are either completely confounded or completely unconfounded with the mean [5], [9], [11]. Many other types of fractionally replicated factorial experiments may be obtained by adding fractional replicates together. For example,

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add fractions ks^{-m}, s^{-r}, s^{-t}, etc. together to form the desired combinations of a factorial experiment. Using this procedure response surface designs, diallel crossing systems, proportional factorials, etc. may easily be generated.

Irregular fractions may be selected in such a way that no main effect or interaction need be completely confounded with the mean [1]. Although such designs lead to correlated estimates, (e.g. see [1], [3], and [14]), it will be shown in this paper how to adjust the treatment design matrix to obtain an orthogonal form for the estimates, thereby facilitating the computations. In addition, some generalizations of the theory on irregular fractional replicates, a new relationship between an irregular fractional replicate and the full replicate and a measure of efficiency of a design are presented. Two theorems in matrix transformations used for the above results are given together with their proofs. The relationship between the method presented here and ordinary missing plot techniques is pointed out.

2. Two useful theorems. Let Y represent a column vector of n stochastic variates, $y_1, y_2, \cdots, y_n$, let $\mathbf{B}$ represent a column vector of p unknown parameters, $b_1, b_2, \cdots, b_p$, and let the known treatment design matrix X be composed of n rows and p columns. Then, the observational equations may be represented as

$$Y = X\mathbf{B} + e, \tag{1}$$

where e is an $n \times 1$ column vector of error components, $e_1, e_2, \cdots, e_n$, and where $E(Y) = X\mathbf{B}$. If $n \geqq p$, the least squares estimates of $\mathbf{B}$ are given by $\mathbf{B}^+ = [X'X]^{-1}X'Y$ as in ordinary regression theory.

Now let $p = n$, let the treatment design matrix be a $p \times p$ square matrix and let this square matrix of rank p be augmented so as to contain m additional rows with p columns in each and these additional rows are set up corresponding to m additional stochastic variates $y_1', y_2', \cdots, y_m'$. Denote the additional part of the treatment design matrix as $X_m = \lambda'X$, the augmented matrix as X_1, the additional stochastic variates as Y_m, and the augmented stochastic variate matrix as Y_1. Now X_1 will have $p + m$ rows and p columns, and Y_1 will be a $(p + m) \times 1$ column vector. Let the rows of X be denoted by $\alpha_1, \alpha_2, \cdots, \alpha_p$ and those of X_1 by $\alpha_1, \alpha_2, \cdots \alpha_p, \beta_1, \beta_2, \cdots \beta_m$. The least squares estimates corresponding to the observational equations, $Y_1 = X_1\mathbf{B}_1 + e_1$ will be given by $[X_1'X_1]^{-1}X_1'Y_1 = \mathbf{B}_1^+$, say.

THEOREM 1. *If* $\beta_k = \sum_{i=1}^{p} \lambda_{ik}\alpha_i$ *and* $y_k' = \sum_{i=1}^{p} \lambda_{ik}y_i$, *where* $k = 1, 2, \cdots, m$ *and where* λ_{ik} *are scalars, then* $\mathbf{B}^+ = \mathbf{B}_1^+$.

PROOF. Let λ be a p-row by m-column matrix with elements λ_{ik} such that the augmented treatment design X_1 and the corresponding stochastic vector Y_1 take the form $X_1 = [X \vdots X_m]' = [X \vdots \lambda'X]'$ and $Y_1 = [Y \vdots Y_m]' = [Y \vdots \lambda'Y]'$. The normal equations $[X_1'X_1]\mathbf{B}_1^+ = X_1'Y_1$ may be written as:

$$[X'X + X'\lambda\lambda'X]\mathbf{B}_1^+ = [X' \vdots [\lambda'X]'][Y \vdots \lambda'Y]' = [X'Y + X'\lambda\lambda'Y]$$

or as $[X'[I + \lambda\lambda']X]\mathbf{B}_1^+ = [X'[I + \lambda\lambda']Y]$. In the above, X is square and non-singular. Also, $[I + \lambda\lambda']$ is a square matrix and not equal to zero if solutions exist. Therefore, premultiplying both sides by $[X']^{-1}$, $[I + \lambda\lambda']^{-1}$, and X' in that order, we obtain $X'X\mathbf{B}_1^+ = X'Y$. Hence, $\mathbf{B}_1^+ = \mathbf{B}^+$.

