# ESTIMATION OF THE CROSS-SPECTRUM[1]

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**0. Summary.** Assuming that the wide sense stationary process being sampled has an absolutely continuous spectrum, Parzen [5] has shown the consistency of a general class of estimators for estimating the spectral density at a given frequency. He has shown that the class of estimators he considers, contains as a particular case estimators considered earlier by Grenander and Rosenblatt [2], Barlett [1], Tukey [7], [8], Daniell, and Lomnicki and Zaremba [3].

In this paper we extend Parzen's results to the case of a Stationary Gaussian Vector process whose spectrum is not necessarily absolutely continuous and show that this general class of estimators consistently estimate the co- and quadrature spectral densities at all those frequencies where they exist.

It has been earlier shown by the author [4], using an entirely different approach from the one presented in this paper, that for a normal Stationary process whose spectrum besides the absolutely continuous part contains a step function with a finite number of saltuses, the weighted periodogram estimator, which is a particular case of the general class of estimators considered by Parzen [5], is still a consistent estimate of the spectral density at any point of continuity of the spectrum. Thus this paper also substantially generalizes the earlier result of the author, where he limits himself to a finite number of saltuses.

**1. Introduction and preliminaries.** In what follows we treat the continuous parameter case. A parallel treatment for the discrete parameter case is evident. Parts of these preliminaries are contained in Rosenblatt [6] where he assumes an absolutely continuous spectrum.

Let

$$X'(t) = (x_1(t), x_2(t), \cdots x_p(t)), \qquad -\infty < t < \infty \tag{1.1}$$

be a $p$-dimensional wide sense Stationary process, where the mean value, without loss of generality, is assumed to be identically zero. The process $X(t)$ has the Fourier representation

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dZ(\lambda) \tag{1.2}$$

where $Z'(\lambda) = (z_1(\lambda), z_2(\lambda), \cdots z_p(\lambda))$ is an orthogonal process. That is to say

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$$E\, dZ(\lambda) \equiv 0,$$

$$E\, dZ(\lambda)\, \overline{dZ(\mu)}' = \delta_{\lambda,\mu} \begin{pmatrix} dF_{11}(\lambda) & dF_{12}(\lambda) & \cdots & dF_{1p}(\lambda) \\ dF_{21}(\lambda) & dF_{22}(\lambda) & \cdots & dF_{2p}(\lambda) \\ & \cdots & & \\ dF_{p1}(\lambda) & dF_{p2}(\lambda) & \cdots & dF_{pp}(\lambda) \end{pmatrix} \tag{1.3}$$

and $dZ(\lambda) = \overline{dZ(-\lambda)}$, where the bar above denotes complex conjugate, prime denotes transpose and $\delta_{\lambda\mu}$ stands for the Kronecker $\delta$.

Let

$$R_{\tau+t,\tau} = EX(\tau + t)X'(\tau) = (R(t)) = \begin{pmatrix} R_{11}(t) & R_{12}(t) & \cdots & R_{1p}(t) \\ R_{21}(t) & R_{22}(t) & \cdots & R_{2p}(t) \\ & \cdots & & \\ R_{p1}(t) & R_{p2}(t) & \cdots & R_{pp}(t) \end{pmatrix} \tag{1.4}$$

in view of stationarity of the process $X(t)$, where

$$\begin{aligned} R_{l,m}(t) &= Ex_l(\tau + t)x_m(\tau) = Ex_l(\tau)x_m(\tau - t) \\ &= Ex_m(\tau - t)x_l(\tau) = R_{m,l}(-t). \end{aligned} \tag{1.5}$$

Now (1.2) implies that

$$R_{l,m}(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dF_{l,m}(\lambda), \qquad l, m = 1, 2, \cdots p. \tag{1.6}$$

$$= \int_{-\infty}^{\infty} e^{it\lambda} f_{l,m}(\lambda)\, d\lambda, \tag{1.7}$$

in the absolutely continuous case where $dF_{l,m}(\lambda) = f_{l,m}(\lambda)\, d\lambda$.

In the absolutely continuous case one obtains the following representation for $f_{l,m}(\lambda)$,

$$\begin{aligned} f_{l,m}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{l,m}(t) e^{-it\lambda}\, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{R_{l,m}(t)} e^{it\lambda}\, dt = \overline{f_{l,m}(-\lambda)} = f_{m,l}(-\lambda), \end{aligned} \tag{1.8}$$

since the process is real.

