# RIGHT ANGULAR DESIGNS

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**1. Summary and introduction.** Known PBIB designs with two associate classes were classified according to their association schemes by Bose and Shimamoto [3]. Association schemes for three or more associate classes have been little developed except the rectangular association scheme of Vartak [15] and GD $m$-associate designs of Raghavarao [8] and Hierarchical group divisible designs with $m$-associate classes of Roy [11]. In this paper we define a new association scheme known as right angular scheme which defines four associate class designs. For convenience we call PBIB designs with right angular scheme, right angular designs.

In Section 2 we define the association scheme. Sections 4 and 5 deal with the analysis and construction of right angular designs. In Section 6 we study some combinatorial properties and Sections 3 and 7 give the non-existence of some right angular designs.

**2. Definition of association scheme and parameters of right angular designs.** In right angular design we have $v = 2Sl$ treatments which are arranged in $l$ right angles of equal arms equal to $S$. The angular position of these right angles are kept blank and treatments are written along the arm.

We define the four associates of a particular treatment $\phi$ as follows:

(i) Treatments other than $\phi$ occurring in the same arm with $\phi$ are the first associates.

(ii) Treatments occurring in the different arm to $\phi$ but in the same Right Angle with $\phi$ are the second associates.

(iii) Treatments occurring in the parallel arm to $\phi$, but in other Right Angles are the third associates.

(iv) The remaining treatments are the fourth associates.

It is easy to see that right angular design is a partially balanced design with four associate classes for which [2]

$$n_1 = S - 1, \qquad n_2 = S, \qquad n_3 = S(l-1) = n_4,$$

$$P_{ij}^1 = \begin{bmatrix} S-2 & 0 & 0 & 0 \\ & S & 0 & 0 \\ & & S(l-1) & 0 \\ & & & S(l-1) \end{bmatrix};$$

$$P_{ij}^2 = \begin{bmatrix} 0 & S-1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & S(l-1) \\ & & & 0 \end{bmatrix};$$

Received December 28, 1961; revised February 28, 1963.

(2.1)

$$P_{ij}^{3} = \begin{bmatrix} 0 & 0 & S-1 & 0 \\ & 0 & 0 & S \\ & & S(l-2) & 0 \\ & & & S(l-2) \end{bmatrix};$$

$$P_{ij}^{4} = \begin{bmatrix} 0 & 0 & 0 & S-1 \\ & 0 & S & 0 \\ & & 0 & S(l-2) \\ & & & 0 \end{bmatrix}.$$

**3. Characterization of right angular designs.** Let $n_{ij} = 1$, if the $i$th treatment occurs in the $j$th block; and $n_{ij} = 0$, otherwise. Then the $v \times b$ matrix $N = (n_{ij})$ is known as the incidence matrix of the right angular design. From the definition of the right angular design, we can see that

$$\sum_{j=1}^{b} n_{ij}^{2} = r, \qquad i = 1, 2, \cdots, v; \quad \text{and} \quad \sum n_{ij} n_{i'j} = \lambda_1, \lambda_2, \lambda_3, \text{ or } \lambda_4,$$

according as $i$ and $i'$ are 1st, 2nd, 3rd or 4th associates, $i \neq i' = 1, 2, 3, \cdots, v$. Now by suitably marking the treatments, we have

$$NN' = I_l \times (C - D) + E_{ll} \times D, \tag{3.1}$$

where $C$ and $D$ are $2S \times 2S$ square matrices given by

$$C = \begin{bmatrix} (r - \lambda_1)I_S + \lambda_1 E_{SS} & \lambda_2 E_{SS} \\ \lambda_2 E_{SS} & (r - \lambda_1)I_S + \lambda_1 E_{SS} \end{bmatrix}, \tag{3.2}$$

$$D = \begin{bmatrix} \lambda_3 & \lambda_4 \\ \lambda_4 & \lambda_3 \end{bmatrix} \times E_{SS},$$

where $I_S$ is an identity matrix of order $S$; $E_{SS}$ is a square matrix of order $S$ with positive unit elements everywhere and $\times$ denotes the Kronecker product of matrices. The order of $NN'$ is $2Sl$. $C$ and $D$ are each of order $2S$. Det $(NN')$ can be evaluated in the usual manner and we get

$$|NN'| = \theta_0 \theta_1 \theta_2^{2l(S-1)} \theta_3^{l-1} \theta_4^{l-1} \tag{3.3}$$

where $\theta_0 = rk$, $\theta_1 = r - \lambda_1 + S(\lambda_1 - \lambda_2) + S(l-1)(\lambda_3 - \lambda_4)$, $\theta_2 = r - \lambda_1$, $\theta_3 = r - \lambda_1 + S(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)$, $\theta_4 = r - \lambda_1 + S(\lambda_1 - \lambda_2 + \lambda_4 - \lambda_3)$.

