ELLIPTICAL AND RADIAL TRUNCATION IN NORMAL POPULATIONS

By G. M. Tallis

McMaster Laboratory, C.S.I.R.O., Glebe, N.S.W.

1. Introduction. Over the past few years, considerable attention has been devoted to problems of truncation in normal (and other) parent populations, [see Birnbaum and Meyer (1953), Weiler (1959) and Tallis (1961)]. This work has been useful in the general theory of selection and has provided the basis for a number of selection techniques. It is the purpose of this note to introduce the concept of elliptical truncation in normal populations and to derive the moment generating function, m.g.f., for the resulting distribution. Some applications of the results to selection are given in the last section, where also, combined elliptical and radial truncation is discussed by means of problems in two dimensions.

2. The multinormal distribution under elliptical truncation. Consider the standardised, n-dimensional multinormal distribution

$$\phi(\mathbf{x}) = (2\pi)^{-\frac{1}{2}n}|\mathbf{R}|^{-\frac{1}{2}} \exp(-\tfrac{1}{2}\mathbf{x}'\mathbf{R}^{-1}\mathbf{x}), \tag{1}$$

where $\mathbf{R}$ is positive definite, and define a set E in n-space by

$$E = \{\mathbf{x} \mid a \leqq \mathbf{x}'\mathbf{R}^{-1}\mathbf{x} \leqq b\}, 0 \leqq a < b.$$

That is, E is the set of points which lie inside or on the boundary of the ellipsoid $\mathbf{x}'\mathbf{R}^{-1}\mathbf{x} = b$ and outside or on the boundary of the ellipsoid $\mathbf{x}'\mathbf{R}^{-1}\mathbf{x} = a$.

The problem now is to find the m.g.f. for the n variables in the subspace E.

By definition

$$\alpha m(\mathbf{t}) = (2\pi)^{-\frac{1}{2}n}\,|\mathbf{R}|^{-\frac{1}{2}} \int_E \exp\,(-\tfrac{1}{2}\mathbf{x}'\mathbf{R}^{-1}\mathbf{x} + \mathbf{t}'\mathbf{x})\,d\mathbf{x}, \tag{2}$$

which can be reduced by the non-singular transformation $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ ($\mathbf{PP}' = \mathbf{R}$) to

$$\begin{aligned}\alpha m(\mathbf{t}) &= (2\pi)^{-\frac{1}{2}n} \int_F \exp\,(-\tfrac{1}{2}\mathbf{y}'\mathbf{y} + (\mathbf{P}'\mathbf{t})'\mathbf{y})\,d\mathbf{y} \\ &= (2\pi)^{-\frac{1}{2}n}e^T \int_F \exp\,[-\tfrac{1}{2}(\mathbf{y} - \mathbf{P}'\mathbf{t})'(\mathbf{y} - \mathbf{P}'\mathbf{t})]\,d\mathbf{y},\end{aligned} \tag{3}$$

where $T = \frac{1}{2}\mathbf{t}'\mathbf{R}\mathbf{t}$ and $F = \{\mathbf{y} \mid a \leqq \mathbf{y}'\mathbf{y} \leqq b\}$. From (3) it is clear that the variable $\mathbf{Y}'\mathbf{Y} = W$, say, has a non-central chi-square distribution with parameters n and T. Hence, if $F_{n+2i}(\cdot)$ represents the chi-square distribution function with parameter $n + 2i$,

$$\alpha m(\mathbf{t}) = \sum_{i=0}^{\infty} [F_{n+2i}(b) - F_{n+2i}(a)]T^i/i! \tag{4}$$

since the distribution function of W, $H(w)$, is $H(w) = \sum_{i=0}^{\infty} F_{n+2i}(w)T^i/i!$

Received April 23, 1962; revised April 15, 1963.

It is now obvious that $\alpha = F_n(b) - F_n(a)$ and that the mean vector $\mathbf{u}$ and moment matrix $\mathbf{M}$ are given by $\mathbf{u} = \mathbf{0}$ and $\mathbf{M} = \alpha^{-1}[F_{n+2}(b) - F_{n+2}(a)]\mathbf{R}$. In fact, all odd order moments vanish and even moments of order $2k$ are obtained from those of the multinormal distribution by multiplication by $\alpha^{-1}[F_{n+2k}(b) - F_{n+2k}(a)]$.