Let

$$f_{l,m}(\lambda) = c_{l,m} + iq_{l,m}(\lambda), \qquad l, m = 1, 2, \cdots, p, \tag{1.9}$$

where $q_{l,l} \equiv 0$, $l, = 1, 2, \cdots p$. $(c_{l,m}(\lambda))$ is called the matrix of real co-spectra, and $(q_{l,m}(\lambda))$ is called the matrix of real quadrature-spectra.

Since $f_{l,m}(\lambda) = \overline{f_{l,m}(-\lambda)} = f_{m,l}(-\lambda)$, it follows that the matrix of co-spectra is a symmetric $p \times p$ matrix and has $p(p + 1)/2$ distinct elements, and the matrix of quadrature spectra is a skew-symmetric $p \times p$ matrix and has $p(p - 1)/2$ distinct elements. They together account for the totality of $p^2$ parameters in the cross spectral matrix $(f_{l,m}(\lambda))$, $l, m = 1, 2, \cdots p$.

From (1.8) and (1.9) one obtains

$$\text{(1.10)}\qquad \begin{aligned} c_{l,m}(\lambda) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-it\lambda}(R_{l,m}(t) + R_{m,l}(t))\,dt, \\ &= \frac{1}{2\pi}\int_{0}^{\infty} \cos t\lambda(R_{l,m}(t) + R_{m,l}(t))\,dt, \qquad l, m = 1, 2, \cdots p, \end{aligned}$$

and

$$\text{(1.11)}\qquad \begin{aligned} q_{l,m}(\lambda) &= \frac{1}{4\pi i}\int_{-\infty}^{\infty} e^{-it\lambda}(R_{l,m}(t) - R_{m,l}(t))\,dt, \\ &= \frac{1}{2\pi}\int_{0}^{\infty} \sin t\lambda(R_{l,m}(t) - R_{m,l}(t))\,dt, \qquad l, m = 1, 2, \cdots p. \end{aligned}$$

(1.10) and (1.11) are the spectral representations for the real co- and quadrature spectra.

Let

$$\text{(1.12)}\qquad z_l(\lambda) = z_l^R(\lambda) + iz_l^I(\lambda), \qquad l = 1, 2, \cdots p,$$

where $z_l^R(\lambda)$ and $z_l^I(\lambda)$ are real processes being the real and imaginary parts of the orthogonal process $z_l(\lambda)$. One easily obtains from (1.3) and (1.12) that

$$\text{(1.13)}\qquad \begin{aligned} &E\,dz_l^R(\lambda)\,dz_l^I(\mu) = 0, \quad \text{for all } \lambda, \mu, && l = 1, 2, \cdots p. \\ &E\,dz_l^R(\lambda)\,dz_l^R(\mu) = E\,dz_l^I(\lambda)\,dz_l^I(\mu) = \tfrac{1}{2}\delta_{\lambda,\mu}\,dF_{ll}(\lambda), && l = 1, 2, \cdots p. \\ &E\,dz_l^R(\lambda)\,dz_m^R(\mu) = E\,dz_l^I(\lambda)\,dz_m^I(\mu) = \tfrac{1}{2}\delta_{\lambda,\mu}\,d\mathfrak{C}_{l,m}(\lambda), && l, m = 1, 2, \cdots p. \\ &E\,dz_l^R(\lambda)\,dz_m^I(\mu) = -(\delta_{\lambda,\mu}/2)\,d\varphi_{l,m}(\lambda), \\ &E\,dz_l^I(\lambda)\,dz_m^R(\mu) = (\delta_{\lambda,\mu}/2)\,d\varphi_{l,m}(\lambda) \end{aligned}$$

where $\mathfrak{C}_{l,m}(\lambda)$ is the co-spectrum and $\varphi_{l,m}(\lambda)$ is the quadrature spectrum.

**2. The complex cross-periodogram.** Let $X'(t) = (x_1(t), x_2(t), \cdots x_p(t))$, $0 < t < T$, be a realization (sample) of size $T$ from the real, stationary, Gaussian $p$-dimensional process under consideration.

