THE FIRST PASSAGE TIME DENSITY FOR HOMOGENEOUS SKIP-FREE WALKS ON THE CONTINUUM

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Introduction. In a recent paper of Keilson [7], the term skip-free was introduced to describe a subclass of random walks $X(t)$ on an ordered set of states x. A walk $X(t)$ may be said to be skip-free in the negative direction for example if, in going from x_2 to $x_1 < x_2$, the walk must pass through all intervening states at least once. A variety of birth and death processes, queuing processes and diffusion processes have this property in one or both directions.

For the description of bounded skip-free processes, employment of the Green's function for the associated unbounded process and a technique of compensation as described in the paper cited, is often advantageous. (A comparison of the compensation technique with Wald's identity is given in Section 6.). We will employ this procedure to demonstrate a simple and very useful relationship between the first passage time density and the Green's function for a broad class of additive skip-free processes. The processes are those in continuous time on the continuum $-\infty < x < \infty$, homogeneous in space and time, and having the skip-free property in the negative direction. Contained within this class are the homogeneous diffusion processes, the homogeneous Takács type processes with negative drift and positive increments (cf. Takács [15], Kendall [11]) and mixtures of the two types. The Green's function $G(x, t)$, a probability density in the generalized sense of L. Schwartz, defined by

$$G(x, t) = (d/dx)\Pr\{X(t) \leqq x \mid X(0) = 0\} \tag{1}$$

has the characteristic function

$$g(k, t) = \mathfrak{F}\{G(x, t)\} = \exp\{-Dk^2t - i\alpha kt - \nu t[1 - a^+(k)]\}. \tag{2}$$

In (2), D, α, and ν are non-negative constants, and

$$a^+(k) = \int_0^\infty e^{ikx}\, dF_A(x) \tag{3}$$

where $F_A(x)$ is the distribution governing the finite jumps. The characteristic function of (2) is seen to have infinitely divisible form (cf. Gnedenko and Kolmogorov, [5], Section 16).

For the process $X(t)$ commencing at $X(0) = x_0 > 0$, let τ be the first time at which $X(0) = 0$, and let the corresponding first passage time density (generalized) be denoted by $S(x_0, \tau)$. The basic result of interest is that this density takes the simple form

$$S(x_0, \tau) = (x_0/\tau)G(-x_0, \tau). \tag{4}$$

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A relationship equivalent to (4) but of slightly different form has been given by Kendall [11] for the Takács process in a dam context. The familiar result for the diffusion process with drift (see Section 3) is also of this form. Aside from its theoretical interest, the result is of practical value in that from (2) one may exhibit $S(x_0, \tau)$ directly (cf (4.3)). With the aid of the conjugate distributions of Khinchin [12], one may also deduce the asymptotic behavior of $S(x_0, \tau)$ with τ from the infinitely divisible character of $G(x, t)$ (cf. Section 5). The saddlepoint methodology developed for statistical application by Daniels [3] and Richter [14] is of direct interest.

A relationship identical in structure to (4) has been exhibited [8] for the corresponding class of homogeneous walks $N(t)$ on the lattice in continuous time with a skip-free property in one direction. If a process sample at n with jump rate ν has probability per unit time $\nu\epsilon_k$ of jumping to $n + k$, with $\epsilon_k = 0$ for $k < -1$, and $\epsilon_{-1} \neq 0$, then for a system at $n_0 > 0$ at $t = 0$, the first passage time density for arrival at $n = 0$ is given by

$$(5) \qquad S(n_0, \tau) = (n_0/\tau)\Gamma(-n_0, \tau)$$

where $\Gamma(n, \tau)$ is the Green's function for the unrestricted walk defined by $\Gamma(n, \tau) = \Pr\{N(t) = n \mid N(0) = 0\}$.