3. Applications and extensions.

(a) *General applications in n-dimensions.* A direct application of elliptical truncation to selection would require, for instance, that all individuals in the population satisfying $0 \leqq \mathbf{x}'\mathbf{R}^{-1}\mathbf{x} \leqq a$ be retained and the rest discarded. This procedure would ensure that a proportion α be retained without altering the means of the n variates concerned. Such a situation may be desirable, for instance, if a breeding population applying zero selection pressure to all characters is required as a control group. In practice then, the population from which the selections are made is treated as multinormally distributed with correlation matrix $\bar{\mathbf{R}}$. If the population is accessible and finite, $\bar{\mathbf{R}}$ can be calculated; otherwise $\bar{\mathbf{R}}$ may be an estimate of the true matrix $\mathbf{R}$ which is usually unknown. Now, all individuals with measurement vectors satisfying $\mathbf{x}'\bar{\mathbf{R}}\mathbf{x} > a$ are discarded and, in the remaining group, the desired condition $E(X_i) = 0$ for all i will be approximately satisfied.

Not only can selection be performed without altering the means of the n variates, but the following argument shows that a selected group can be formed such that the covariance matrix also remains unchanged. If selection is carried out in such a way that individuals with measurement vectors, $\mathbf{x}$, satisfying $a \leqq \mathbf{x}'\mathbf{R}^{-1}\mathbf{x} \leqq b$ are retained, then it follows from (4) that if $\mathbf{M}$ is to equal $\mathbf{R}$,

$$F_{n+2}(b) - F_n(b) = F_{n+2}(a) - F_n(a) \tag{5}$$

is a necessary and sufficient condition. Let $G_n(x) = F_{n+2}(x) - F_n(x)$, then since $G_n(0) = 0$ and $G_n(x)$ decreases monotonically and continuously to a minimum at $x = n$ and thereafter increases monotonically to $G_n(\infty) = 0$, it follows that for every $a \varepsilon [0, n]$ there exists a $b \varepsilon [n, \infty]$ such that $G_n(a) = G_n(b)$. Moreover, b is a strictly monotone decreasing and continuous function of a and, as a moves continuously from 0 to n, b moves continuously from ∞ to n. Thus, $F_n(b) - F_n(a)$ takes all values of α from 1 to 0. We have shown, therefore, that selection can in fact be carried out in such a way as to have the first and second moments of the selected group the same as the parent population.

Values of a and b are given for $\alpha = .1(.1).9$ and $n = 2$ in Table 1. The figures in the table were found as the non-trivial, simultaneous solution to the equations $ye^{-y} - xe^{-x} = 0$ and $e^{-x} - e^{-y} = \alpha$, where $x = a/2$ and $y = b/2$.

(b) *Extensions for n = 2.* The following two problems in two dimensions are considerably more interesting than the general applications given in (a).

PROBLEM 1. Let S be the sub-space of the plane defined by $x_1^2 + x_2^2 - 2\rho x_1 x_2 \geqq (1 - \rho^2)a$, where X_1 and X_2 have joint frequency function

$$\begin{aligned} \phi(x_1, x_2; \rho) = {} & (2\pi)^{-1}(1 - \rho^2)^{-\frac{1}{2}} \\ & \times \exp\{-[2(1 - \rho^2)]^{-1}(x_1^2 + x_2^2 - 2\rho x_1 x_2)\}. \end{aligned} \tag{6}$$

TABLE 1

Values of a and b for n = 2 and various α

α	a	b	α	a	b
0.1	1.740	2.285	0.6	0.684	4.411
0.2	1.500	2.601	0.7	0.506	5.144
0.3	1.277	2.956	0.8	0.335	6.161
0.4	1.068	3.361	0.9	0.171	8.632
0.5	0.871	3.836			

Consider the sub-space S' of S enclosed in the sector $\theta = \theta_1$, $\theta = \theta_2$, $\theta_2 > \theta_1$, and let $\Pr\{(X_1, X_2) \varepsilon S'\} = \alpha$. Then it is required to determine the values of θ_1, θ_2 and a, $(\bar{\theta}_1, \bar{\theta}_2, \bar{a})$ which maximise $[\beta_1 E(X_1) + \beta_2 E(X_2)]$, where β_1 and β_2 are arbitrary real numbers.

Such a maximisation is desirable, for instance, when animals are selected for breeding. In this case, the β's are the appropriate regression functions of the X's on the particular genotype considered and the problem posed above is analogous to the one discussed by Young and Weiler (1960). These authors investigated the problem of the maximisation of $[\beta_1 E(X_1) + \beta_2 E(X_2)]$ under rectangular truncation in X_1 and X_2 and published several charts for this purpose. From the point of view of maximisation, the system of combined radial and elliptical truncation is much more easily handled, since the maxima can be obtained directly from a single table such as Table 2. With rectangular truncation, maximisation in general can only be achieved iteratively with the aid of a complicated six-dimensional chart.