Let

$$\text{(2.1)}\qquad G_{l,m}^T(t) = R_{l,m}^T(t) + R_{m,l}^T(t), \qquad H_{l,m}^T(t) = R_{l,m}^T(t) - R_{m,l}^T(t),$$

where

$$\text{(2.2)}\qquad \begin{aligned} R_{l,m}^T(t) &= (1/T)\int_0^{T-t} x_l(j+t)x_m(j)\,dj, \qquad 0 < t < T, \\ R_{m,l}^T(t) &= (1/T)\int_0^{T-t} x_m(j+t)x_l(j)\,dj, \qquad 0 < t < T. \end{aligned}$$

Let us define the complex cross-periodogram by

$$f_{l,m}^T(\lambda) = \frac{1}{2\pi T} \int_0^T e^{-it\lambda} x_l(t)\, dt \overline{\int_0^T e^{-it\lambda} x_m(t)\, dt}, \qquad l, m = 1, 2, \cdots p. \tag{2.3}$$

Thus $f_{l,m}^T(\lambda) = (1/2\pi)\int_{-T}^{T} e^{-i\lambda t} R_{l,m}^T(t)\, dt$, after a straight-forward simplification. In a similar way, we obtain $f_{m,l}^T(\lambda) = (1/2\pi)\int_{-T}^{T} R_{m,l}^T(t) e^{-i\lambda t}\, dt$.

It is easily seen that

$$f_{l,m}^T(\lambda) = c_{l,m}^T(\lambda) + i q_{l,m}^T(\lambda), \tag{2.4}$$

where

$$c_{l,m}^T(\lambda) = (1/4\pi) \int_{-T}^{T} e^{-it\lambda} G_{l,m}^T(t)\, dt \tag{2.5}$$

and

$$q_{l,m}^T(\lambda) = (1/4\pi i) \int_{-T}^{T} e^{-it\lambda} H_{l,m}^T(t)\, dt. \tag{2.6}$$

$c_{l,m}^T(\lambda)$ being the real part of the complex cross-periodogram, it is called the co-periodogram, and $q_{l,m}^T(\lambda)$, the imaginary part, is called the quadrature periodogram. Now $x_l(t) = \int_{-\infty}^{\infty} e^{it\mu}\, dz_l(\mu)$; therefore

$$\int_0^T e^{-it\lambda} x_l(t)\, dt = \int_0^T \int_{-\infty}^{\infty} e^{it(\mu-\lambda)}\, dz_l(\mu)\, dt = \frac{1}{i} \int_{-\infty}^{\infty} \frac{e^{iT(\mu-\lambda)} - 1}{\mu - \lambda}\, dz_l(\mu).$$

Hence

$$\begin{aligned} f_{l,m}^T(\lambda) &= \frac{1}{2\pi T} \int_0^T e^{-it\lambda} x_l(t)\, dt \overline{\int_0^T e^{-it\lambda} x_m(t)\, dt} = \frac{1}{2\pi T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \\ &\cdot \frac{\{\exp[iT(\mu_1 - \lambda)] - 1\}\{\exp[-iT(\mu_2 - \lambda)] - 1\}}{(\mu_1 - \lambda)(\mu_2 - \lambda)}\, dz_l(\mu)\, \overline{dz_m(\mu_2)} \end{aligned} \tag{2.7}$$

Let

$$dz_l(\mu_1) = dz_l^R(\mu_1) + i\, dz_l^I(\mu_1), \qquad dz_m(\mu_2) = dz_m^R(\mu_2) + i\, dz_m^I(\mu_2). \tag{2.8}$$

Let

$$\begin{aligned} &g(\mu_1, \mu_2, \lambda) \\ &= \{\exp[iT(\mu_1 - \lambda)] - 1\}\{\exp[-iT(\mu_2 - \lambda)] - 1\}/(\mu_1 - \lambda)(\mu_2 - \lambda) \\ &= g_R(\mu_1, \mu_2, \lambda) + i g_I(\mu_1, \mu_2, \lambda), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} g_R(\mu_1, \mu_2, \lambda) &= \{[\cos T(\mu_1 - \lambda) - 1][\cos T(\mu_2 - \lambda) - 1] \\ &\qquad + \sin T(\mu_1 - \lambda) \sin T(\mu_2 - \lambda)\}/(\mu_1 - \lambda)(\mu_2 - \lambda), \\ g_I(\mu_1, \mu_2, \lambda) &= \{\sin T(\mu_1 - \lambda)[\cos T(\mu_2 - \lambda) - 1] \\ &\qquad - \sin T(\mu_2 - \lambda)[\cos T(\mu_1 - \lambda) - 1]\}/(\mu_1 - \lambda)(\mu_2 - \lambda). \end{aligned} \tag{2.10}$$