LEMMA. *For the augmented design matrix X_1 and for the observational equations $Y_1 = X_1\mathbf{B}_1 + e_1$, the residual sum of squares is zero.*

The residual sum of squares is given by

$$Y_1'Y_1 - \mathbf{B}_1^{+\prime}X_1'Y_1 \tag{2}$$

which reduces to

$$[Y'Y + Y'\lambda\lambda'Y] - \mathbf{B}^{+\prime}[X' \vdots X'\lambda][Y \vdots \lambda'Y]'$$

or to

$$[Y'Y - \mathbf{B}^{+\prime}X'Y] + [Y'\lambda\lambda'Y - \mathbf{B}^{+\prime}X'\lambda\lambda'Y] = [Y'Y - \mathbf{B}^{+\prime}X'Y]$$
$$+ [Y'\lambda\lambda'Y - [X^{-1}Y]'X'\lambda\lambda'Y] = 0 + [Y'\lambda\lambda'Y - Y'\lambda\lambda'Y] = 0$$

since the residual sum of squares for the observational equations (1) is zero when X is a $p \times p$ square matrix. The introduction of the new variates, y_1', y_2', $\cdots$, y_m' does not alter the residual sum of squares.

Let the square matrix X of rank p be transformed to X_2 in such a manner that the rows of X_2 are dependent on the rows of X such that the vector space generated by the row vectors of X_2 is the same as that generated by the row vectors of X. Then both X and X_2 have the same rank p. X_2 may be written as $FX = X_2$, where F is a square matrix obtained by premultiplication of elementary matrices required to transform X to X_2 and involving only addition or subtraction of rows.

THEOREM 2. *If the treatment design matrix X and the stochastic vector Y are transformed to X_2 and Y_2, respectively, in such a manner that $X_2 = FX$ and $Y_2 = FY$, where these matrices are as defined above, then the least squares estimates $\mathbf{B}_2^+$ from the observational equations $Y_2 = X_2\mathbf{B}_2 + e_2$ will be the same as $\mathbf{B}^+$ from the observational equations $Y = X\mathbf{B} + e$.*

PROOF. We have $X_2'X_2\mathbf{B}_2^+ = X_2'Y_2$, and premultiplying both sides by

$$[X_2]^{-1}[X_2']^{-1}$$

we obtain:

$$\mathbf{B}_2^+ = [X_2]^{-1}[X_2']^{-1}X_2'Y_2 = [X_2]^{-1}Y_2 = X^{-1}F^{-1}FY = X^{-1}Y.$$

But, this is equal to $\mathbf{B}^+$ and hence $\mathbf{B}_2^+ = \mathbf{B}^+$. Likewise, it can be shown that the residual sum of squares is zero for this case.

3. Formation of irregular fractional plans. It is well known that the total number of treatments or combinations s^n of n factors at s levels may be denoted by points of an n-dimensional lattice with n axes as x_1, x_2, $\cdots$, x_n, where each axis will have s points given by the elements of the Galois Field GF(s). For the

n factors $A, B, C, \cdots, T$, the notation $ABC \cdots T_i$, for the n-factor interaction, will be used to denote the set of treatments for which $x_1 + x_2 + \cdots + x_n = i \pmod{s}$. For $s = 2$, the treatments in a level of a main effect would be denoted as A_0 and A_1, say, and A would represent the main effect; the treatments in a level of a two-factor interaction would be AB_0 and AB_1, say, and AB would represent the two-factor interaction; etc.

A fractional replicate of a complete factorial may be formed by taking only those treatments corresponding to a given level of one or more main effects and/or interactions. For example, the eight one-eighth replicates of a 2^3 factorial are, for I_f^* ($f = 1, 2, \cdots, 8$) denoting the identifying contrast and for the equals sign associated with I_f^* meaning "completely confounded with" (using textbook notation, e.g., see [10], [17]):

