CHEBYSHEV POLYNOMIAL AND OTHER NEW APPROXIMATIONS TO MILLS' RATIO

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0. Summary. For various but sound practical reasons it has become desirable to approximate to previously tabulated mathematical functions by polynomials or rational fractions. In this paper Chebyshev polynomials are used to approximate Mills' Ratio over two separate ranges [0, 1], [1, ∞] of the argument. Some new asymptotic expansions for this ratio are also obtained by an extended use of the symbolic operator method, revealing incidentally that Ruben's (1962) expansion is a special but not necessarily superior case.

1. Introduction. Following accepted notation we define Mills' Ratio to be $R(t) = \int_t^\infty p(x)dx/p(t)$, where $p(x) = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$. Shenton (1954) defined another ratio $\bar{R}(t)$, related to Mills' Ratio, as $\bar{R}(t) = \int_0^t p(x)dx/p(t)$ and noted that $R(t) + \bar{R}(t) = [2p(t)]^{-1}$. Inequalities for $R(t)$ have been given by Gordon (1941), Birnbaum (1950), Murty (1952), Sampford (1953), Boyd (1959), and indirectly through the normal probability integral by Williams (1946), Pólya (1949), Tate (1953), Chu (1955), and Haldane (1961).

Shenton (1954) has given continued fraction expansions and hence inequalities for $\bar{R}(t)$. Ruben (1962) produced a new asymptotic expansion for $R(t)$ based on some results of Franklin and Friedman (1957).

For large values of t the usual approximating form for $R(t)$ has been the asymptotic series due to Laplace (1812):

$$R(t) = 1/t - 1/t^3 + 1\cdot3/t^5 - 1\cdot3\cdot5/t^7 \cdots, \qquad t > 0.$$

This form is of limited use except when t is fairly large. However certain continued fraction forms have been found to be better, for example, the original Laplace form

$$R(t) = \frac{1}{t+}\,\frac{1}{t+}\,\frac{2}{t+}\,\frac{3}{t+}\cdots,$$

and Shenton's (1954)

$$\bar{R}(t) = \frac{t}{1-}\,\frac{t^2}{3+}\,\frac{2t^2}{5-}\,\frac{3t^2}{7+}\,\frac{4t^2}{9-}\,\frac{5t^2}{11+}.$$

The Laplace C.F. is fairly accurate when t is moderate to large, but the rate of convergence falls off rapidly as t becomes small. The Shenton C.F. on the other hand converges rapidly for small t, the rate of convergence falling off slowly as t

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increases. Thus, depending on the value of t, there appears to be a variety of different approximating forms from which to choose.

Clenshaw (1957, 1962) has, *inter alia*, developed the use of Chebyshev polynomials as series approximations to many mathematical functions which satisfy linear differential equations. Among the many advantages claimed for using such series are that they are convergent (subject to weak restrictions), are readily calculable, and that bounds for the error are immediately known when truncating at any term of the series. The succeeding terms of the series are easily determined from the preceding terms by means of a recurrence relation. Further, by applying simple rules it is possible to manipulate such series approximations as with an explicit mathematical function, for example, differentiating or integrating it, an advantage not shared by continued fraction forms.

2. Chebyshev polynomial expansion for $R(t)$ in the range [0, 1]. We shall find it convenient first of all to give an account of Clenshaw's method for finding the Chebyshev series representation in $[-1, 1]$ of a function $f(t)$ satisfying an ordinary linear differential equation, before applying it to the present problem. In terms of accepted notation $T_r(t)$ denotes the rth degree, Chebyshev polynomial, and the first seven such polynomials are $T_0(t) = 1$, $T_1(t) = t$, $T_2(t) = 2t^2 - 1$, $T_3(t) = 4t^3 - 3t$, $T_4(t) = 8t^4 - 8t^2 + 1$, $T_5(t) = 16t^5 - 20t^3 + 5t$, $T_6(t) = 32t^6 - 48t^4 + 18t^2 - 1$.