Taking the real and imaginary parts of $f_{l,m}^{T}(\lambda)$ given by (2.7), we obtain

$$c_{l,m}^{T}(\lambda) = \frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{g_R(\mu_1, \mu_2, \lambda)[dz_l^R(\mu_1)\, dz_m^R(\mu_2) + dz_l^I(\mu_1)\, dz_m^I(\mu_2)] + g_I(\mu_1, \mu_2, \lambda)[dz_l^R(\mu_1)\, dz_m^I(\mu_2) - dz_l^I(\mu_1)\, dz_m^R(\mu_2)]\}, \tag{2.11}$$

and

$$q_{l,m}^{T}(\lambda) = \frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{g_R(\mu_1, \mu_2, \lambda)[dz_l^I(\mu_1)\, dz_m^R(\mu_2) - dz_l^R(\mu_1)\, dz_m^I(\mu_2)] + g_I(\mu_1, \mu_2, \lambda)[dz_l^R(\mu_1)\, dz_m^R(\mu_2) + dz_l^I(\mu_1)\, dz_m^I(\mu_2)]\}, \tag{2.12}$$

as the representations for the co- and quadrature periodograms respectively.

**3. A class of estimators and their variances.** Let $\lambda_0$ be a point of continuity of the cross spectrum $F_{l,m}(\lambda)$ of the $l$th and $m$th components of the process $X(t)$. Let $c_{l,m}(\lambda_0)$ and $q_{l,m}(\lambda_0)$ be the co- and quadrature spectral densities at $\lambda_0$. The general class of estimates considered by Parzen [5] for estimating the spectral density at a given frequency $\lambda_0$ as applied to our case yields the following estimates for the co- and quadrature-spectral densities.

$$c_{l,m}^{T*}(\lambda_0) = \int_{-\infty}^{\infty} K_T(\lambda - \lambda_0) c_{l,m}^{T}(\lambda)\, d\lambda, \tag{3.1}$$

and

$$q_{l,m}^{T*}(\lambda_0) = \int_{-\infty}^{\infty} K_T^1(\lambda - \lambda_0) q_{l,m}^{T}(\lambda)\, d\lambda, \qquad l, m = 1, 2, \cdots p, \tag{3.2}$$

where $K_T(\lambda)$ and $K_T^1(\lambda)$ are general spectral windows in the sense of Parzen [5]. We will firstly obtain the variance of the estimator $c_{l,m}^{T*}(\lambda_0)$ given by (3.1). Now

$$\begin{aligned} c_{l,m}^{T*}(\lambda_0) &= \frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[dz_l^R(\mu_1)\, dz_m^R(\mu_2) \\ &\qquad + dz_l^I(\mu_1)\, dz_m^I(\mu_2)]\int_{-\infty}^{\infty} K_T(\lambda - \lambda_0) g_R(\mu_1, \mu_2, \lambda)\, d\lambda \\ &\quad + \frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[dz_l^R(\mu_1)\, dz_m^I(\mu_2) \\ &\qquad - dz_l^I(\mu_1)\, dz_m^R(\mu_2)]\int_{-\infty}^{\infty} K_T(\lambda - \lambda_0) g_I(\mu_1, \mu_2, \lambda)\, d\lambda \\ &= \frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_R^{\lambda_0}(\mu_1, \mu_2)\, dz_l^R(\mu_1)\, dz_m^R(\mu_2) \\ &\quad + \frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_R^{\lambda_0}(\mu_1, \mu_2)\, dz_l^I(\mu_1)\, dz_m^I(\mu_2) \\ &\quad + \frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_I^{\lambda_0}(\mu_1, \mu_2)\, dz_l^R(\mu_1)\, dz_m^I(\mu_2) \\ &\quad - \frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_I^{\lambda_0}(\mu_1, \mu_2)\, dz_l^I(\mu_1)\, dz_m^R(\mu_2), \end{aligned} \tag{3.3}$$