	Treatment
$I = A_0 = B_0 = AB_0 = C_0 = AC_0 = BC_0 = ABC_0$	$000 = (1)$
$I = A_1 = B_0 = AB_1 = C_0 = AC_1 = BC_0 = ABC_1$	$100 = a$
$I = A_0 = B_1 = AB_1 = C_0 = AC_0 = BC_1 = ABC_1$	$010 = b$
$I = A_1 = B_1 = AB_0 = C_0 = AC_1 = BC_1 = ABC_0$	$110 = ab$
$I^* = A_0 = B_0 = AB_0 = C_1 = AC_1 = BC_1 = ABC_1$	$001 = c$
$I_6^* = A_1 = B_0 = AB_1 = C_1 = AC_0 = BC_1 = ABC_0$	$101 = ac$
$I_7^* = A_0 = B_1 = AB_1 = C_1 = AC_1 = BC_0 = ABC_0$	$011 = bc$
$I_8^* = A_1 = B_1 = AB_0 = C_1 = AC_0 = BC_0 = ABC_1$	$111 = abc$

where the letter and number notation in the last column is that used by Fisher [13] and Yates [20]. Utilizing this notation certain simplifying algebraic relationships are possible among treatments and effects in factorial experiments; this relationship will be illustrated in the examples below.

Likewise, a set of four one-quarter replicates of a 2^3 factorial would be:

	Treatments
$I_1^* = A_0 = BC_0 = ABC_0$	$000 + 011 = (1) + bc$
$I_2^* = A_0 = BC_1 = ABC_1$	$010 + 001 = b + c$
$I_3^* = A_1 = BC_0 = ABC_1$	$100 + 111 = a + abc$
$I_4^* = A_1 = BC_1 = ABC_0$	$110 + 101 = ab + ac$

The two half replicates for which ABC is completely confounded with the mean are

	Treatments
$I_1^* = ABC_0$	$000 + 110 + 101 + 011 = (1) + ab + ac + bc$
$I_2^* = ABC_1$	$100 + 010 + 001 + 111 = a + b + c + abc$

The fact that half replicates could also be formed by adding appropriate quarter replicates or one-eighth replicates was utilized by Banerjee [3] and others [1], [7], [14] to form irregular fractional plans with ks^{-m} treatments.

Thus, a $\frac{5}{8}$ or, a $\frac{7}{8}$ replicate of a 2^3 may be formed by adding any 5 or any 7 one-eighth replicates. Likewise, a $\frac{3}{4}$ replicate may be obtained by adding any 3 quarter replicate plans. Doing this it will be noted that no main effect or interaction need be completely confounded with the mean. This fact was proved by Addelman [1] for $k \geqq m + 1$. He did not state, however, if this holds for any k of the fractions, nor did he state whether there is a value of k for which this theorem will hold for all ks^{-m} fractions. In the eight one-eighth replicates Addelman's [1] theorem indicates that $k \geqq 3 + 1 = 4$ in order that no main effect or interaction be completely confounded with the mean, but it is obvious that not any four will do. The four will need to be taken in such a way that three one-eighth replicates come from the first four one-eighth replicates listed above and the fourth from the last four one-eighth replicates, or *vice versa*. Also, it is clear that any five of the eight will meet this requirement.

4. Estimates from irregular fractional plans. Expression of the yields as linear functions of the main effects and interactions in matrix notation is $Y = X\mathbf{B} + e$, where Y is the column vector of the yields, X the treatment design matrix, $\mathbf{B}$ the column vector for the mean, the main effects and interactions, and e the error vector. It has to be remembered, however, that in the column vector, $\mathbf{B}$, the coefficient of the mean is 1, whereas those of the main effects and interactions are $\pm\frac{1}{2}$, for the 2^n factorial.

When the treatment design matrix X corresponds to the full replicate or to a regular fraction of a 2^n factorial experiment, the information matrix, $[X'X]$, and its inverse, $[X'X]^{-1}$, reduce to diagonal forms. But when the design matrix corresponds to an irregular fractional plan, the information matrix ceases to be diagonal, and, in such a situation, the linear functions of the yields estimating the effects cease to be orthogonal. Let $\mathbf{B} = LY$, where $L = [X'X]^{-1}X'$. In order that the estimates $\mathbf{B}^+$ be available in orthogonal linear functions of the yields, we should have

$$LL' = [[X'X]^{-1}X'][[X'X]^{-1}X']' = [X'X]^{-1}X'X[X'X]^{-1} = [X'X]^{-1}$$

reduced to a diagonal form. The constitution of $[X'X]^{-1}$ will, therefore, reveal whether the linear functions, LY, will be orthogonal, or not.