Now Clenshaw's (1957, 1962) procedure assumes that the prescribed function $f(t)$ is continuous and of bounded variation in the range $[-1, 1]$. This being so the function and its derivatives can separately be expressed as series of Chebyshev Polynomials as follows:

$$f(t) = \tfrac{1}{2}a_0T_0(t) + a_1T_1(t) + a_2T_2(t) \cdots, = \sum_{r=0}^{\infty}{}' a_rT_r(t)$$

$$f^{(s)}(t) = \tfrac{1}{2}a_0^{(s)}T_0(t) + a_1^{(s)}T_1(t) + a_2^{(s)}T_2(t) \cdots, = \sum_{r=0}^{\infty}{}' a_r^{(s)}T_r(t), \qquad s = 1, 2, 3, \cdots q$$

where $f^{(s)}(t)$ is the sth derivative of $f(t)$ with respect to t, and $f^q(t)$ is the highest order derivative of $f(t)$ which appears in the linear differential equation. From a property of the $T_r(t)$ it can easily be shown that

$$(1) \qquad 2ra_r^{(s)} = a_{r-1}^{(s+1)} - a_{r+1}^{(s+1)}, \ (r \geqq 1).$$

Let $C_r(g)$ denote the coefficient of $T_r(t)$ in the expansion of $g(t)$ when $r > 0$, and twice this coefficient when $r = 0$, so that $C_r(f) = a_r$. Then from the relation

$$(2) \qquad T_{r+1}(t) - 2tT_r(t) + T_{r-1}(t) = 0,$$

it can be shown that $C_r(tf^{(s)}) = \tfrac{1}{2}(a_{|r-1|}^{(s)} + a_{r+1}^{(s)})$, $r = 0, 1, 2, 3 \cdots$; $s = 0, 1, 2 \cdots q$ and hence

$$(3) \qquad C_r(t^pf^{(s)}) = 2^{-p}\sum_{j=0}^{p}\binom{p}{j}a_{|r-p+2j|}^{(s)}.$$

Now from the definition of $R(t)$ we have

$$R'(t) - tR(t) + 1 = 0, R(0) = (\pi/2)^{\frac{1}{2}}. \tag{4}$$

Operating with C_r on the differential Equation (4) we have $C_r(R') - C_r(tR) = 0$, $r = 1, 2, 3 \cdots$, or

$$a_r' - \tfrac{1}{2}(a_{r-1} + a_{r+1}) = 0. \tag{5}$$

Now $2ra_r = a_{r-1}' - a_{r+1}'$, and (5) implies $a_{r-1}' - a_{r+1}' - \frac{1}{2}(a_{r-2} + a_r - a_r - a_{r+2}) = 0$, hence $4ra_r = a_{r-2} - a_{r+2}$. Re-arrangement of this recurrence relation for the coefficients gives

$$a_{r-2} = 4ra_r + a_{r+2}, \qquad r = 2, 3, 4 \cdots. \tag{6}$$

Choosing an integer N large enough to suppose $a_r = 0$ for $r > N$, the precision of the series expansion can be controlled at any level. Having chosen N we can utilise the recurrence relation (6) to determine the coefficients in two sets, the first in terms of the last non-zero coefficient a_N, and the other in terms of a_{N-1}.

We have chosen $N = 11$, and utilised the initial condition $R(0) = \frac{1}{2}a_0 - a_2 + a_4 - a_6 + a_8 - a_{10} = (\pi/2)^{\frac{1}{2}}$, and (6) to obtain a_{10}. Also from (4) we can write $\frac{1}{2}C_0(R') - \frac{1}{2}C_0(tR) + 1 = 0$ and $C_2(R') - C_2(tR) = 0$ from which it follows using (1) and (3), $3a_1 + a_3 = -4$. Thus we can determine a_{11}.

The final values of the coefficients so obtained are given in Table 1. As a check we can determine $R(1)$ from $R(1) = \frac{1}{2}a_0 + a_1 + a_2 + \cdots + a_{10} + a_{11} = .655680$, which agrees with the true $R(1)$ up to the sixth decimal place.

Finally then we have

$$\text{(A)} \quad R(t) = 1.6345308\, T_0(t) - 1.2974425\, T_1(t) + .4054731\, T_2(t) \cdots - .0000001\, T_{11}(t).$$

It is now a simple matter to determine $R(t)$ for any t in [0, 1], knowing $T_0(t)$, $T_1(t)$, and using (2) to calculate the remaining values of the polynomials for the prescribed t.

Table 2 shows the number of terms of the series required to obtain specified decimal precision in the range [0, 1] of t.