By replacing $r$ by $(r - Z)$ in det $(NN')$ we can easily see that $rk$ and $\theta_i$'s $(i = 1, 2, 3, 4)$ are the distinct characteristic roots of $NN'$. We know from the result of Connor and Clatworthy [4] that the characteristic roots of $NN'$ cannot be negative for an existing design. Thus, we have the following theorem.

THEOREM 3.1. *A necessary condition for the existence of a right angular design is that* $\theta_i \geqq 0$ $(i = 1, 2, 3, 4)$.

The designs with the following parameters violate the above necessary condition and hence are impossible.

| $v$ | $b$ | $r$ | $k$ | $l$ | $S$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 8 | 4 | 4 | 2 | 2 | 2 | 3 | 2 | 0 |
| 12 | 12 | 4 | 4 | 2 | 3 | 0 | 3 | 1 | 0 |
| 16 | 12 | 6 | 8 | 4 | 2 | 0 | 3 | 4 | 2 |
| 20 | 15 | 6 | 8 | 2 | 5 | 3 | 2 | 3 | 1 |
| 30 | 30 | 10 | 10 | 5 | 3 | 6 | 6 | 5 | 0 |
| 30 | 20 | 10 | 15 | 5 | 3 | 10 | 8 | 6 | 2 |
| 28 | 28 | 7 | 7 | 7 | 2 | 6 | 6 | 2 | 0 |
| 54 | 54 | 20 | 20 | 9 | 3 | 10 | 0 | 8 | 7 |

**4. Analysis.** Let us assume the model to be

$$y_{ij} = m + t_i + b_j + e_{ij}, \tag{4.1}$$

where $y_{ij}$ is the yield of the plot in the $j$th block to which the $i$th treatment is applied, $m$ is the general effect, $t_i$ is the effect of the $i$th treatment, $b_j$ is the effect of the $j$th block and the $e_{ij}$ are the independent normal variates with mean zero and variance $\sigma^2$. Let $T_i$ be the total yield of all plots having the $i$th treatment, $B_j$ be the total yield of the $j$th block and $\hat{t}_i$ be a solution for $t_i$ in the normal equations. Further denote the column vectors $\{T_1, T_2, \cdots, T_v\}$, $\{B_1, B_2, \cdots, B_b\}$, $\{t_1, t_2, \cdots, t_v\}$ and $\{\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_v\}$ by $T$, $B$, $t$ and $\hat{t}$ respectively. It is well known that the reduced normal equations are

$$Q = C\hat{t} \tag{4.2}$$

where $$Q = T - NB/k; \qquad C = rI_v - NN'/k.$$

Following [12] the matrix $C + aE_{vv}$ is nonsingular where $a$ is any nonzero real number and $t = (C + aE_{vv})^{-1}Q$ is a solution of $Q = Ct$. Raghavarao [10] has simplified the analysis of group divisible and $L_2$ designs, with the help of characteristic roots and characteristic vectors. Here we proceed the same way to simplify the analysis of right angular designs. Characteristic roots of $C + aE_{vv}$ would be $\phi_0 = av$, $\phi_1$, $\phi_2$, $\phi_3$ and $\phi_4$ where $\phi_i = (r - \theta_i/k)$ $i = 1, 2, 3, 4$ and normalised orthogonal vectors corresponding to $\phi_0$, $\phi_1$, $\phi_2$, $\phi_3$ and $\phi_4$ are

$$v^{-\frac{1}{2}}E_{v1}; \qquad \begin{bmatrix} (2l)^{-\frac{1}{2}} \\ -(2l)^{-\frac{1}{2}} \\ (2l)^{-\frac{1}{2}} \\ \vdots \\ \cdot \\ (2l)^{-\frac{1}{2}} \\ -(2l)^{-\frac{1}{2}} \end{bmatrix} \times S^{-\frac{1}{2}}E_{S,1};$$