1. The process. Consider the additive process $X(t)$ characterized by probability ν_0 per unit time of an increment s and $dX(t)/dt = -\alpha$ between increment epochs. The increments are independent and governed by the distribution $F_{A_0}(s)$ with generalized density $A_0(s)$. The process is defined by $X(t) = X(0) - \alpha t + \sum_{t_i<t} s_i$ where s_i is the ith increment occurring at time t_i. The density (generalized) of $X(t)$ denoted by $W(x, t)$ is found from continuity considerations to be governed by the equation

$$(1.1) \qquad \partial W(x, t)/\partial t = \alpha(\partial/\partial x)W - \nu W + \nu \int W(x', t)A_0(x - x')\, dx'$$

and initial condition

$$(1.2) \qquad W(x, 0) = \delta(x - x_0).$$

When Equation (1.1) is Fourier transformed and (1.2) is employed, we find, denoting transforms by lower case letters,

$$(1.3) \qquad w(k, t) = \exp(ikx_0)\exp\{-\nu_0 t[1 - a_0(k)] - i\alpha kt\}.$$

Suppose further that two types of increments occur independently, the first with frequency $\gamma^2\nu_1$ and density $\gamma A_1(\gamma s)$ for which $\int sA_1(s)ds = 0$, and the second with frequency ν_2 and density $A_2(s)$, so that $\nu_0 = \gamma^2\nu_1 + \nu_2$ and $A_0(s) = \{(\gamma^2\nu_1/\nu_0)\gamma A_1(\gamma s) + (\nu_2/\nu_0)A_2(s)\}$. Then $\nu_0[1 - A_0(s)] = \gamma^2\nu_1[1 - \gamma A_1(\gamma s)] + \nu_2[1 - A_2(s)]$ and

$$(1.4) \qquad w(k, t) = \exp\{ikx_0 - i\alpha kt - \gamma^2\nu_1 t[1 - a_1(k/\gamma)] - \nu_2 t[1 - a_2(k)]\}.$$

If we now let $\gamma \to \infty$, component 1 gives rise to a diffusive contribution and we obtain in the limit

$$w(k, t) = \exp\{ikx_0 - i\alpha kt - Dk^2t - \nu t[1 - a(k)]\} \tag{1.5}$$

where $\nu = \nu_2$, $a(k) = a_2(k)$, and $D = \frac{1}{2}\nu_1 \int s^2 A_1(s) ds$. The corresponding equation (cf. Feller [4]) is

$$\partial W(x, t)/\partial t = \alpha(\partial W/\partial x) + D(\partial^2 W/\partial x^2) - \nu W + \nu \int W(x', t) A(x - x')\, dx'. \tag{1.6}$$

Suppose further that $F_A(0+) = 0$, i.e., that increments s_2 are positive. We emphasize this character by employing a positive superscript, viz $A(s) = A^+(s)$. The limiting diffusive process with positive increments for the second component has the skip-free property in the negative direction, and the compensation method [7] may be employed to determine the first passage time density.

2. The first passage problem. Consider now Equation (1.6) for positive increments modified by addition of an inhomogeneous source term localized at $x = 0$,

$$\frac{\partial W_F}{\partial t} - \alpha \frac{\partial W_F}{\partial x} - \frac{D\partial^2 W_F}{\partial x^2} + \nu W_F - \nu \int_{-\infty}^{\infty} W_F(x', t) A^+(x - x')\, dx' = C(t)\delta(x) \tag{2.1}$$

with $W_F(x, 0) = \delta(x - x_0)$ where $x_0 > 0$. When $C(t) = 0$, the solution of (2.1) subject to the stated initial distribution is the Green's function

$$G(x - x_0, t) = \frac{1}{2\pi} \int \exp\{-ik(x - x_0) - i\alpha kt - Dk^2 t - \nu t[1 - a^+(k)]\}\, dk. \tag{2.2}$$

For $C(t) \neq 0$, we have, denoting convolution by an asterisk,

$$W_F(x, t) = G(x - x_0, t) + C(t) * G(x, t). \tag{2.3}$$

The compensating function $C(t)$ will be chosen to be such that $W_F(0-, t) = 0$. The structure of Equation (2.1) is such that this condition, together with the boundary condition at infinity $W_F(-\infty, t) = 0$, insures that $W_F(x, t) = 0$ for all $x < 0$, whence $(\partial W_F/\partial x)(0-, t) = 0$. From (2.1) we have from integration over a vanishingly small interval about $x = 0$.

$$C(t) = -D(\partial W_F/\partial x)(0+, t) - \alpha W_F(0+, t). \tag{2.4}$$

We also have from integration over the interval $0+ < x < \infty$

$$-S(x_0, t) = \frac{d}{dt} \int_{0+}^{\infty} W_F(x, t)\, dx = -\frac{D\partial W_F}{\partial x}(0+, t) - \alpha W_F(0+, t). \tag{2.5}$$