TABLE 1

r	a_r	r	a_r
0	3.2690 615	1	−1.2974 425
2	.4054 731	3	−.1076 724
4	.0252 763	5	−.0053 740
6	.0010 518	7	−.0001 917
8	.0000 328	9	−.0000 053
10	.0000 008	11	−.0000 001

TABLE 2
Number of terms required to achieve specified precision in range [0, 1]

t	decimal places			
	3	4	5	6
.2	6	8	10	11
.4	7	8	10	11
.6	7	8	9	11
.8	7	8	10	11
1.0	7	8	9	11

3. In range [1, ∞]. In this case we seek a representation for $R(t)$ of the form

$$R(t) = (1/t)[\tfrac{1}{2}b_0 + b_1T_1(1/t) + \cdots] = (1/t)\sum_{r=0}^{\infty}{}' b_rT_r(1/t), 1 \leqq t < \infty.$$

It will be convenient to put $u = 1/t$ and consider the function $y(u)$ where

$$y(u) = (1/u)R(1/u) = [\tfrac{1}{2}b_0 + b_1T_1(u) + b_2T_2(u) + \cdots], 0 \leqq u < 1.$$

From (4) we can make the transformation and write down the differential equation for $y(u)$ as

$$u^3y' + (u^2 + 1)y = 1. \tag{7}$$

Boundary conditions for $y(u)$ are $y(1) = .655680$ and $y(0) = 1$. Inspection of the boundary conditions and (7) show that $y(u)$ can be represented as a series in even powers of u. Proceeding as in Section 2 we have $C_r(u^3y') + C_r[(u^2 + 1)y] = C_r(1)$, $r = 0, 1, 2, \cdots$, then using (4) there follows

$$2^{-3}[b'_{r-3} + 3b'_{r-1} + 3b'_{r+1} + b'_{r+3}] + 2^{-2}[b_{r-2} + 2b_r + b_{r+2}] + b_r = 0,$$
$$r = 2, 4, \cdots,$$

which implies

$$2(r - 3)b_{r-3} + 6(r - 1)b_{r-1} + 6(r + 1)b_{r+1} + 2(r + 3)b_{r+3}$$
$$+ 2[b_{r-3} - b_{r-1} + 2(b_{r-1} - b_{r+1}) + b_{r+1} - b_{r+3}] + 8(b_{r-1} - b_{r+1}) = 0$$

or

$$(r - 2)b_{r-3} + (3r + 2)b_{r-1} + (3r - 2)b_{r+1} + (r + 2)b_{r+3} = 0. \tag{8}$$

In this case we have chosen $N = 26$ and by using (8) obtained the coefficients b_{2r} in terms of b_{26}.

Now $y(1) = \frac{1}{2}b_0 + b_2 + b_4 \cdots + b_{26}$, and since $y(1) = .655680$ we can obtain b_{2r}. These are tabulated in Table 3.

As a check we have $y(0) = \frac{1}{2}b_0 - b_2 + b_4 - b_6 + \cdots - b_{26} = 1.0000003$

TABLE 3

r	b_r	r	b_r
0	1.5792 832	14	−.0001 771
2	−.1611 570	16	.0000 754
4	.0345 335	18	−.0000 331
6	−.0096 581	20	.0000 149
8	.0031 367	22	−.0000 070
10	−.0011 253	24	.0000 038
12	.0004 341	26	−.0000 027

TABLE 4

Number of terms required to achieve specified precision in range [1, ∞]

t	decimal places			
	3	4	5	6
2	5	7	10	14
3	5	6	11	13
4	3	7	9	12
5	4	8	10	11
10	4	5	7	12

whereas the true value is unity, an error of three units in the seventh place of decimals.

Finally then we have

$$\text{(B)}\quad R(t) = (1/t)[.7896416\, T_0(1/t) - .1611570\, T_2(1/t) + .0345335\, T_4(1/t) \cdots - .0000027\, T_{26}(1/t)].$$

Once again one can determine $R(t)$ for any t in $[1, \infty]$ knowing $T_0(1/t)$, $T_2(1/t)$, and the recurrence relation $(4x^2 - 2)T_r(x) = T_{r+2}(x) + T_{r-2}(x)$.

Table 4 gives some information on the number of terms of the series to be included to give specified decimal precision in the range $[1, \infty]$ of t.

4. Some new asymptotic expansions, I. A heuristic derivation of Ruben's expansion. It was noted in the introduction that Ruben (1962) had produced an asymptotic expansion for $R(t)$ using Shenton's integral form

$$R(t) = \int_0^\infty e^{-tx} e^{-\frac{1}{2}x^2}\, dx, \tag{9}$$

and a theorem due to Franklin and Friedman (1957) concerned with asymptotic expansions of Laplace Transforms.