Addelman [1] illustrated the use of a $\frac{3}{4}$ replicate of a 2^4-experiment by taking an aggregate of three quarter replicates as given by:

Identity relationship	Fractional replicates		
	1	2	3
ABC	0	0	1
ABD	0	1	0
CD	0	1	1

In this case, the information matrix comes out in the following form:

$$(3)\quad [X'X] = \begin{bmatrix} [P] & & & 0 \\ & [P] & & \\ & & [P] & \\ 0 & & & [P] \end{bmatrix}, \qquad \text{where } [P] = \begin{bmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{bmatrix}.$$

The inverse matrix will be obtained as

$$(4)\quad [X'X]^{-1} = \begin{bmatrix} [Q] & & & 0 \\ & [Q] & & \\ & & [Q] & \\ 0 & & & [Q] \end{bmatrix}, \qquad \text{where } [Q] = \tfrac{1}{32}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

The constitution of the above matrix will be clear when it is recalled that the 12 effects of the $\frac{3}{4}$ replicate of a 2^4 experiment may be divided into four groups of three each, and that the three in a group will be correlated among themselves, but will be orthogonal to the rest in the other groups. The effects may be estimated as $\mathbf{B}^+ = [X'X]^{-1}X'Y$, which, in this case, will not be available as orthogonal linear functions of Y, as $[X'X]^{-1}$ is not of the diagonal form. The variance and the covariance factors for the estimates will be obtained from the diagonal and off-diagonal elements of $[X'X]^{-1}$. For example, the variance of each of the estimates comes out to be $\sigma^2/2$, whereas the covariance between any two in a group of three is $\sigma^2/4$.

5. Estimates as orthogonal linear functions. The form of the matrix given by Equation (4) indicates that the estimates of the effects within a group are correlated. The treatment design matrix may, however, be reoriented in the form of either Theorem 1 or 2 to furnish orthogonal linear functions of the yields. The procedure is illustrated below with three examples. The notation of Theorems 1 and 2 is retained here.

EXAMPLE 1. Consider the $\frac{7}{8}$ replicate of a 2^3 factorial obtained by deleting treatment 111. If one of the effects is set equal to zero, e.g. ABC, then $X\mathbf{B}^+ = Y$ is of the form:

$$\begin{bmatrix} + & - & - & + & - & + & + \\ + & + & - & - & - & - & + \\ + & - & + & - & - & + & - \\ + & + & + & + & - & - & - \\ + & - & - & + & + & - & - \\ + & + & - & - & + & + & - \\ + & - & + & - & + & - & + \end{bmatrix} \cdot \begin{bmatrix} M \\ A/2 \\ B/2 \\ AB/2 \\ C/2 \\ AC/2 \\ BC/2 \end{bmatrix} = \begin{bmatrix} 000 = (1) \\ 100 = a \\ 010 = b \\ 110 = ab \\ 001 = c \\ 101 = ac \\ 011 = bc \end{bmatrix}.$$

If X is augmented by $\lambda'X$ where $\lambda' = (+--+-++)$, and Y is augmented by $\lambda'Y$ we obtain $X_1\mathbf{B}_1^+ = Y_1$ as

$$\begin{bmatrix} + & - & - & + & - & + & + \\ + & + & - & - & - & - & + \\ + & - & + & - & - & + & - \\ + & + & + & + & - & - & - \\ + & - & - & + & + & - & - \\ + & + & - & - & + & + & - \\ + & - & + & - & + & - & + \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ + & + & + & + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} M \\ A/2 \\ B/2 \\ AB/2 \\ C/2 \\ AC/2 \\ BC/2 \end{bmatrix} = \begin{bmatrix} 000 \\ 100 \\ 010 \\ 110 \\ 001 \\ 101 \\ 011 \\ \cdots \\ 111^+ \end{bmatrix}.$$

Since $[X_1'X_1]$ is a diagonal matrix with 8's down the diagonal then we may write $\mathbf{B}^+$ as

$$\begin{bmatrix} M \\ A/2 \\ B/2 \\ AB/2 \\ C/2 \\ AC/2 \\ BC/2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} +000 + 100 + 010 + 110 + 001 + 101 + 011 + 111^+ \\ -000 + 100 - 010 + 110 - 001 + 101 - 011 + 111^+ \\ -000 - 100 + 010 + 110 - 001 - 101 + 011 + 111^+ \\ +000 - 100 - 010 + 110 + 001 - 101 - 011 + 111^+ \\ -000 - 100 - 010 - 110 + 001 + 101 + 011 + 111^+ \\ +000 - 100 + 010 - 110 - 001 + 101 - 011 + 111^+ \\ +000 + 100 - 010 - 110 - 001 - 101 + 011 + 111^+ \end{bmatrix}.$$