A comparison of (2.4) with (2.5) gives the identification

$$C(t) = \frac{d}{dt} \int_{0+}^{\infty} W_F(x, t)\, ds = -S(x_0, t). \tag{2.6}$$

When $D > 0$, as we shall assume for the moment, it may be seen from (2.2) that $G(x, t)$ and all its derivatives are continuous in x for $t > 0$. $W_F(x, t)$ will also be seen to be continuous for all x when $t > 0$, but $\partial W_F/\partial x$ will have a discontinuity at $x = 0$. From (2.3), (2.6), and Laplace transformation, we obtain[1]

$$s(x_0, p) = \mathfrak{L}\{G(-x_0, t)\}/\mathfrak{L}\{G(0, t)\} = \gamma(-x_0, p)/\gamma(0, p) \tag{2.7}$$

where

$$\gamma(x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}\, dk}{p + ik\alpha + Dk^2 + \nu[1 - a^+(k)]}. \tag{2.8}$$

When $\mathrm{Re}(p) > 0$, the integral is absolutely convergent, $\gamma(x, p)$ is analytic in p and vanishes at ∞. To evaluate the integral, we observe that the denominator of (2.8) has precisely one simple zero in the upper half-plane $\mathrm{Im}(k) > 0$ when $\mathrm{Re}(p) > 0$. For consider the function

$$\psi(k, p) = 1 - \{\nu/(\nu + p + ik\alpha + Dk^2)\}a^+(k). \tag{2.9}$$

When k is real and $\mathrm{Re}(p) > 0$, the expression in curly brackets is of magnitude less than unity, and $|a^+(k)| \leqq 1$, so that $\mathrm{Re}\{\psi(k, p)\} > 0$. At infinity in the upper half plane, $\psi(k, p) = 1$, a direct consequence of the half-line character of $A^+(s)$. From the Principle of the Argument [2] we infer that $\psi(k, p)$ has as many zeros as singularities in the upper half plane. Since $a^+(k)$ is analytic in the upper half plane and $(\nu + p + ik\alpha + Dk^2)$ has precisely one zero in the upper half plane, as may readily be seen, $\psi(k, p)$ and the denominator of the integrand in (2.8) have one simple zero there. If we denote this zero by $\sigma(p)$, and the residue for $x_0 = 0$ at this zero by $\zeta(p)$, we have from (2.8)

$$\gamma(-x_0, p) = i\zeta(p) \exp(i\sigma(p)x_0); \qquad \gamma(0, p) = i\zeta(p) \tag{2.10}$$

so that (2.7) becomes,

$$s(x_0, p) = \exp(i\sigma(p)x_0). \tag{2.11}$$

It follows that $s(x_0 + x_1, p) = s(x_0, p)s(x_1, p)$ as is required from simple probabilistic considerations. Consider now the function

$$H(t) = G(-x_0, t)/t. \tag{2.12}$$

We note that $G(-x_0, 0) = 0$ and that $H(t)$ will have the Laplace transform, $h(p) = \int_p^\infty \gamma(-x_0, p')dp'$, so that $h'(p) = -\gamma(-x_0, p)$. To achieve the identification

$$S(x_0, \tau) = \frac{x_0}{\tau} G(-x_0, \tau) \tag{2.13}$$

[1] Because the walk has a skip-free character, Equation (2.7) may also be obtained from a renewal argument, i.e. $G(-y, t) = S(x_0, t) * G(x_0 - y, t)$ when $y \geqq x$, and 2.7 follows.

we need only employ (2.10) and (2.11) and show that $\zeta(p) = -\sigma'(p)$. This may be seen in the following way. The residue at $k = \sigma(p)$ for the integral of (2.8), when $x_0 = 0$, is given from L'Hospital's rule by

$$\zeta(p) = \lim_{k\to\sigma(p)}(k - \sigma(p))/(p + ik\alpha + Dk^2 + \nu[1 - a^+(k)])$$
$$= [i\alpha + 2D\sigma(p) - \nu a^{+\prime}(\sigma(p))]^{-1}.$$

We also have, since $\sigma(p)$ is the root of the denominator, $p + i\sigma(p)\alpha + D\sigma^2(p) + \nu[1 - a^+(\sigma(p))] = 0$. If we differentiate the latter equation with respect to p, we find $1 + [i\alpha + 2D\sigma(p) - \nu a^{+\prime}(\sigma(p))]\sigma'(p) = 0$, whence $\zeta(p) = -\sigma'(p)$ as required.