Ruben's expansion is as follows:

$$R(t) = f_0(1/t)/t + f_1(2/t)/t^3 + 2!f_2(3/t)/t^5 + \cdots (N-1)!f_{N-1}(N/t)/t^{2N-1} + O(t^{-2N-1}),\ t > 0,$$

where

$$f_0(x) = e^{-\frac{1}{2}x^2}, \qquad f_k(x) = \frac{d}{dx}\left\{\frac{f_{k-1}(x) - f_{k-1}(k/t)}{x - k/t}\right\}, \qquad k = 1, 2, \cdots.$$

We shall obtain this expansion by using the method of operators and go on in Section 5 to obtain other expansions by a similar procedure.

Before proceeding it may be advantageous to restate some relationships between finite difference operators.

For example,

$$\begin{aligned} Ef(x) = f(x+1), Df(x) = (d/dx)f(x), E = 1 + \Delta, \mu = \tfrac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}}),\\ \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}, E = e^D, \end{aligned} \tag{10}$$

hence

$$\mu\delta = \tfrac{1}{2}[E - E^{-1}] = \tfrac{1}{2}(e^D - e^{-D}) = \sinh D \tag{11}$$

or $D = \sinh^{-1}\mu\delta$.

It is clear that by integrating by parts the integral in Equation (9), we have formally

$$\begin{aligned} \int_0^\infty e^{-tx}f(x)\,dx &= f(0)/t + f'(0)/t^2 + f''(0)/t^3 + \cdots \\ &= t^{-1}(1 + D/t + D^2/t^2 + \cdots)f(0) \\ &= t^{-1}(1 - D/t)^{-1}f(0), \end{aligned} \tag{12}$$

where $f(x) = e^{-\frac{1}{2}x^2}$.

Carrying out the differentiation we have that

$$R(t) = t^{-1}(1 - 1/t^2 + 1\cdot3/t^4 - 1\cdot3\cdot5/t^6 + \cdots),$$

which is the well-known Laplace asymptotic expansion. Now we may re-write the right hand side of (12) as

$$\begin{aligned} t^{-1}\left(\frac{E^{1/t}}{E^{1/t}(1 - D/t)}\right)f(0) &= t^{-1}\frac{E^{1/t}}{[1 - (1 + E^{1/t}(D/t - 1)]}f(0) \\ &= (E^{1/t}/t)[1 + (1 + E^{1/t}(D/t - 1)) \\ &\qquad + (1 + E^{1/t}(D/t - 1))^2 + \cdots]f(0) \cdots. \end{aligned} \tag{13}$$

The first term of this expansion is

$$(E^{1/t}/t)f(0) = f(1/t)/t = e^{-1/2t^2}/t,$$

the second term is

$$\begin{aligned} t^{-1}(1 + E^{1/t}(D/t - 1))E^{1/t}f(0) &= t^{-1}[f(1/t) + (D/t - 1)f(2/t)], \\ &= t^{-1}[f(1/t) - (2/t^2)f(2/t) - f(2/t)], \\ &= t^{-3}[t^2f(1/t) - (2 + t^2)f(2/t)], \end{aligned}$$

and similarly the third term is

$$(2/t^5)[\tfrac{1}{2}t^4 f(1/t) - (2t^2 + t^4)f(2/t) + \tfrac{1}{2}(9 + 5t^2 + t^4)f(3/t)].$$

These three terms are identically equal to the first three terms of Ruben's expansion. Each term in the expansion (13) is obtained from the preceding term by operating with the operator $[1 + E^{1/t}(D/t - 1)]$. This operator appears to be directly related to the recurrence form that Ruben (following Franklin and Friedman) employs to obtain successive terms in his expansion.

5. Some new asymptotic expansions, II. The $\mu\delta$- and Δ-operator expansions. We shall now consider symbolic operator expansions for an integral which is a transform of the integral in Equation (9).

Let $x = z/t$: then Equation (9) becomes

$$R(t) = \frac{1}{t}\int_0^\infty e^{-z}\exp(-z^2/2t^2)dz.$$

Proceeding as before, we have

$$\begin{aligned}\int_0^\infty e^{-z}h(z)\,dz &= h(0) + h'(0) + h''(0)\cdots \\ &= (1 + D + D^2 + \cdots)h(0) \\ &= (1 - D)^{-1}h(0),\end{aligned} \tag{14}$$

where $h(z) = \exp(-z^2/2t^2)$. Using relation (11) we may write

$$\begin{aligned}\int_0^\infty e^{-z}h(z)\,dz &= \frac{1}{1 - \sinh^{-1}\mu\delta}\,h(0) \\ &= [1 + \mu\delta + (\mu\delta)^2 + (5/6)(\mu\delta)^3 + (2/3)(\mu\delta)^4 \\ &\quad + (23/40)(\mu\delta)^5 + (23/45)(\mu\delta)^6 + (241/560)(\mu\delta)^7 \\ &\quad + (37/105)(\mu\delta)^8\cdots]h(0).\end{aligned} \tag{15}$$