$$I_{2l} \times \begin{bmatrix} 2^{-\frac{1}{2}} & (2\cdot 3)^{-\frac{1}{2}} & & [S(S-1)]^{-\frac{1}{2}} \\ -2^{-\frac{1}{2}} & (2\cdot 3)^{-\frac{1}{2}} & & [S(S-1)]^{-\frac{1}{2}} \\ 0 & -2(2\cdot 3)^{-\frac{1}{2}} & & \cdot \\ & & \cdots & \\ \vdots & 0 & & \vdots \\ & \vdots & & \\ 0 & 0 & & -(S-1)[S(S-1)]^{-\frac{1}{2}} \end{bmatrix}; \tag{4.3}$$

$$\begin{bmatrix} 2^{-\frac{1}{2}} & (2\cdot3)^{-\frac{1}{2}} & [l(l-1)]^{-\frac{1}{2}} \\ -2^{-\frac{1}{2}} & (2\cdot3)^{-\frac{1}{2}} & [l(l-1)]^{-\frac{1}{2}} \\ 0 & -2(2\cdot3)^{-\frac{1}{2}} & \cdots \\ \vdots & 0 & \vdots \\ 0 & 0 & -(l-1)[l(l-1)]^{-\frac{1}{2}} \end{bmatrix} \times (2S)^{-\frac{1}{2}}E_{2S,1};$$

$$\begin{bmatrix} 2^{-\frac{1}{2}} & (2\cdot3)^{-\frac{1}{2}} & [l(l-1)]^{-\frac{1}{2}} \\ -2^{-\frac{1}{2}} & (2\cdot3)^{-\frac{1}{2}} \cdots & [l(l-1)]^{-\frac{1}{2}} \\ 0 & -2(2\cdot3)^{-\frac{1}{2}} & \vdots \\ & 0 & \\ 0 & 0 & -(l-1)[l(l-1)]^{-\frac{1}{2}} \end{bmatrix} \times \begin{bmatrix} (2S)^{-\frac{1}{2}}E_{S,1} \\ -(2S)^{-\frac{1}{2}}E_{S,1} \end{bmatrix}.$$

It can be shown that

$$(C + aE_{vv}) = \sum_{i=0}^{4} A_i\phi_i, \tag{4.4}$$

where $A_0$, $A_1$, $A_2$, $A_3$ and $A_4$ are mutually orthogonal symmetric idempotent matrices corresponding to characteristic roots $\phi_0$, $\phi_1$, $\phi_2$, $\phi_3$ and $\phi_4$. Hence, we get

$$\begin{aligned}(C + aE_{vv})^{-1} = \sum_{i=0}^{4} \frac{A_i}{\phi_i} &= E_{ll} \times \left\{\frac{E_{2S,2S}}{av^2} + \frac{1}{v\phi_1}\begin{bmatrix} E_{SS} & -E_{SS} \\ -E_{SS} & E_{SS}\end{bmatrix}\right. \\ &- \frac{E_{2S,2S}}{v\phi_3} - \left.\frac{1}{v\phi_4}\begin{bmatrix} E_{SS} & -E_{SS} \\ -E_{SS} & E_{SS}\end{bmatrix}\right\} + I_l \times \left\{\frac{I_{2S}}{\phi_2} - \frac{I_2 \times E_{SS}}{\phi_2 S}\right. \\ &+ \left.\frac{E_{2S,2S}}{2S\phi_3} + \frac{1}{2S\phi_4}\begin{bmatrix} E_{SS} & -E_{SS} \\ -E_{SS} & E_{SS}\end{bmatrix}\right\}.\end{aligned} \tag{4.5}$$

Hence the solution of the reduced normal equations can be written as

$$\begin{aligned}\hat{t}_i = \left\{\frac{(S-1)Q_i - H_i}{S\phi_2} + \frac{(\phi_3 + \phi_4)(Q_i + H_i) + J_i(\phi_4 - \phi_3)}{2S\phi_3\phi_4}\right. \\ \left. + \frac{(\phi_1 - \phi_4)(J_i + Z_i)}{Sl\phi_1\phi_4}\right\}\end{aligned} \tag{4.6}$$

where $Q_i$ denotes the $i$th adjusted treatment total and $H_i$, $J_i$, and $Z_i$ denote the sum of the adjusted 1st, 2nd, and 4th associate treatment totals of the $i$th treatment. From (4.5) it can be seen that the variance of the elementary treatment contrasts and average variance of all elementary treatment contrasts is given in terms of characteristic roots of $C$ matrix as

$V(t_i - t_i') = \dfrac{2\sigma^2}{\phi_2}$ when both the treatments are 1st associates.