3. Agreement with older results. Consider first the pure diffusion processes with $D > 0$ and $\nu = 0$. If we subject the equation $\partial W/\partial t = D(\partial^2 W/\partial x^2) + \alpha(\partial W/\partial x)$ governing the density $W(x, t)$ to the transformation $W(x, t) = \exp[-(\alpha^2/4D)t - (\alpha/2D)(x - x_0)]\, W^*(x, t)$, we find that W^* obeys the equation $\partial W^*/\partial t = D(\partial^2 W^*/\partial x^2)$. The conditions $W(x, 0) = \delta(x - x_0)$ and $W(0, t) = 0$ become respectively $W^*(x, 0) = \delta(x - x_0)$ and $W^*(0, t) = 0$. Thereby one finds the familiar results $G(x, t) = (4\pi Dt)^{-\frac{1}{2}} \exp\{-(x + \alpha t)^2/4Dt\}$, and for the first passage process, $W_F(x, t) = (4\pi Dt)^{-\frac{1}{2}} \exp\{-(\alpha^2/4D)t - (\alpha/2D)(x - x_0)\}[\exp\{-(x - x_0)^2/4Dt\} - \exp\{-(x + x_0)^2/4Dt\}]$. The first passage time density is given by $S(x_0, \tau) = D(\partial W_F/\partial x)]_{x=0} = (x_0/\tau)(4\pi D\tau)^{-\frac{1}{2}} \cdot \exp\{-(x_0 - \alpha\tau)^2/4D\tau\}$ and Equation (2.13) is confirmed.

As a second example consider the non-diffusive process with $D = 0$. This process which may be regarded as a limiting form of the class of processes we have treated, has been employed by Takács [15] to discuss the virtual waiting time of an arrival to an $M/G/1$ queue with service time density $A^+(s)$. If $X(t)$ has the density $A^+(x_0)$ at $t = 0$, the first passage time density is given from (2.11) by

$$s(p) = \int A^+(x_0) \exp(i\sigma(p)x_0)\, dx_0 = a^+(\sigma(p)). \tag{3.1}$$

$\sigma(p)$, however, is the limit approached by the root $\sigma_D(p)$ in the upper half plane as $D \to 0$, and satisfies the equation

$$p + i\alpha\sigma(p) + \nu - \nu a^+(\sigma(p)) = 0. \tag{3.2}$$

Consequently we have from (3.1), when $\alpha = 1$ as needed for the Takács process,

$$s(p) = a^+(i[p + \nu - \nu s(p)]) \tag{3.3}$$

and this is the transcendental equation governing the server busy period density $S(t)$. The basic result provides the alternate representation

$$S(t) = \int A^+(x_0)G(-x_0, \tau)(x_0/\tau)\, dx_0 \tag{3.4}$$

which may be reduced to the form given by Takács on p. 58 of his recent book [16]. A slightly different form of Equation (4) has been given by Kendall [11] for the Takács process.

4. The Green's function. The simplicity of the unrestricted process permits an immediate evaluation of the Green's function. When $D = \alpha = 0$, $g_0(k, t) = \exp\{-\nu t[1 - a^+(k)]\} = \sum\{e^{-\nu t}[(\nu t)^n/n!][a^+(k)]^n\}$ and

$$G_0(x, t) = \sum_1^\infty \{e^{-\nu t}[(\nu t)^n/n!]\}A^{+(n)}(x) + e^{-\nu t}\delta(x) \tag{4.1}$$

where $A^{+(n)}(x)$ is the n-fold convolution of $A^+(x)$ with itself and $\delta(x)$ is the Dirac delta function, i.e., the density associated with the improper distribution about $x = 0$. The distribution is given by

$$F_0(x, t) = \sum_1^\infty \{e^{-\nu t}[(\nu t)^n/n!]\}F^{(n)}(x) + e^{-\nu t}U(x). \tag{4.1$'$}$$

The convergence of (4.1′) is assured since $F^{(n)}(x) \leqq 1$.