In order to find $(\bar{\theta}_1, \bar{\theta}_2, \bar{a})$, make the transformation $\mathbf{x} = \mathbf{P}\mathbf{y}$ where

$$\mathbf{P} = \begin{bmatrix} 2^{-\frac{1}{2}}(1-\rho)^{\frac{1}{2}} & 2^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}} \\ -2^{-\frac{1}{2}}(1-\rho)^{\frac{1}{2}} & 2^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}} \end{bmatrix}.$$

Now $\beta_1 E(X_1) + \beta_2 E(X_2) = \gamma_1 E(Y_1) + \gamma_2 E(Y_2)$, $\gamma_1 = (\beta_1 - \beta_2)2^{-\frac{1}{2}}(1-\rho)^{\frac{1}{2}}$, $\gamma_2 = (\beta_1 + \beta_2)2^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}}$, and the new angle θ_i', $(i = 1, 2)$, are given by the formula

$$\tan\theta_i' = [(1-\rho)/(1+\rho)]^{\frac{1}{2}}[(1+\tan\theta_i)/(1-\tan\theta_i)].$$

Another transformation, this time orthogonal, subsequently simplifies the problem. Let $\mathbf{z} = \mathbf{H}\mathbf{y}$, where

$$\mathbf{H} = \begin{bmatrix} \gamma_1(\gamma_1^2+\gamma_2^2)^{-\frac{1}{2}} & \gamma_2(\gamma_1^2+\gamma_2^2)^{-\frac{1}{2}} \\ -\gamma_2(\gamma_1^2+\gamma_2^2)^{-\frac{1}{2}} & \gamma_1(\gamma_1^2+\gamma_2^2)^{-\frac{1}{2}} \end{bmatrix}.$$

Upon making the above two transformations in (6) and letting $z_1 = r\cos\theta$, $z_2 = r\sin\theta$ we obtain finally

$$\alpha E[\beta_1 X_1 + \beta_2 X_2] = (2\pi)^{-1}(\gamma_1^2 + \gamma_2^2)^{\frac{1}{2}} \int_{\theta_1''}^{\theta_2''} \int_a^\infty \cos\theta r^2 e^{-r^2/2}\, dr\, d\theta \tag{7}$$
$$= (\gamma_1^2 + \gamma_2^2)^{\frac{1}{2}} E(z_1) = F(\theta_1'', \theta_2'', a), \text{ say,}$$

and by assumption

$$(2\pi)^{-1} \int_\theta^{\theta_2''} \int_a^\infty r e^{-r^2/2}\, dr\, d\theta = (2\pi)^{-1}(\theta_2'' - \theta_1'')e^{-a^2/2} = \alpha.$$

Let

$$H(\theta_1'', \theta_2'', a) = (2\pi)^{-1}(\theta_2'' - \theta_1'')e^{-a^2/2} - \alpha = 0 \tag{8}$$

and $G = F + \lambda H$, then

$$\text{(a)}\quad \partial G/\partial\theta_1'' = (2\pi)^{-1}(\gamma_1^2 + \gamma_2^2)^{\frac{1}{2}} \left(\int_a^\infty r^2 e^{-r^2/2}\, dr\right) \cos\theta_1'' + (2\pi)^{-1}\lambda\, e^{-a_2/2} = 0$$

$$\text{(9)}\quad \text{(b)}\quad \partial G/\partial\theta_2'' = (2\pi)^{-1}(\gamma_1^2 + \gamma_2^2)^{\frac{1}{2}} \left(\int_a^\infty r^2 e^{-r^2/2}\, dr\right) \cos\theta_2'' + (2\pi)^{-1}\lambda e^{-a^2/2} = 0$$

$$\text{(c)}\quad \partial G/\partial a = (2\pi)^{-1}(\gamma_1^2 + \gamma_2^2)^{\frac{1}{2}} a^2 e^{-a^2/2}(\sin\theta_2'' - \sin\theta_1'') + (2\pi)^{-1}\lambda(\theta_2'' - \theta_1'')a e^{-a^2/2} = 0$$

Substract 9(b) from 9(a) to obtain $\cos\theta_1'' = \cos\theta_2''$ or $\theta_2'' = -\theta_1''$, and for $a > 0$, divide 9(c) through by $a(\theta_2'' - \theta_1'')$ and subtract it from 9(b) to give

$$\{1 + [1 - \Phi(\bar{a})]/\bar{a}\phi(\bar{a})\} \cos\bar{\theta}_2'' - \sin\bar{\theta}_2''/\bar{\theta}_2'' = 0. \tag{10}$$

By using the relation $\phi(a) = \alpha/\theta_2'' \cdot (\pi/2)^{\frac{1}{2}}$, (10) can be solved iteratively for $\bar{\theta}_2''$. The quantities $\bar{\theta}_1''$ and $\bar{a}$ are obtained immediately and back substitution gives $\bar{\theta}_1$ and $\bar{\theta}_2$. If $a = 0$, it is found that $\theta_2'' = -\theta_1''$, as previously, and $\theta_2'' = \alpha\pi$.