Since $h(z)$ is an even function of z it follows that $(\mu\delta)^{2r+1}h(0) = 0$, $r = 0, 1, 2, \cdots$. Hence (15) becomes $\int_0^\infty e^{-z}h(z)dz = [1 + (\mu\delta)^2 + (2/3)(\mu\delta)^4 + (23/45)(\mu\delta)^6 + (37/105)(\mu\delta)^8 + \cdots]h(0)$. Finally then

$$\begin{aligned}R(t) = t^{-1}\{&1 + \tfrac{1}{2}(e^{-2/t^2} - 1) + (1/12)(e^{-8/t^2} - 4e^{-2/t^2} + 3) \\ &+ (23/1440)(e^{-18/t^2} - 6e^{-8/t^2} + 15e^{-2/t^2} - 10) \\ &+ (37/13{,}440)(e^{-32/t^2} - 8e^{-18/t^2} + 28e^{-8/t^2} - 56e^{-2/t^2} + 35)\ldots\}.\end{aligned} \tag{D}$$

No attempt will be made at this stage to investigate the asymptotic nature or precision of this series but a table of estimates term by term will be obtained over a fairly wide range of t. (See Table 5).

As an adjunct to the previous approach a different expansion may be obtained using the forward difference operator Δ.

Using relation (10) we have $e^D = E = 1 + \Delta$ or $D = \ln(1 + \Delta)$. We may now express (14) in a slightly different form

$$(1 - D)^{-1}h(0) = [1 - \ln(1 + \Delta)]^{-1}h(0)$$
$$= [1 + \Delta + \Delta^2/2 + \Delta^3/3 + \Delta^4/6 + (7/60)\,\Delta^5 + \cdots]h(0).$$

Carrying out the differentiation on $h(z) = \exp(-z^2/2t^2)$ and putting $z = 0$, we have

$$\begin{aligned} R(t) = t^{-1}\{1 &+ (e^{-1/2t^2} - 1) + \tfrac{1}{2}(e^{-2/t^2} - 2e^{-1/2t^2} + 1) \\ &+ \tfrac{1}{3}(e^{-9/2t^2} - 3e^{-2/t^2} + 3e^{-1/2t^2} - 1) \\ &+ \tfrac{1}{6}(e^{-8/t^2} - 4e^{-9/2t^2} + 6e^{-2/t^2} - 4e^{-1/2t^2} + 1) \cdots\}. \end{aligned} \tag{E}$$

It is perhaps worth noting that the first two terms of this expansion taken together produce the first term of Ruben's expansion. However it will be shown that the first two terms of the expansion (D) using the operator $(\mu\delta)$ gives a better 'first' approximation to $R(t)$, at least over the range of t considered. (See Table 5).

One advantage of the expressions (D), (E), and the succeeding one of the next section is that the terms contain only exponentials. There are no products of polynomials and exponentials as there are in Ruben's expansions.

6. Some new asymptotic expansions, III. Laguerre-Gauss expansion. An alternative expansion may be obtained, again from the integral form (9) by using a Laguerre-Gauss quadrature formula as described in Hildebrand (1956, p. 325).

$$\int_0^\infty e^{-x}f(x)\,dx = \sum_{k=1}^{m} H_{k,m}f(x_{k,m}) + (m!)^2/(2m)!\,f^{[2m]}(\xi), \tag{16}$$

where the $H_{k,m}$, $x_{k,m}$ are known functions of k, m, and $0 < \xi < \infty$. Transforming (16) by putting $z = tx$ we have

$$R(t) = t^{-1}\int_0^\infty e^{-z}e^{-z^2/2t^2}\,dz = t^{-1}\left[\sum_{k=1}^{m} H_{k,m}\exp(-z_{k,m}^2/2t^2) + \text{error}\right] \tag{F}$$

Now as has been stated, the error term involves a certain order derivative of $f(z)$, where $f(z) = \exp(z^2/2t^2)$.