$= \dfrac{2}{lS}\left\{\dfrac{\phi_1 l(S-1) + \phi_2}{\phi_1\phi_2} + \dfrac{l-1}{\phi_4}\right\}\sigma^2$ when both the treatments are 2nd associates.

$$= 2\left\{\frac{\phi_3 + \phi_4}{2S\phi_3\phi_4} + \frac{S-1}{S\phi_2}\right\}\sigma^2 \text{ when both the treatments are 3rd associates.}$$

$$= 2\left\{\frac{l(S-1)\phi_1 + \phi_2}{lS\phi_1\phi_2} + \frac{(l-2)\phi_3 + l\phi_4}{v\phi_3\phi_4}\right\}\sigma^2 \text{ when both the treatments are the 4th associates.}$$

So average variance of all elementary treatment contrasts is given as

$$\text{A.V.} = \frac{2\sigma^2}{v-1}\left\{\frac{2(lS-1)\phi_1 + \phi_2}{\phi_1\phi_2} + \frac{(l-1)(\phi_3+\phi_4)}{\phi_3\phi_4}\right\}.$$

**5. Some methods of constructing right angular designs.** By addition of blocks, Raghavarao[ 9] has given the construction of GD 3-associate designs. Proceedin the same way the following methods have been derived by adding blocks to the plans of the group divisible designs.

5.1. *If a* GD *design exists with parameters*

$$v' = mn, \quad n = 2S, \quad m = l, \quad b' = lr', \quad r', \quad k' = 2S, \quad \lambda_1', \quad \lambda_2', \tag{5.1}$$

*then a right angular design with parameters*

$$v = 2Sl, \quad b = rl, \quad r = r' + 2(l-1), \quad k = k', \quad \lambda_1 = \lambda_1' + 2(l-1), \quad \lambda_2 = \lambda_1', \quad \lambda_3 = \lambda_2' + 2, \quad \lambda_4 = \lambda_2', \tag{5.2}$$

*can be constructed as follows:*

Denote the $k$th treatment in the first $S$ treatments of the $i$th group by $2ik$ $\{i = 1, 2, \cdots, l, k = 1, 2, \cdots, S\}$ and $k$th treatment in the next $S$ treatments of the $i$th group by $1ik$, and write down the plan of the GD design accordingly. To the $b'$ blocks thus formed add $2l$ groups of $(l-1)$ blocks each, where a group of $(l-1)$ blocks is formed by adding third associate arms, one at a time to any chosen arm.

5.2. *If a GD design exists with parameters*

$$v' = mn, \quad n = lS, \quad m = 2, \quad b' = lr', \quad r', \quad k' = 2S, \quad \lambda_1', \quad \lambda_2', \tag{5.3}$$

*then a right angular design with parameters*

$$v = 2Sl, \quad b = lr, \quad r = r' + 1, \quad k = k' \quad \lambda_1 = \lambda_1' + 1, \quad \lambda_2 = \lambda_2' + 1, \quad \lambda_3 = \lambda_1', \quad \lambda_4 = \lambda_2', \tag{5.4}$$

*can be constructed as follows:*

Divide each group into $l$ subgroups. Denote the $k$th treatment in the $i$th subgroup of the 1st group by $2ik$ $\{i = 1, 2, \cdots, l, k = 1, 2, \cdots, S\}$ and $k$th treatment of the $i$th subgroup of the 2nd group by $1ik$, and write down the plan of the GD design, accordingly. To the $b'$ blocks thus formed add $l$ blocks, where the $i$th block contains treatments from the $i$th right angle.