From the preceding equations we note that

$$111^+ = M + A/2 + B/2 + AB/2 + C/2 + AC/2 + BC/2 \text{ for } ABC = 0.$$

Summing these effects we note that

$$\begin{aligned} 111^+ &= 000 - 100 - 010 + 110 - 001 + 101 + 011 \\ &= (a-1)(b+c-1) + ac = (b-1)(a+c-1) + ac \\ &= (c-1)(a+b-1) + ab. \end{aligned}$$

An easier method for obtaining 111^+ is from $\lambda'Y = 111^+$. Since we now have the value of 111^+ the estimates of the effects are obtained in the same manner as for the complete factorial with 111^+ substituted for the yield of 111. The analogy here with missing plot techniques is obvious. It should be noted that $\lambda'X$ in X_1 is simply the remaining rows of the treatment design matrix for the complete factorial and that λ is the negative of the remaining columns of the treatment design matrix for the complete factorial. These relationships hold for the 2^n factorials.

EXAMPLE 2. Consider now the $\frac{3}{4}$ replicate obtained by adding the following three quarter replicates together:

$$\begin{aligned} I_1^* &= B_0 = C_0 = BC_0 \\ I_2^* &= B_0 = C_1 = BC_1 \\ I_3^* &= B_1 = C_0 = BC_1 \end{aligned}$$

and augmenting the X matrix by utilizing the remaining rows of the complete factorial. The following results for $[X \vdots \lambda' X]' \mathbf{B}_1^+ = X_1 \mathbf{B}_1^+ = Y_1$:

$$\begin{bmatrix} + & - & - & + & - & + \\ + & + & - & - & - & - \\ + & - & + & - & - & + \\ + & + & + & + & - & - \\ + & - & - & + & + & - \\ + & + & - & - & + & + \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ + & - & + & - & + & - \\ + & + & + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} M \\ A/2 \\ B/2 \\ AB/2 \\ C/2 \\ AC/2 \end{bmatrix} = \begin{bmatrix} 000 \\ 100 \\ 010 \\ 110 \\ 001 \\ 101 \\ \cdots \\ 011^+ \\ 111^+ \end{bmatrix},$$

where

$$\lambda' = \begin{bmatrix} -0 + 0 + 0 \\ 0 - 0 + 0 + \end{bmatrix}$$

and where $\lambda' Y$ results in the following values of 011^+ and 111^+:

$$011^+ = 110 + 101 - 100 = ab + ac - a$$

$$111^+ = 010 + 001 - 000 = b + c - (1).$$

With these "missing plot" solutions substituted for the missing yields, the solutions for the above effects are obtained in an identical manner to that for a complete factorial where BC and ABC are set equal to zero. If any other effects had been equated to zero then the design matrix X would change but otherwise the solution goes through as above.

We note that the following transformation could have been used: $FX_1 = X_2$ and $FY_1 = Y_2$, where F is the square matrix used in Theorem 2 and is defined below:

$$FX_1 = \left[\begin{array}{cccccc:cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} \end{array}\right]$$

$$\cdot X_1 = \begin{bmatrix} + & - & - & + & - & + \\ + & + & - & - & - & - \\ + & - & + & - & - & + \\ + & + & + & + & - & - \\ + & - & - & + & + & - \\ + & + & - & - & + & + \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2^{\frac{1}{2}} & 0 & 2^{\frac{1}{2}} & 0 & 2^{\frac{1}{2}} & 0 \\ 0 & -2^{\frac{1}{2}} & 0 & 2^{\frac{1}{2}} & 0 & 2^{\frac{1}{2}} \end{bmatrix} = X_2$$

and

$$Y_2' = (000\ 100\ 010\ 110\ 001\ 101\ z_1'z_2'],$$

where

$$z_1' = (011^+ + 111^+)/2^{\frac{1}{2}}, \qquad z_2' = (-011^+ + 111^+)/2^{\frac{1}{2}}$$

and $2^{\frac{1}{2}} = (\text{no. of treatments omitted})^{\frac{1}{2}}$. Alternatively, X_2 could have been obtained using

$$\lambda' = \begin{bmatrix} - & - & + & + & + & + \\ + & - & - & + & - & + \end{bmatrix}.$$

This λ, incidentally, is obtained by taking the negative coefficients from the deleted columns for the X matrix for the complete factorial. Also, $\lambda'Y$ yields the values for $z_1' = 011^+ + 111^+$ and $z_2' = -011^+ + 111^+$. It should be noted here that the rows of X_m and Y_m are divided by $(\text{no. of treatments omitted})^{\frac{1}{2}}$ to obtain X_2 and Y_2, and that the estimates of the parameters in **B** obtained using X, X_1, or X_2 are identical.