which is equal to $I_1 + I_2 + I_3 - I_4$, say, where

$$(3.4)\quad \begin{aligned} W_R^{\lambda_0}(\mu_1, \mu_2) &= \int_{-\infty}^{\infty} K_T(\lambda - \lambda_0) g_R(\mu_1, \mu_2, \lambda)\, d\lambda, \\ W_I^{\lambda_0}(\mu_1, \mu_2) &= \int_{-\infty}^{\infty} K_T(\lambda - \lambda_0) g_I(\mu_1, \mu_2, \lambda)\, d\lambda. \end{aligned}$$

Taking the variance on both sides of (3.3), we obtain

$$(3.5)\quad \begin{aligned} \text{Var}\,[c_{l,m}^{T*}(\lambda_0)] = {} & \text{Var}\,[I_1] + \text{Var}\,[I_2] + \text{Var}\,[I_3] + \text{Var}\,[I_4] \\ & + 2\,\text{cov}\,[I_1, I_2] + 2\,\text{cov}\,[I_1, I_3] - 2\,\text{cov}\,[I_1, I_4] \\ & + 2\,\text{cov}\,[I_2, I_3] - 2\,\text{cov}\,[I_2, I_4] - 2\,\text{cov}\,[I_3, I_4]. \end{aligned}$$

We will now demonstrate in detail the calculation of the variance of $I_1$, under the assumption of normality of the process $X(t)$. Now

$$(3.6)\quad \begin{aligned} \text{Var}\,[I_1] &= \frac{1}{4\pi^2 T^2} \text{Var}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_R^{\lambda_0}(\mu_1, \mu_2)\, dz_l^R(\mu_1)\, dz_m^R(\mu_2)\right] \\ &= \frac{1}{4\pi^2 T^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_R^{\lambda_0}(\mu_1, \mu_2) W_R^{\lambda_0}(\mu_3, \mu_4) \\ &\qquad \cdot \text{cov}\,[dz_l^R(\mu_1)\, dz_m^R(\mu_2), dz_l^R(\mu_3)\, dz_m^R(\mu_4)]. \end{aligned}$$

Now, it is well known that if $Z_1, Z_2, Z_3$, and $Z_4$ have a joint normal distribution with zero means then

$$(3.7)\quad \text{cov}\,[Z_1 Z_2, Z_3 Z_4] = \text{cov}\,[Z_1, Z_3]\,\text{cov}\,[Z_2, Z_4] + \text{cov}\,[Z_1, Z_4]\,\text{cov}\,[Z_2, Z_3].$$

Using (3.7) and (1.13) we obtain that

$$(3.8)\quad \begin{aligned} &\text{cov}\,[dz_l^R(\mu_1)\, dz_m^R(\mu_2), dz_l^R(\mu_3)\, dz_m^R(\mu_4)] \\ &\quad = [\tfrac{1}{2}\delta_{\mu_1,\mu_3}\, dF_{ll}(\mu_1)][\tfrac{1}{2}\delta_{\mu_2,\mu_4}\, dF_{m,m}(\mu_2)] \\ &\qquad + [\tfrac{1}{2}\delta_{\mu_1,\mu_4}\, d\mathcal{C}_{l,m}(\mu_1)][\tfrac{1}{2}\delta_{\mu_2,\mu_3}\, d\mathcal{C}_{l,m}(\mu_2)]. \end{aligned}$$

Substituting (3.8) in (3.6) and noting that $W_R^{\lambda_0}(\mu_1, \mu_2) = W_R^{\lambda_0}(\mu_2, \mu_1)$, we obtain

$$(3.9)\quad \begin{aligned} \text{Var}\,[I_1] = \frac{1}{16\pi^2 T^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [W_R^{\lambda_0}(\mu_1, \mu_2)]^2 [dF_{ll}(\mu_1)\, dF_{m,m}(\mu_2) \\ + d\mathcal{C}_{l,m}(\mu_1)\, d\mathcal{C}_{l,m}(\mu_2)]. \end{aligned}$$