5.3. *If a* GD *design exists with parameters*

$$v' = mn, \quad m = 2l, \quad n = S, \quad b' = lr', \quad r', \quad k' = 2S, \quad \lambda_1', \quad \lambda_2', \tag{5.5}$$

*then a right angular design with parameters*

$$v = 2Sl, \quad b = lr, \quad r = r' + 2(l-1) + 1, \quad k = k', \quad \lambda_1 = \lambda_1' + 2(l-1) + 1, \quad \lambda_2 = \lambda_2' + 1, \quad \lambda_3 = \lambda_2' + 2, \quad \lambda_4 = \lambda_2', \tag{5.6}$$

*can be constructed as follows:*

Form $l$ pairs of groups from the $2l$ groups. Denote the $k$th treatment in the first group of the $i$th pair by $2ik$ $\{i = 1, 2, \cdots, l, k = 1, 2, \cdots, S\}$, and the $k$th treatment in the 2nd group of the $i$th pair by $1ik$ and write down the plan of the GD design accordingly. To the $b'$ blocks thus formed add $2l(l-1)$ blocks following the procedure of 5.1. Further add $l$ blocks where $i$th block contains treatments from the $i$th right angle.

5.4. *If a* GD *design exists with parameters*

$$v' = mn, \quad m = l, \quad n = 2S, \quad b', \quad r', \quad k' = S + 1, \quad \lambda_1', \quad \lambda_2', \tag{5.7}$$

*then a right angular design with parameters*

$$v = 2Sl, \quad b = b' + 2lS(l-1), \quad r = r' + (S+1)(l-1), \quad k = k', \quad \lambda_1 = \lambda_1' + S(l-1), \quad \lambda_2 = \lambda_1', \quad \lambda_3 = \lambda_2' + 2, \quad \lambda_4 = \lambda_2', \tag{5.8}$$

*can be constructed as follows:*

Write down the plan of the GD design following the procedure of 5.1. To the $b'$ blocks thus formed add $2l$ groups of $S(l-1)$ blocks each, where a group of $S(l-1)$ blocks is formed by adding the third associate treatments of any chosen arm, one at a time to it.

5.5. *If a* GD *design exists with parameters*

$$v' = mn, \quad n = lS, \quad m = 2, \quad b', \quad r', \quad k' = S + 1, \quad \lambda_1', \quad \lambda_2', \tag{5.9}$$

*then a right angular design with parameters*

$$v = 2Sl, \quad b = b' + v', \quad r = r' + S + 1, \quad k = k', \quad \lambda_1 = \lambda_1' + S, \quad \lambda_2 = \lambda_2' + 2, \quad \lambda_3 = \lambda_1', \quad \lambda_4 = \lambda_2', \tag{5.10}$$

*can be constructed as follows:*

Write down the plan of the GD design following the procedure of 5.2. To the $b'$ blocks thus formed add $2l$ groups of $S$ blocks each, where a group of $S$ blocks is formed by adding 2nd associate treatments of any chosen arm, one at a time to it.

5.6. *If a* GD *design exists with parameters*

$$v' = mn, \quad m = 2l, \quad n = S, \quad b', \quad r', \quad k' = S + 1, \quad \lambda_1', \quad \lambda_2', \tag{5.11}$$

*then a right angular design with parameters*

$$v = 2Sl, \quad b = b' + 2\cdot v' + v'(l - 1), \quad r = r' + 2(S + 1) + (S + 1)(l - 1), \quad k = k', \quad \lambda_1 = \lambda_1' + 2S + S(l - 1), \quad \lambda_2 = \lambda_2' + 2\cdot 2, \quad \lambda_3 = \lambda_2' + 2, \quad \lambda_4 = \lambda_2', \tag{5.12}$$

*can be constructed as follows*:

Write down the plan of the GD design following the procedure of 5.3. To the $b'$ blocks thus formed add $2lS(l - 1)$ blocks following the procedure of 5.4. Further add 2 groups of $2lS$ blocks each, where blocks of the group are formed by adding 2nd associate treatments of any chosen arm one at a time to it.

