ON THE INTEGRABILITY, CONTINUITY AND DIFFERENTIABILITY OF A FAMILY OF FUNCTIONS INTRODUCED BY L. TAKÁCS

By A. M. Hasofer

University of Tasmania

1. Introduction. L. Takács has shown [5] that a single-server queue with non-homogeneous Poisson input of density $\lambda(t)$, where $\lambda(t)$ is a bounded and continuous function of t, and service time χ whose distribution function is $P\{\chi \leqq x\} = H(x)$ can be described by a Markov stochastic process $\eta(t)$ with continuous parameter and continuous state space, representing the total length of time required to serve the customers queueing at time t, but not including the customer who arrives at time t, if any.

Moreover, Takács has stated that the transition probability

$$W(t, x) = P\{\eta(t) \leqq x \mid \eta(0) = 0\}$$

is continuous in x for $x > 0$ and all t, and that it has a right-hand derivative in x for $x \geqq 0$ and a left-hand derivative for $x > 0$. Takács has used $F(t, x)$ where I have used $W(t, x)$, but I shall define a slightly more general version of $F(t, x)$ later.

On page 108 of [5], Takács has presented an argument for the continuity of $W(t, x)$. However, I believe the argument to be incomplete on two counts:

(a) Formula (11) (see [5], p. 108), contains an integral in t whose integrand depends on $W(t, x)$, and no argument has been produced to justify the integrability of $W(t, x)$ with respect to t. A rigorous proof of the integrability of $W(t, x)$ in t is given in this paper. (See Lemma 2.)

(b) Assuming Takács' formula (11) to be true, it does not seem to follow that $W(t, x)$ is continuous in either t or x.

Further, as pointed out by E. Reich (see [4] p. 143, Footnote 2), the existence of the right-hand and the left-hand derivatives of $W(t, x)$ with respect to x has been assumed by Takács without justification.

The purpose of this paper is to establish conditions on the service-time distribution for the continuity and differentiability of $W(t, x)$. These conditions will be established for the more general absolute probability distribution

$$F(t,x) = \int_{0-}^{\infty} f(t; y, x)\, d_x F(0, x) \tag{1}$$

where $f(t; y, x) = P\{\eta(t) \leqq x \mid \eta(0) = y\}$ and $F(0, x)$ is the probability distribution of $\eta(0)$. It will also be shown that $F(t, x)$ satisfies the integro-differential equation

$$\frac{\partial}{\partial x} F(t, x) = \frac{\partial}{\partial t} F(t, x) + \lambda(t) \int_{0-}^{x+} [F(t, x - y) - F(t, x)]\, dH(y)$$

when it is continuous and differentiable.

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It is to be noted that the conditions given do not cover the important case of constant service time. On the other hand, it has been shown directly by Takács ([6], p. 53) under no restrictions on the service time that the Laplace transform of $F(t, x)$: $\phi(t, s) = \int_{0-}^{+\infty} e^{-sx}\, d_x F(t, x)$ satisfies the differential equation:

$$\frac{\partial}{\partial t}\phi(t, s) = [s - \lambda(t)\{1 - \psi(s)\}] - sF(t, 0)$$

where $\psi(s) = \int_{0-}^{+\infty} e^{-sx}\, dH(x)$.

Since it is proved in Lemma 2 that $F(t, 0)$ is integrable in t, is follows that the differential equation has the unique solution:

$$\phi(t, s) = e^{st-[1-\psi(s)]\Lambda(t)}\left\{\phi(0, s) - s\int_0^t e^{-su+[1-\psi(s)]\Lambda(u)}F(u, 0)\, du\right\}$$

where $\Lambda(t) = \int_0^t \lambda(u)\, du$, without any restriction on the service time.

Since most of the results on the queue with constant service time have been obtained, not from the integro-differential equation, but from the Laplace transform formula, the above argument provides a justification of these results.

2. Basic formulae and properties. Takács has shown ([5] p. 106) that $F(t, x)$ satisfies the following difference equation:

$$F(t + h, x) - F(t, x + h) = \lambda(t)h[G(t, x) - F(t, x)] + o(h) \tag{2}$$

where

$$G(t, x) = \int_{0-}^{x+} H(x - y)\, d_y F(t, y) = \int_{0-}^{x+} F(t, x - y)\, dH(y),$$

for $t \geqq 0, x \geqq 0, h > 0$.