Repeating a similar calculation for the rest of the terms on the right hand side of (3.5) and combining them all, we finally obtain that

$$(3.10)\quad \begin{aligned} \text{Var}\,[c_{l,m}^{T*}(\lambda_0)] = \frac{1}{8\pi^2 T^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{[W_R^{\lambda_0}(\mu_1, \mu_2)]^2 + [W_I^{\lambda_0}(\mu_1, \mu_2)]^2\} \\ \cdot \{dF_{l,l}(\mu_1)\, dF_{m,m}(\mu_2) + d\mathcal{C}_{l,m}(\mu_1)\, d\mathcal{C}_{l,m}(\mu_2) - d\varphi_{l,m}(\mu_1)\, d\varphi_{l,m}(\mu_2)\}. \end{aligned}$$

In a similar way one obtains after a straight forward but laborious computation that

$$\text{(3.11)} \quad \begin{aligned} \operatorname{Var}[q_{l,m}^{T*}(\lambda_0)] = \frac{1}{8\pi^2 T^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{[W_R^{1\lambda_0}(\mu_1, \mu_2)]^2 + [W_I^{1\lambda_0}(\mu_1, \mu_2)]^2\} \\ \cdot \{dF_{l,l}(\mu_1)\, dF_{m,m}(\mu_2) + d\varphi_{l,m}(\mu_1)\, d\varphi_{l,m}(\mu_2) - d\mathcal{C}_{l,m}(\mu_1)\, d\mathcal{C}_{l,m}(\mu_2)\}, \end{aligned}$$

where

$$\text{(3.12)} \quad \begin{aligned} W_R^{1\lambda_0}(\mu_1, \mu_2) &= \int_{-\infty}^{\infty} K_T^1(\lambda - \lambda_0) g_R(\mu_1, \mu_2, \lambda)\, d\lambda, \\ W_I^{1\lambda_0}(\mu_1, \mu_2) &= \int_{-\infty}^{\infty} K_T^1(\lambda - \lambda_0) g_I(\mu_1, \mu_2, \lambda)\, d\lambda. \end{aligned}$$

Formulae (3.10) and (3.11) may be called the spectral representations of the variances of the estimators for the co- and quadrature-spectral densities respectively.

**4. Consistency of the estimators.** To establish the consistency of the estimators $c_{l,m}^{T*}(\lambda_0)$ and $q_{l,m}^{T*}(\lambda_0)$ for estimating $c_{l,m}(\lambda_0)$ and $q_{l,m}(\lambda_0)$ at every point of continuity $\lambda_0$ of the spectra, we will in Step I establish the asymptotic unbiasedness of the estimators and in Step II show that the variances of the estimators tend to zero as the sample size $T$ tends to $\infty$.

STEP I. *Proof of the asymptotic unbiasedness of $c_{l,m}^{T*}(\lambda_0)$ and $q_{l,m}^{T*}(\lambda_0)$ at a point of continuity $\lambda_0$.*

We will be establishing the asymptotic unbiasedness of $c_{l,m}^{T*}(\lambda_0)$ and $q_{l,m}^{T*}(\lambda_0)$ as estimates of $c_{l,m}(\lambda_0)$ and $q_{l,m}(\lambda_0)$ respectively at a point of continuity $\lambda_0$ of the spectrum by showing that

$$\text{(4.1)} \qquad \lim_{T\to\infty} E[f_{l,m}^{T*}(\lambda_0)] = f_{l,m}(\lambda_0).$$

We have

$$\text{(4.2)} \qquad f_{l,m}^{T*}(\lambda_0) = \int_{-\infty}^{\infty} K_T(\lambda - \lambda_0) f_{l,m}^{T}(\lambda)\, d\lambda = c_{l,m}^{T*}(\lambda_0) + i q_{l,m}^{T*}(\lambda_0),$$

where

$$\text{(4.3)} \quad \begin{aligned} f_{l,m}^{T}(\lambda) = \frac{1}{2\pi T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \\ \cdot \frac{\{\exp[iT(\mu_1 - \lambda)] - 1\}\{\exp[-iT(\mu_2 - \lambda)] - 1\}}{(\mu_1 - \lambda)(\mu_2 - \lambda)}\, dz_l(\mu_1)\overline{dz_m(\mu_2)}. \end{aligned}$$