**6. Some combinatorial properties of right angular designs.** Considering special cases of the design, we have

THEOREM 6.1. *If in a right angular design,* $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$, *then the design reduces to group divisible design and if* $\lambda_1 \neq \lambda_2$ *and* $\lambda_2 \neq \lambda_3 = \lambda_4$, *the design reduces to* GD 3*-associate design.*

We now prove the following

THEOREM 6.2. *In a right angular design* (i) *if* $\theta_1 = 0$, *then* $k/2$ *is an integer and every block contains* $k/2$ *treatments from parallel arms.* (ii) *If* $\theta_3 = 0$, *then* $k/l$ *is an integer and every block contains* $k/l$ *treatments from each of the* $l$ *right angles.* (iii) *If* $\theta_3 = \theta_4 = 0$, *then* $k/l$ *is an integer and every block contains* $k/l$ *treatments from each paired fourth associate arms and* $\lambda_2$ *equals* $\lambda_4$. (iv) *If* $\theta_1 = \theta_4 = 0$, *then* $k/2$ *is an integer and every block contains* $k/2$ *treatments each from two groups of* $l$ *arms, where each group consists of* $(l - 1)$ *parallel arms and one opposite arm from remaining right angle and the design reduces to* GD 3*-associate design.* (v) *If* $\theta_1 = \theta_3 = \theta_4 = 0$, *then* $k/2l$ *is an integer and every block contains* $k/2l$ *treatments from each of the* $2l$ *arms of the* $l$ *right angles, and the design reduces to group divisible design.*

PROOF. Let $e_j^i$ be the number of treatments occurring in the $j$th block from $i$th parallel arms $(i = 1, 2)$. Then

$$\sum_{j=1}^{b} e_j^i = lSr, \qquad \sum_{j=1}^{b} e_j^i(e_j^i - 1) = lS(S - 1)\lambda_1 + S^2 l(l - 1)\lambda_3. \tag{6.1}$$

Define $e_.^i = b^{-1}\sum_{j=1}^{b} e_j^i = lSr/b = k/2$. Then

$$\sum_{j=1}^{b} (e_j^i - e_.^i)^2 = lS[r + (S - 1)\lambda_1 + S(l - 1)\lambda_3] - bk^2/4 = (lS/2)\{\theta_1\} = 0. \tag{6.2}$$

Hence $e_j^i = e_.^i$ $(j = 1, 2, \cdots, b)$. But $e_j^i$ is an integer hence $k/2$ is an integer and the result is proved.

Applying the same method as above, other parts of the theorem can be proved.

The designs with the following parameters violate the conditions of the above theorem and hence are non-existing.

| $v$ | $b$ | $r$ | $k$ | $l$ | $S$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 12 | 8 | 6 | 9 | 2 | 3 | 6 | 5 | 3 | 4 |
| 24 | 16 | 6 | 9 | 4 | 3 | 6 | 0 | 1 | 3 |
| 56 | 32 | 12 | 21 | 4 | 7 | 5 | 3 | 4 | 5 |
| 12 | 9 | 6 | 8 | 3 | 2 | 6 | 2 | 5 | 3 |
| 12 | 16 | 4 | 3 | 2 | 3 | 1 | 0 | 0 | 2 |
| 20 | 32 | 8 | 5 | 2 | 5 | 3 | 0 | 0 | 4 |
| 20 | 32 | 24 | 15 | 2 | 5 | 24 | 12 | 20 | 16 |
| 24 | 18 | 6 | 8 | 3 | 4 | 2 | 1 | 3 | 1 |

Another necessary condition for the existence of a right angular design with $\theta_3 = 0$ can be obtained by the application of a theorem proved in [14]. For reference we reproduce the theorem below:

THEOREM 6.3. *Let a* PBIB *design with* $S + 1$ *associate classes and* $b = v - \alpha$ *have distinct positive rational roots* $\theta_0 = rk, \theta_1, \theta_2, \cdots, \theta_S$ *with multiplicities* $\alpha_0 = 1, \alpha_1, \alpha_2, \cdots, \alpha_S$ *and zero a root with multiplicity* $\alpha$. *Then necessary conditions for existence of the design are*

$$\left\{\prod_{i=0}^{S} \theta_i^{\alpha_i}\right\} |Q| \sim 1 \tag{6.3}$$

*and further if* (6.3) *is satisfied then,*

$$\begin{aligned}(\theta_0, -v)_p \left(v, \prod_{i=1}^{S} \theta_i^{\alpha_i}\right)_p \left\{\prod_{i=1}^{S} (-1, \theta_i)^{\frac{1}{2}\alpha_i(\alpha_i+3)}\right\} \\ \cdot \left\{\prod_{i<j=1}^{S} (\theta_i^{\alpha_i}, \theta_j^{\alpha_j})_p\right\}\left\{\prod_{i<j=1}^{S} (\theta_i^{\alpha_i}, |Q_j|)_p\right\} \\ \cdot \left\{\prod_{i<j=1}^{S} (\theta_j^{\alpha_j}, |Q_i|)_p\right\}\left\{\prod_{i=1}^{S} (\theta_i, |Q_i|)_p^{\alpha_i - 1}\right\} C_p(Q) = +1,\end{aligned} \tag{6.4}$$