EXAMPLE 3. Delete treatments 21 and 22 from a 3^2 factorial to obtain the normal equations for a $\frac{7}{9}$ replicate of a 3^2 factorial. After augmentation of the X and Y matrices and setting A_QB_L and A_QB_Q equal to zero we obtain:

$$\begin{bmatrix} + & - & + & - & + & + & - \\ + & - & + & 0 & -2 & 0 & 2 \\ + & - & + & + & + & - & - \\ + & 0 & -2 & - & + & 0 & 0 \\ + & 0 & -2 & 0 & -2 & 0 & 0 \\ + & 0 & -2 & + & + & 0 & 0 \\ + & + & + & - & + & - & + \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ + & + & + & 0 & -2 & 0 & -2 \\ + & + & + & + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} M \\ A_L \\ A_Q \\ B_L \\ B_Q \\ A_LB_L \\ A_LB_Q \end{bmatrix} = \begin{bmatrix} 00 \\ 01 \\ 02 \\ 10 \\ 11 \\ 12 \\ 20 \\ \cdots \\ 21^+ \\ 22^+ \end{bmatrix},$$

where

$$\lambda' = \begin{bmatrix} + & - & 0 & -2 & 2 & 0 & + \\ + & 0 & - & -2 & 0 & 2 & + \end{bmatrix}$$

and where the divisors of the effects are omitted. In the above, $\lambda'X$ is the same as the last two columns of the matrix of coefficients on page 196 of [10], after adding in a set of pluses for the effect M; λ' is equal to the coefficients for

$$(A_QB_Q - A_QB_L)/2$$

and $-A_QB_L$ in the first seven columns of the matrix.

If, on the other hand, λ was obtained from the last two columns and first seven rows of the matrix on page 196 of [10] (after adding in a set of pluses for the first row), this would change the augmented part, X_m, of the matrix X_1. Either procedure leads to the same estimates for the effects, which are:

$$\begin{bmatrix} M \\ A_L \\ A_Q \\ B_L \\ B_Q \\ A_LB_L \\ A_LB_Q \end{bmatrix} = \tfrac{1}{36} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\cdot \begin{bmatrix} +00 & +01 & +02 & +10 & +11 & +12 & +20 & +21^+ & +22^+ \\ -00 & -01 & -02 & & & & +20 & +21^+ & +22^+ \\ 00 & +01 & +02 & -2(10 & +11 & +12) & +20 & +21^+ & +22^+ \\ -00 & & +02 & -10 & & +12 & -20 & & +22^+ \\ 00 & -2(01) & +02 & +10 & -2(11) & +12 & +20 & -2(21^+) & +22^+ \\ 00 & & -02 & & & & -20 & & +22^+ \\ -00 & +2(01) & -02 & & & & +20 & -2(21^+) & +22^+ \end{bmatrix},$$

where the "missing plot" solutions 21^+ and 22^+ are obtained from $\lambda'Y$ as

$$21^+ = 00 - 01 - 2(10) + 2(11) + 20$$

and

$$22^+ = 00 - 02 - 2(10) + 2(12) + 20.$$

Following the procedure for Example 3, we note that estimates of the effects for a fractional replicate of a $p \times q \times k \times \cdots$ factorial, for $p, q, k, \cdots$ any integer, may be obtained as orthogonal linear combinations of the observations in a straightforward manner.

6. A measure of efficiency. Since a ks^{-m} fraction of an s^n complete factorial may be chosen in several ways, a measure of efficiency of alternative schemes is desirable. To do this we shall use the relative efficiency to two designs as implied by Wald [19] and used by Banerjee [4] for weighing designs. Doing this we denote the relative efficiency of two designs for fractional replicates as the pth root of the two determinants: $\det [X'X]$ for design $1/\det [X'X]$ for design 2. The efficiency factor for a prescribed fractional replicate may be taken as the pth root of the ratio of the determinant $[X'X]$ for the design and the maximum

value of the determinant of $[X'X]$ over the set of possible fractional replicates of which the prescribed fractional replicate is a member.

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