When $\alpha > 0$ and $D = 0$, we have the Takács type processes with $g_\alpha(k, t) = \exp(-i\alpha kt)\exp\{-\nu t[1 - a^+(k)]\}$, for which

$$\begin{aligned} G_\alpha(x, t) &= G_0(x + \alpha t, t) \\ &= \sum_1^\infty \{e^{-\nu t}[(\nu t)^n/n!]\}\ A^{+(n)}(x + \alpha t) + e^{-\nu t}\delta(x + \alpha t). \end{aligned} \tag{4.2}$$

For the diffusive processes with $D \neq 0$, we have from (1.5) that $g(k, t) = e^{-Dk^2t}g_\alpha(k, t)$ whence

$$G(x, t) = \{\exp[-x^2/(4Dt)]/(4\pi Dt)^{\frac{1}{2}}\} * G_\alpha(x, t) \tag{4.3}$$

the asterisk denoting convolution.

The formal solutions given in (4.1), (4.2) and (4.3) may not afford rapid convergence. The infinitely divisible character of the unrestricted process lends itself, however to asymptotic study as we will now show.

5. Asymptotic behavior of the first passage time density. The central limit behavior for the infinitely divisible processes has been thoroughly explored, and a summary may be found in Gnedenko and Kolmogorov (1954). Such behavior however is associated with deviations of the process $X(t)$ from its mean. For the first passage density $S(x_0, \tau) = (x_0/\tau)\ G(-x_0, \tau)$, one is interested in the asymptotic behavior of $G(x, t)$ for x fixed as $\tau \rightarrow \infty$. A preliminary conjugate transformation of the type introduced by Khintchine enables one, however, to employ the ordinary central limit theory, when $A(x)$ has suitable convergence properties.

Consider the process $X(t)$ of section one with initial process density $W(x, 0) = W_0(x)$, and consider the conjugate density

$$W^*(x, t) = e^{\xi x}W(x, t) \Big/ \int e^{\xi x}W(x, t)\,dx. \tag{5.1}$$

The c.f. for this density is given by

$$w^*(k, t) = \{g(k - i\xi, t)/g(-i\xi, t)\}\{w_0(k - i\xi)/w_0(-i\xi)\} \tag{5.2}$$

where $g(k, t) = \exp\{-Dk^2t - i\alpha kt - \nu t[1 - a^+(k)]\}$. From (5.2) we then have

$$w^*(k, t) = \exp\{-Dk^2t - i(\alpha - 2D\xi)kt - \nu^*t[1 - a^{+*}(k)]\}w_0^*(k) \tag{5.3}$$

where $w_0^*(k) = w_0(k - i\xi)/w_0(-i\xi)$, $\nu^* = \nu a^+(-i\xi)$, and $a^{+*}(k) = a^+(k - i\xi)/a^+(-i\xi)$. We see that (5.3) corresponds to the density for a conjugate process $X^*(t)$ of the skip-free homogeneous type with $D^* = D$, $\alpha^* = \alpha - 2D\xi$, $\nu^* = \nu a(-i\xi)$, and $A^{+*}(x) = e^{\xi x}A^+(x)/\{\int e^{\xi x}A^+(x)dx\}$, etc., provided only that the integrals defining $a^+(-i\xi)$ and $w_0^+(-i\xi)$ converge. When $W_0(x) = \delta(x)$, $W_0^*(x) = \delta(x)$ and only $a^+(-i\xi)$ is of concern. For the asymptotic behavior of interest, one wants $X^*(t)$ to have zero mean, so that a real value ξ is needed for which $[(d/dk)w^*(k, t)]_{k=0} = 0$. This may be seen to be equivalent to the requirement that

$$f(\xi) = 2D\xi - \alpha + \nu \int x\, A^+(x)e^{\xi x}\, dx = 0 \tag{5.4}$$

for some real ξ_0 lying in the convergence strip of $\int A(x)e^{\xi x}dx$. Since $f(\xi)$ is a monotonic function of ξ, we require that the singularities ξ_- and ξ_+ defining the strip be such that

$$\lim_{\xi\to\xi_-} f(\xi) \leqq 0 \leqq \lim_{\xi\to\xi_+} f(\xi). \tag{5.5}$$

The monotonic behavior of $f(\xi)$ assures that ξ_0 when available will be unique. When $A^+(x)$ has finite support, the two limits are $-\infty$ and $+\infty$ respectively and the existence of ξ_0 is assured. From (5.3) we find that for $X^*(0) = 0$,

$$\sigma^{*2}(t) = E\{X_{\xi_0}^{*2}(t)\} = 2Dt + \nu t \int_0^\infty A^+(x) \exp(\xi_0 x)x^2\, dx \tag{5.6}$$