*where* $Q, Q_1, Q_2, \cdots, Q_S$ *denote the Gramians corresponding to zero root and* $\theta_1, \theta_2, \cdots, \theta_S$ *the distinct positive rational roots, and* $C_p(Q)$ *is the Hasse-Minkowski invariant of* $Q$.

In right angular design the characteristic root $\theta_3 = 0$ of $NN'$ has multiplicity $l - 1$. A set of orthogonal characteristic vectors corresponding to the zero root of $NN'$ are the column vectors

$$(6.5)\qquad \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -1 & 1 & & 1 \\ 0 & -2 & & \vdots \\ \vdots & 0 & & \\ & \vdots & & \\ 0 & & & 1 \\ \vdots & & & \\ 0 & 0 & & -(l-1) \end{bmatrix} \times E_{2S,1},$$

where $E_{2S,1}$ is an $2S$th order column vector with positive unit elements everywhere. It can be easily seen that for this set of characteristic vectors

$$(6.6)\qquad Q = (2S)^{l-1} \prod_{j=1}^{l-1} j(j+1).$$

With the help of (6.5) and [1] the value of $C_p(Q)$ can be evaluated as

$$(6.7)\qquad C_p(Q) = (-1, -1)_p (2S, l)_p^{l-2} (2S, -1)_p^{\frac{1}{2}l(l-1)}.$$

By application of the above theorem and necessary simplifications we have the following.

THEOREM 6.4. *Necessary conditions for the existence of a right angular design with* $\theta_3 = 0$ *and* $b = l(2S-1)+1$ *are*

(i) *If* $l$ *is even then* $\theta_0\theta_1\theta_4 v$ *should be a perfect square and further*

$$(\theta_4, -\theta_1)(\theta_0, -v) = +1 \quad \text{if} \quad l \equiv 0 (\text{mod } 4),$$

$$(\theta_0, -v)_p(\theta_1, \theta_4)_p(-1, 2S)_p = +1 \quad \text{if} \quad l \equiv 2 (\text{mod } 4).$$

(ii) *If* $l$ *is odd then* $\theta_0\theta_1 l$ *should be a perfect square and further*

$$(l, \theta_1\theta_4)_p(-1, l\theta_0\theta_2^{S-1})_p = +1 \quad \text{if} \quad l \equiv 1 (\text{mod } 4),$$

$$(l, \theta_1\theta_4)_p(-1, \theta_0\theta_2^{S-1}\cdot\theta_4\cdot 2S)_p = +1 \quad \text{if} \quad l \equiv 3 (\text{mod } 4).$$

The designs with the following parameters violate the above necessary conditions and hence are impossible.

| $v$ | $b$ | $r$ | $k$ | $l$ | $S$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 16 | 8 | 10 | 5 | 2 | 6 | 1 | 5 | 3 |
| 24 | 22 | 11 | 12 | 3 | 4 | 7 | 3 | 6 | 5 |
| 28 | 22 | 11 | 14 | 7 | 2 | 9 | 1 | 6 | 5 |
| 30 | 26 | 13 | 15 | 5 | 3 | 10 | 2 | 7 | 6 |
| 48 | 45 | 15 | 16 | 4 | 6 | 9 | 0 | 6 | 4 |

**7. Necessary conditions for the existence of a class of symmetrical Right Angular Designs.** In this section, we consider symmetrical Right Angular Designs where $\theta_i$'s ($i = 1, 2, 3, 4$) are positive. Since the design is symmetrical one, det $(NN')$ is a perfect square (cf. Connor and Clatworthy and Shrikhande [13]). Thus, we have the following theorem.