Taking expectations on both sides of (4.2) we obtain

$$\text{(4.4)} \qquad E[f_{l,m}^{T*}(\lambda_0)] = \int_{-\infty}^{\infty} K_T(\lambda - \lambda_0) E[f_{l,m}^{T}(\lambda)]\, d\lambda.$$

Now

$$E[f_{l,m}^T(\lambda)] = \frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdot \frac{\{\exp[iT(\mu_1-\lambda)]-1\}\{\exp[-iT(\mu_2-\lambda)]-1\}}{(\mu_1-\lambda)(\mu_2-\lambda)}\, dF_{l,m}(\mu_1)\delta_{\mu_1,\mu_2} \tag{4.5}$$

$$= \frac{1}{2\pi T}\int_{-\infty}^{\infty} \frac{\{\exp[iT(\mu_1-\lambda)]-1\}\{\exp[-iT(\mu_1-\lambda)]-1\}}{(\mu_1-\lambda)^2}\, dF_{l,m}(\mu_1)$$

$$= \frac{1}{\pi T}\int_{-\infty}^{\infty} \frac{1-\cos T(\mu_1-\lambda)}{(\mu_1-\lambda)^2}\, dF_{l,m}(\mu_1).$$

Now to prove (4.1) and for more general applications of the above nature we need the following extension of Parzen's lemma.

LEMMA. *Let $K(\omega)$ be a function satisfying $K(\omega) = K(-\omega)$, $K(\omega) \geqq 0$, $\int_{-\infty}^{\infty} K(\omega)\, d\omega = 1$ and further let the symmetric function $K(\omega)$ be a monotonically decreasing function of $\omega \geqq 0$. Let $B_n$ be a sequence of constants tending to infinity as $n \to \infty$. Let $A_n(\omega) = B_nK(B_n(\omega - \omega_0))$. Then at a point of continuity $\omega_0$ of the spectrum $F(\omega)$*

$$\lim_{n\to\infty} J(A_n) = \lim_{n\to\infty}\int_{-\infty}^{\infty} A_n(\omega)\, dF(\omega) = f(\omega_0). \tag{4.6}$$

PROOF.

$$J(A_n) = \int_{-\infty}^{\infty} B_n K(B_n(\omega-\omega_0))\, dF(\omega)$$

$$= \int_{|\omega-\omega_0|<\epsilon} B_n K(B_n(\omega-\omega_0)) f(\omega)\, d\omega + \int_{|\omega-\omega_0|\geqq\epsilon} B_n K(B_n(\omega-\omega_0))\, dF(\omega), \tag{4.7}$$

which equals $I_1 + I_2$, say. Put $B_n(\omega - \omega_0) = x$; then

$$I_1 = \int_{|x|<B_n\epsilon} K(x) f(\omega_0 + x/B_n)\, dx.$$

In view of the continuity of $f(\omega)$ at $\omega_0$ we have

$$\lim_{n\to\infty} I_1 = f(\omega_0). \tag{4.8}$$

We will now show that $I_2$ tends to zero as $n \to \infty$. For

$$I_2 = \int_{-\infty}^{\omega_0-\epsilon} B_n K(B_n(\omega-\omega_0))\, dF(\omega) + \int_{\omega_0+\epsilon}^{\infty} B_n K(B_n(\omega-\omega_0))\, dF(\omega)$$

$$\leqq B_n K(-B_n\epsilon)\int_{-\infty}^{\omega_0-\epsilon} dF(\omega) + B_n K(B_n\epsilon)\int_{\omega_0+\epsilon}^{\infty} dF(\omega),$$

and in view of the monotonicity and symmetry we have

$$I_2 < B_n K(B_n \epsilon). \tag{4.9}$$

Now in view of $\int_{-\infty}^{\infty} K(\omega)\, d\omega = 1$ and the monotonicity we have $\int_n^{2n} K(x)\, dx \geqq nK(n)$, and hence

$$\lim_{n\to\infty} nK(n) = 0. \tag{4.10}$$

In view of (4.10) the right hand side of (4.9) goes to zero as $n \to \infty$ for any given $\epsilon > 0$. Hence

$$\lim_{n\to\infty} I_2 = 0, \tag{4.11}$$

combining (4.8) and (4.11) we obtain $\lim_{n\to\infty} J(A_n) = f(\omega_0)$, which proves the lemma.