If the constraints (8) and $\theta_2'' = -\theta_1''$ are introduced into (7), F becomes a function of a only and

$$F(a) = K \sin(\pi\alpha e^{a^2/2}) \int_a^\infty r^2 e^{-r^2/2}\, dr \tag{11}$$

TABLE 2

Values of θ_2'', a and $E(z_1)$ for various α

α	θ_2''	a	$E(z_1)$	α	θ_2''	a	$E(z_1)$
.1	0.877	1.433	1.722	.6	1.885	0	0.632
.2	1.044	1.008	1.375	.7	2.199	0	0.461
.3	1.196	0.691	1.144	.8	2.513	0	0.293
.4	1.357	0.393	0.960	.9	2.827	0	0.137
.5	1.571	0.000	0.798				

where $K = \pi^{-1}(\gamma_1^2 + \gamma_2^2)^{\frac{1}{2}}$. For $\alpha < \frac{1}{2}$, (10) has a unique root, $0 < \bar{a} < (-2 \ln 2\alpha)^{\frac{1}{2}}$, and from an inspection of (11), $F(\bar{a})$ is clearly a maximum. When $\alpha \geqq \frac{1}{2}$ (10) has no solution, but $F(a)$ is monotone decreasing in a and hence attains its maximum when $a = 0$. Points of maximisation and values of $E(z_1)$ are given for various values of α in Table 2.

A referee has pointed out that (7) can be maximised readily without using the Lagrange procedure. First make the transformation $\epsilon = 2^{-1}(\theta_1'' + \theta_2'')$ and $\Delta = 2^{-1}(\theta_1'' - \theta_2'')$ to show that, for all Δ and a, (7) is maximum when $\epsilon = 0$. By introducing the constraint $\Delta = \pi\alpha e^{a^2/2}$, (7) can be written in the form (11) and the extreme points investigated in the usual manner. Both methods lead to the same result.

Problem 2. It was shown above that, from an original population, a control population can be constructed so that no changes in means or second order moments occur, provided the radii a and b are suitably chosen. However, the problem of simultaneously establishing a control group of proportion α and a selection group of proportion $\delta < 1 - \alpha$ from a single base population often arises. In this case it may be desirable to leave the first and second moments in the control group the same as the base population and, at the same time, maximise $E[\beta_1 X_1 + \beta_2 X_2]$ in the selection group to obtain the greatest possible selection differential using a single sector.

The control group is established by means of the elliptical truncation $a \leqq \mathbf{x}'\mathbf{R}^{-1}\mathbf{x} \leqq b$, where a and b are determined from Table 1. In order to find the region from which the selection group is formed, notice that

$$\delta E[\beta_1 X_1 + \beta_2 X_2] = (2\pi)^{-1}(\gamma_1^2 + \gamma_2^2)^{\frac{1}{2}} \int_{\theta_1''}^{\theta_2''} \cos\theta \left(\int_0^a + \int_b^\infty\right) r^2 e^{-r^2/2}\, dr\, d\theta$$

and $H(\theta_1'', \theta_2'') = (2\pi)^{-1}(\theta_2'' - \theta_1'')(1 - \exp(-a^2/2) + \exp(-b^2/2)) - \delta = 0$. It is found immediately that $\bar{\theta}_2'' = -\bar{\theta}_1''$, as previously, and

$$\bar{\theta}_2'' = \delta\pi/(1 - e^{-a^2/2} + e^{-b^2/2})$$

$$E(z_1) = (2/\pi)^{\frac{1}{2}} \sin(\bar{\theta}_2'')[b\phi(b) - a\phi(a) + 1 + \Phi(a) - \Phi(b)].$$

Thus, all those individuals lying in the sector $(\bar{\theta}_1'', \bar{\theta}_2'')$ and outside the control group form the selection group.

4. Acknowledgement. The author gratefully acknowledges the valuable suggestions of Mr. George Brown and the referees.

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