whence

$$G_{\xi_0}^*(x, t) \sim \exp\{-\tfrac{1}{2}x^2[\sigma^{*2}(t)]^{-1}\}/[2\pi\sigma^{*2}(t)]^{\frac{1}{2}} \tag{5.7}$$

and, from (5.1) and (1.5)

$$G(x, t) \sim \frac{\exp\left\{-\alpha\xi_0 t + D\xi_0^2 t - \nu t\left[1 - \int A^+(x)e^{\xi_0 x}\, dx\right] - \xi_0 x - \tfrac{1}{2}x^2[\sigma^{*2}(t)]^{-1}\right\}}{[2\pi\sigma^{*2}(t)]^{\frac{1}{2}}} \tag{5.8}$$

Superior approximations to $G(x, t)$ may be obtained via the saddlepoint procedures developed by Daniels [3] and Richter [14]. An account of the application of these procedures to the walks in continuous time arising in the theory of queues may be found in Keilson [8].

6. The compensation method and Wald's identity. A proof of Equation (4) of the introduction may also be obtained from Wald's identity (cf. Bartlett [1], Kemperman [10] and Miller [13]). As normally employed on a semi-infinite interval, however, this identity requires that a system ultimately be absorbed with probability one. The compensation method does not require this. It permits discussion moreover of nonstationary versions of our basic skip-free process leading to a Volterra integral equation for the first passage time density (cf. Keilson [9]) and is in this sense, a more general procedure. Wald's identity for the finite interval and discrete time may be related to the compensation method by an argument similar to that of Miller in the following way.

Let X_i be a sequence of independent identically distributed random variables with probability density $A(x)$ and let $S_n = S_0 + \sum_1^n X_i$ where S_0 lies in the closed interval $[\alpha, \beta]$. Let $W_n(x)$ be the joint probability density that $S_n = x$ and that the boundaries of the interval have neither been reached nor crossed. Let $B_n(x)$ be the joint probability density that absorption occurs at n and that $S_n = x$. Then for all x, on or outside the interval, continuity of probability gives

$$(6.1) \quad W_0(x) = \delta(x - S_0) \text{ and } W_m(x) = \int_\alpha^\beta W_{m-1}(x')A(x - x')\,dx' - B_m(x).$$

Let the g.f. of $W_m(x)$ be $G(u, x)$ and the g.f. of $B_m(x)$ be $B(u, x)$. If Fourier transforms are denoted by lower case letters, (6.1) becomes

$$(6.2) \qquad g(u, k)[1 - ua(k)] = \exp(ikS_0) - b(u, k).$$

If X has a finite non-zero 2'nd moment the function $g(u, k)$ is an entire function of k and an analytic function of u, for u inside a circle about zero of radius $R > 1$. As proof we note that $\int W_n(x)dx < \theta^n$ for some θ less than unity and n sufficiently large (cf. Lemma I, Section 2.1 of Bartlett), whence

$$(6.3) \qquad \left| \int_\alpha^\beta W_n(x)e^{i(U+iV)x}\,dx \right| \leq e^{|V|\beta} \int_\alpha^\beta W_n(x)\,dx < \theta^n e^{|V|\beta}.$$

The analyticity follows from the uniform convergence in k in any bounded region for $|u| < R < \theta^{-1}$. Suppose now that $a(k)$ has continuation into a convergence strip for which $|a(k)| > \theta$ for some k. For such k and $u = [a(k)]^{-1}$, (6.2) gives $\exp(ikS_0) = b([a(k)]^{-1}, k) = \sum_n \int [a(k)]^{-n} B_n(x)e^{ikx}dx$; i.e.,

$$(6.4) \qquad E\{e^{ikx}[a(k)]^{-n}\} = \exp(ikS_0).$$

Equation (6.4) is Wald's identity. The densities $B_n(x)$ play the role of compensating functions. If the walk is skip-free in one direction, the functions $B_n(x)$ at the corresponding side of the interval become localized at the boundary.

In the above, the distribution of X must be non-singular but may have a discrete component. The discussion carries through if the probability densities are regarded as generalized functions. The derivation of Wald's identity for continuous time is similar. It may be noted that the differentiability of Wald's identity follows from the analytic structure of (6.2). In particular differentiation at $u = 1$ and $k = 0$ gives rise to the moment formulae of Johnson [6].

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