Writing now $t - h$ for t and $x - h$ for x, we obtain

$$\begin{aligned} &F(t - h, x) - F(t, x - h) \\ &\qquad = -\lambda(t - h)h[G(t - h, x - h) - F(t - h, x - h)] + o(h). \end{aligned} \tag{3}$$

This formula is valid for $h > 0$, $x \geqq h$, $t \geqq h$.

LEMMA 1.

$$F(t + 0, x) = F(t, x) \tag{4}$$

$$F(t - 0, x) = F(t, x - 0). \tag{5}$$

PROOF. Let h tend to zero in formulae (2) and (3) and use the right continuity of $F(t, x)$ in x.

LEMMA 2. *$F(t, x)$ is Riemann-integrable in t.*

PROOF. It follows from Lemma 1 that all the discontinuities of $F(t, x)$ as a function of t are ordinary and this implies Riemann-integrability (c.f. [1] p. 439).

LEMMA 3. *$F(t - u, x + u)$ is a continuous function of u for $x \geqq 0$, $t \geqq 0$, $-x \leqq u \leqq t$.*

PROOF. In formula (2) replace t by $t - u$ and x by $x + u - h$. We obtain:

$$F(t - u + h, x + u - h) - F(t - u, x + u) = \lambda(t - u)h[G(t - u, x + u - h) - F(t - u, x + u - h)] + o(h)$$

for $-x + h \leqq u \leqq t$.

Similarly, writing $t - u - h$ for t and $x + u$ for x in (2), we obtain

$$F(t - u - h, x + u + h) - F(t - u, x + u) = -\lambda(t - u)h[G(t - u - h, x + u) - F(t - u - h, x + u)] + o(h)$$

for $-x \leqq u \leqq t - h$.

The result follows by letting h tend to zero.

LEMMA 4. *$G(t - u, x + u)$ is a continuous function of u.*

PROOF. This follows from the continuity of $F(t - u, x + u)$ in u by letting h tend to zero in the equation

$$(6)\quad G(t - u - h, x + u + h) = \int_{-\infty}^{+\infty} F(t - u - h, x + u + h - y)\, dH(y)$$

and using Lebesgue's dominated convergence theorem, (c.f. [2] p. 125).

LEMMA 5. *The transition probability $f(t; y, x)$ satisfies the equation*

$$(7)\quad f(t; y, x) = e^{-\Lambda(t)}\left[U(x - y - t) + \int_0^t \lambda(t - u)e^{\Lambda(t-u)}g(t - u; y, x + u)\, du\right]$$

where $\Lambda(t) = \int_0^t \lambda(u)\, du$, $g(t; y, x) = \int_{0-}^{x+} f(t; y, x - z)\, dH(z)$, and $U(x)$ *is the Heaviside unit function,* i.e., $U(x) = 1$ if $x \geqq 0$, *and* $U(x) = 0$ *if* $x < 0$.

PROOF. We first note that the function $f(t; y, x)$ is a special form of the function $F(t, x)$, obtained when $F(0, x) = U(x - y)$, and that $g(t; y, x)$ is also a special form of $G(t, x)$. If follows that Lemma 4 applies and that $g(t - u; y, x + u)$ is an integrable function of u for $-x \leqq u \leqq t$.

We now note that given that $\eta(0) = y$, the event $\{\eta(t) \leqq x\}$ can occur in two exhaustive and mutually exclusive ways: (a) There is no arrival in $(0, t)$ and $y - t \leqq x$. The probability of this event is $e^{-\Lambda(t)}U(x - y + t)$. (b) The last arrival occurs at $t - u$ and $\eta(t - u) + \chi - t \leqq x$. The probability of this event is

$$\int_0^t g(t - u; y, x + u)\lambda(t - u)e^{-\Lambda(t)+\Lambda(t-u)}\, du.$$

Adding these two probabilities, we obtain the result.