THEOREM 7.1. *A necessary condition for the existence of a symmetrical Right Angular design whose* $\theta_i$'s $(i = 1, 2, 3, 4)$ *are positive is that*

$$\theta_1 \theta_3^{l-1} \cdot \theta_4^{l-1} \tag{7.1}$$

*is a perfect square.*

The following corollary is obvious.

COROLLARY 7.1.1. *The above class of designs have* $\theta_1$ *as a perfect square if* $l$ *is odd and they have* $\theta_1\theta_3\theta_4$ *as a perfect square if* $l$ *is even.*

The designs with the following parameters violate the above necessary condition and hence are impossible.

| $v = b$ | $r = k$ | $l$ | $S$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|---|---|---|---|
| 12 | 7 | 3 | 2 | 6 | 2 | 5 | 3 |
| 18 | 9 | 3 | 3 | 6 | 4 | 5 | 3 |
| 24 | 8 | 4 | 3 | 7 | 2 | 3 | 1 |
| 30 | 10 | 5 | 3 | 9 | 4 | 3 | 2 |
| 36 | 9 | 6 | 3 | 6 | 0 | 3 | 1 |

We now apply the Hasse-Minkowski invariant for the above class of designs and obtain a necessary condition for the existence of them.

A good resume of the Legendre symbol, Hilbert norm residue symbol and the Hasse-Minkowski invariant can be had from [5], [7], [13].

$C_p(NN')$ can be calculated in the usual way [6] and after simplification, we get

$$\begin{aligned} C_p(NN') &= (-1, \theta_1)_p(-1, \theta_3\theta_4)_p^{\frac{1}{2}l(l-1)}(-1, \theta_2)_p^{l(S-1)} \\ &\quad \cdot (\theta_3\theta_4, l(2S)^{l-1})_p^{l-2} \cdot (\theta_3^{l-1}, \theta_4^{l-1} \cdot l(2S)^{l-1})_p \cdot (l(2S)^{l-1}, \theta_4^{l-1}), \end{aligned} \tag{7.2}$$

for all odd primes $p$, where $C_p(NN')$ is the Hasse-Minkowski invariant of $NN'$. We know that $C_p(NN')$ should be equal to $+1$, for all odd primes and for all existing designs [13].

Thus, we have

THEOREM 7.2. *A necessary condition for the existence of symmetrical right angular designs whose* $\theta_i$'s $(i = 1, 2, 3, 4)$ *are positive is that the right hand side of* (7.2) *is equal to* $+1$, *for all odd primes.*

COROLLARY 7.2.1. *A necessary condition for the existence of symmetrical right angular designs whose* $\theta_i$'s $(i = 1, 2, 3, 4)$ *are positive is that*

$$\begin{aligned} (-1, \theta_1)_p(\theta_3, \theta_4 v)_p(v, \theta_4) &= +1, &\text{if}\quad l &\equiv 0 \pmod 4, \\ (-1, \theta_2^{S-1})_p(\theta_3\theta_4, l)_p &= +1, &\text{if}\quad l &\equiv 1 \pmod 4, \\ (v, \theta_4)_p(\theta_3, v\theta_4)_p &= +1, &\text{if}\quad l &\equiv 2 \pmod 4, \\ (-1, \theta_2^{S-1} \cdot \theta_3\theta_4)_p(\theta_3\theta_4, l)_p &= +1, &\text{if}\quad l &\equiv 3 \pmod 4. \end{aligned}$$

The designs with the following parameters do not satisfy the corollary and hence are nonexistent.

| $v = b$ | $r = k$ | $l$ | $S$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|---|---|---|---|
| 20 | 10 | 5 | 2 | 8 | 5 | 5 | 4 |
| 24 | 10 | 6 | 2 | 6 | 2 | 5 | 3 |
| 28 | 13 | 7 | 2 | 12 | 6 | 6 | 5 |
| 32 | 10 | 8 | 2 | 8 | 6 | 3 | 2 |

**8. Remarks and acknowledgement.** In the definition of Right Angular Association Scheme, we have taken arms at right angles to each other. However, the right angles are not really essential, as it can be seen that equal angles of any size would do just as well. Keeping this in mind, generalisation of above association scheme can be defined. The same will be dealt with in the succeeding paper.

The author is indebted to his guide Professor M. C. Chakrabarti, to Dr. D. Raghavarao for suggesting the problem and to referee for his stimulating comments.

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