Let $v = z/t$, then

$$f(z) = \phi(v) = e^{-v^2/2}.$$

Successive differentiation of this function gives

$$(d^n/dv^n)\phi(v) = (-1)^n H_n(v)\phi(v),$$

where $H_n(v)$ is the Hermite polynomial of degree n in v. Note that $H_0(v) = 1$. Since $(d^n/dv^n)\phi(v) = t^n(d^n/dz^n)f(z)$, we have

$$(d^n/dz^n)f(z) = (-1)^n(1/t^n)H_n(z/t)f(z).$$

Thus the error term in (F) becomes

$$(17) \qquad [(m!)^2/(2m)!](1/t^{2m})H_{2m}(\xi/t)e^{-\xi^2/2t^2}.$$

Now it is a property of the Hermite polynomials that

$$\max_\xi |H_{2m}(\xi/t)e^{-\xi^2/2t^2}| \leqq (2m)!/2^m m!,$$

hence (17) is numerically less than $2^{-m}m!\ t^{-2m}$, and the error in $R(t)$ is less than $2^{-m}m!\ t^{-2m-1}$.

It is perhaps worth pointing out that both Ruben's and the Laguerre-Gauss expansions have the same order of precision, i.e. $O(1/t^{2m+1})$. However in the latter case we have an exact numerical bound to the error involved. In all cases the amount of calculation involved in the evaluation of Ruben's expansion is considerably greater than that using the Laguerre-Gauss form for the same precision.

TABLE 5

A numerical comparison of the various expansions

t	Expansion	$g_1(t)$	$g_2(t)$	$g_3(t)$	$g_4(t)$	$g_5(t)$	True
	C	.441249	.42760	.41559			
2	D	.5	.401633	.431183	.417583	.424105	.421369
	E	.5	.441249	.401633	.427058	.422551	
	F	.441249	.425912	.420105	.421260	.421450	
	C	.315320	.30441	.30424			
3	D	.333333	.300126	.305884	.304202	.304767	.304590
	E	.333333	.315320	.300126	.304684	.305085	
	F	.315320	.304690	.304450	.304608	.304591	
	C	.242308	.23641	.23663			
4	D	.25	.235312	.236907	.236599	.236668	.236652
	E	.25	.242308	.235312	.236566	.236784	
	F	.242308	.236546	.236638	.236655	.236652	
	C	.196040	.19269	.19281			
5	D	.2	.192312	.192873	.192798	.192810	.192808
	E	.2	.196040	.192312	.192746	.192845	
	F	.196040	.192741	.192807	.192808	.192808	
	C	.099501	.099022	.099029			
10	D	.1	.099010	.099029	.099029	.099029	.099029
	E	.1	.099501	.099010	.099025	.099029	
	F	.099501	.099025	.099029	.099029	.099029	

C) Ruben; D) $\mu\delta$-series; E) Δ-series; F) Laguerre-Gauss. $g_i(t)$ denotes the sum of the first i terms in each series.

The expansions for $R(t)$ when m is 1, 2, 3, 4, 5 are as follows:

m	Approximation to $R(t)$	Error less than
1	$t^{-1}e^{-1/2t^2}$	$1/2t^3$
2	$t^{-1}[.853553 \exp(-.171573/t^2) + .146447 \exp(-5.828429/t^2)]$	$1/2t^5$
3	$t^{-1}[.711093 \exp(-.086434/t^2) + .278518 \exp(-2.631860/t^2) + .010389 \exp(-19.78170/t^2)]$	$3/4t^7$
	$t^{-1}[.603154 \exp(-.052019/t^2) + .357419 \exp(-1.523841/t^2) + .038888 \exp(-10.2905/t^2) + .000539 \exp(-44.1337/t^2)]$	$3/2t^9$
5	$t^{-1}[.521756 \exp(-.034732/t^2) + .398667 \exp(-.998854/t^2) + .075942 \exp(-6.46714/t^2) + .003612 \exp(-25.1044/t^2) + .000023 \exp(-79.8949/t^2)]$	$15/4t^{11}$

The Laguerre-Gauss expansion for $m = 1$ is the same as the first term of Ruben's expansion.

In order to compare the precision of the Ruben, $\mu\delta$, Δ, and Laguerre-Gauss expansions, we have tabulated in Table 5 the various estimates, term by term, for a specimen set of values of t.

In overall terms the most precise series to use in approximating to $R(t)$ would appear to be the Laguerre-Gauss' expansion, but in terms of simplicity of form the $\mu\delta$-series would be reasonable. Ranking in terms of a combination of these criteria, Ruben's form might be placed third and the Δ-series a poor fourth.

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