LEMMA 6. *$F(t, x)$ satisfies the equation*

$$(8)\quad F(t, x) = e^{-\Lambda(t)}\left[F(0, x + t) + \int_0^t \lambda(t - u)e^{\Lambda(t-u)}G(t - u, x + u)\, du\right].$$

PROOF. This follows easily from the relations

$$F(t, x) = \int_{0-}^{+\infty} f(t; y, x)\, d_y F(0, y)$$

$$G(t, x) = \int_{0-}^{+\infty} g(t; y, x)\, d_y F(0, y)$$

$$F(0, x + t) = \int_{0-}^{+\infty} U(x - y + t)\, d_y F(0, y).$$

The required interchange of integrals is easily justified by using Fubini's theorem as all integrands are positive and bounded.

COROLLARY. $F(t, 0) \geqq e^{-\Lambda(t)} F(0, t)$.

This shows that $F(t, x)$ has in general a discontinuity at the origin.

3. Continuity of $F(t, x)$.

THEOREM 1. *If $H(x)$ is absolutely continuous and $F(0, x)$ is continuous for all $x > y$, then $F(t, x)$ is a continuous function of both t and x for all $x \geqq 0$, $t \geqq 0$, $x + t > y$.*

PROOF. We first notice that if $H(x)$ is absolutely continuous, $G(t, x)$ is continuous in x for all t and x.

This follows from standard properties of the convolution operation (c.f. [3] p. 45, Th. 3.3.2).

Let now $I(t, x) = \int_0^t G(t - u, x + u)\lambda(t - u)e^{\Lambda(t-u)}\, du$. Then

$$F(t, x) = e^{-\Lambda(t)}[F(0, x + t) + I(t, x)] \tag{9}$$

and $I(t, x + h) = \int_0^t G(t - u, x + u + h)\lambda(t - u)e^{\Lambda(t-u)}\, du$. The integrand obviously satisfies the conditions of Lebesgue's dominated convergence theorem, and we conclude $\lim_{h\to 0} I(t, x + h) = I(t, x)$. $I(t, x)$ is therefore continuous in x. The continuity of $F(t, x)$ in x follows from (9) and its continuity in t from Lemma 1.

4. Differentiability of $F(t, x)$.

THEOREM 2. *If $H(x)$ has a bounded derivative for all x and if $F(0, x)$ has a bounded derivative for $x > y$, then $F(t, x)$ has a bounded derivative in both x and t for $x > 0$, $t > 0$, $x + t > y$, and satisfies the equation*

$$(\partial/\partial t)F(t, x) = (\partial/\partial x)F(t, x) + \lambda(t)[G(t, x) - F(t, x)].$$

PROOF. Applying Lebesgue's dominated convergence theorem successively to

$$[G(t, x + h) - G(t, x)]/h = \int_{-\infty}^{+\infty} \{[H(x + h - y) - H(x - y)]/h\}\, d_y F(t, y)$$

and

$$[I(t, x + h) - I(t, x)]/h$$
$$= \int_0^t \{[G(t - u, x + u + h) - G(t - u, x + u)]/h\}\lambda(t - u)e^{\Lambda(t-u)}\, du,$$

we conclude that first $G(t, x)$ and then $I(t, x)$ have bounded derivatives in x. The existence of $\partial F/\partial x$ follows from (9).

From formulae (1) and (3) we now deduce, for $h > 0$

$$[F(t + h, x) - F(t, x)]/h = [F(t, x + h) - F(t, x)]/h + \lambda(t)[G(t, x) - F(t, x)] + O(h),$$

$$[F(t - h, x) - F(t, x)]/(-h) = [F(t, x - h) - F(t, x)]/(-h) + \lambda(t - h)[G(t - h, x - h) - F(t - h, x - h)] + O(h).$$

Letting $h \downarrow 0$, we find

$$\lim_{h\downarrow 0} [F(t + h, x) - F(t, x)]/h = \lim_{h\downarrow 0} [F(t - h, x) - F(t, x)]/(-h) = (\partial/\partial x)F(t, x) + \lambda(t)[G(t, x) - F(t, x)],$$

the continuity of $G(t, x)$ in t following from the application of Lebesgue's dominated convergence theorem to $G(t + h, x) = \int_{-\infty}^{+\infty} F(t + h, x - y)\, dH(y)$. This completes the proof of the theorem.

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