In order to show the asymptotic unbiasedness, we have to prove now that the right hand side of (4.5) tends to $f_{l,m}(\lambda)$ at a point of continuity $\lambda$ of the spectrum. In view of the lemma we have only to show that the function $K(\omega) = \pi^{-1}(1 - \cos \omega)/\omega^2$ satisfies the conditions of the lemma.

Clearly $K(\omega)$ satisfies all the conditions except the monotonicity condition. While $K(\omega)$ is oscillatingly decreasing it is easy to see that $K(\omega)$ is dominated by a $K_1(\omega)$ which is monotonically decreasing and satisfies all the other conditions imposed on $K(\omega)$ by observing that $K(\omega) = \pi^{-1}(1 - \cos \omega)/\omega^2 \leqq c/(1 + \omega^2) = K_1(\omega)$, where $c$ is a constant. This completes the proof of the asymptotic unbiasedness.

STEP II. *Consistency of $c_{l,m}^{T*}(\lambda_0)$ and $q_{l,m}^{T*}(\lambda_0)$ as estimates of the co- and quadrature spectral densities at a point of continuity $\lambda_0$ of the spectrum.*

We will only show the consistency of $c_{l,m}^{T*}(\lambda_0)$ and the consistency of $q_{l,m}^{T*}(\lambda_0)$ follows in a similar manner. We will prove the consistency by showing that

$$\lim_{T\to\infty} \operatorname{Var}\, [c_{l,m}^{T*}(\lambda_0)] = 0, \tag{4.12}$$

at a point of continuity $\lambda_0$ and (4.12) together with the asymptotic unbiasedness proved in Step I yields the consistency. We have from Section 3 that

$$\begin{aligned} \operatorname{Var}\, [c_{l,m}^{T*}(\lambda_0)] = \frac{1}{8\pi^2 T^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{(W_R^{\lambda_0}(\mu_1, \mu_2))^2 + (W_I^{\lambda_0}(\mu_1, \mu_2))^2\} \\ \cdot \{dF_{l,l}(\mu_1)\, dF_{m,m}(\mu_2) + d\mathcal{C}_{l,m}(\mu_1)\, d\mathcal{C}_{l,m}(\mu_2) - d\varphi_{l,m}(\mu_1)\, d\varphi_{l,m}(\mu_2)\}. \end{aligned} \tag{4.13}$$

In order to show that the right hand side (4.13) goes to zero, it is enough to show that for any $\mu_1, \mu_2$

$$\lim_{T\to\infty} (1/T^2)[(W_R^{\lambda_0}(\mu_1, \mu_2))^2 + (W_I^{\lambda_0}(\mu_1, \mu_2))^2] = 0.$$

It is easily seen that

$$(1/T^2)[W_R^{\lambda_0}(\mu_1, \mu_2))^2 + (W_I^{\lambda_0}(\mu_1, \mu_2))^2] = \psi_T(\lambda_0)\overline{\psi_T(\lambda_0)},$$

where

$$\psi_T(\lambda_0) = (1/T) \int_{-\infty}^{\infty} B_T K(B_T(\lambda - \lambda_0))\varphi_T(\lambda)\, d\lambda,$$

and

$$\varphi_T(\lambda) = \{\exp[-iT(\lambda - \mu_1)] - 1\}\{\exp[iT(\lambda - \mu_2)] - 1\}/(\lambda - \mu_1)(\lambda - \mu_2).$$

Considering two different cases $\mu_1 = \mu_2$ and $\mu_1 \neq \mu_2$, and assuming that $B_T$ tends to infinity more slowly than $T$ in such a way that $B_T/T \to 0$ as $T \to \infty$, a straight forward calculation leads us to

$$\lim_{T\to\infty} \psi_T(\lambda_0) = 0 \tag{4.14}$$

at a point of continuity $\lambda_0$. (4.14) then implies $\lim_{T\to\infty} \overline{\psi_T(\lambda_0)} = 0$, which together imply $\lim_{T\to\infty} \text{Var}\,[c_{l,m}^{T*}(\lambda_0)] = 0$, at a point of continuity $\lambda_0$ of the spectrum. This completes the proof of consistency.

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