ON THE SAMPLE SIZE AND COVERAGE FOR THE JIŘINA SEQUENTIAL PROCEDURE

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0. Summary. A short exposition of the use of the generalized Jiřina sequential tolerance limits procedure, and the necessary assumptions, is given. A method of obtaining the confidence level of the coverage for various values of the parameters from existing tables is included. Formulae for the expectation and variance of the random sample size are derived and usable approximations obtained. That the random sample size, appropriately scaled by one of the parameters, has an asymptotic distribution as that parameter increases is proved and the Laplace transform of this distribution is found. Also formulae for the asymptotic mean and variance are found and methods for their calculation exhibited. Some applications are given in the following paper.

1. Ordering the observations. Suppose that we have a finite (or countable) set of (measurable) functions $\{\omega_i\}$ mapping a sample space $\mathfrak{X}$ which we may think of as being a Euclidian n-space, into a space which is partially ordered by the relation $<$.

We define a balance $(\prec, \sim, \succ)$ on $\mathfrak{X}$ by taking $x \sim y$ iff $\omega_i(x) \nless \omega_i(y)$ and $\omega_i(x) \ngtr \omega_i(y)$ for $i = 1, 2, \cdots$ $x \prec y$ iff $\omega_j(x) < \omega_j(y)$ where j is the least index such that $\omega_i(x) \nless \omega_i(y)$ and $\omega_i(x) \ngtr \omega_i(y)$. Thus, a balance allows us to say for any two points x and y in $\mathfrak{X}$ exactly one of "x equivalent with y", "x less than y", or "y less than x" must hold.

We assume that the balance is *continuous* for a random variable (r.v.) X, that is, $\Pr[X \sim x] = 0$ for each $x \varepsilon \mathfrak{X}$. Any triplet of binary relations which are mutually exclusive and exhaustive for which the equivalence classes induced by one such relation are such that each has (probability) measure zero will do for our purposes. But the balance we have defined above fits most applications in which one has a qualitative way of deciding with probability one for any two sample values which is the worst (or best).

In the above definition of a balance we have followed that of Kemperman [4] in an earlier publication.

We describe the operation of the Jiřina procedure. Take η independent observations for the r.v. X, call them $X_1, \cdots, X_\eta$. During the first stage we determine an acceptable region, call it R_1, which is one of the statistically equivalent blocks.

During the jth stage $j = 2, 3, \cdots$ continue sampling as long as

$$X_{t+i} \, \varepsilon \, R_{j-1} \quad \text{and} \quad i < k, \tag{1.1}$$

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where t is the number of observations drawn during the preceding $(j - 1)$ stages. If (1.1) holds for $i = k$, terminate the procedure and set $D = R_{j-1}$. If $X_{t+i} \not\varepsilon R_{j-1}$ and $i \leqq k$ determine a new region from the ordered sample of $(t + i)$ observations, call it R_j, by omitting exactly η of the $(t + i + 1)$ statistically equivalent blocks in such a way that $R_{j-1} \subset R_j$. We then continue to do our sampling for the $(j + 1)$st stage.

It is known that this procedure terminates with probability one, call the region determined D, and if we let $Q = \Pr[X \varepsilon D]$ be the *coverage*, then

$$\Pr[Q > \beta] = (1 - \beta)^{\eta} \exp\left\{\eta \sum_{j=1}^{k} \beta^j/j\right\} \quad \text{for } 0 < \beta < 1. \tag{1.2}$$

In his original paper [2], Jiřina has shown that (1.2) holds for the usual ordering on the real line with the acceptable region the interval between the sth and the $(n + \eta - r - 1)$st order observations where $r + s = \eta$, i.e., we obtain upper and lower tolerance limits by omitting the upper r and the lower s blocks. However, the generalization that we have stated is immediate and is given in [5].

Jiřina's original paper [2] was in Russian, however, this paper has recently been translated and has appeared in [3].

2. Discussion of applications. The sequential tolerance region procedure for independent identically distributed random variables defined above (as well as others defined in [5]) are equivalent to standard non-sequential procedures in which the sample size is chosen at random, according to a specific distribution, before taking the sample. The procedures are equivalent in the sense that they have the same distribution of the coverage and of the sample size (see [5]). This is because for a given sample size the rank statistic which gives the order in which the set of observations appears and the order statistic which gives the values of the set of observations are independent.

This raises a question as to the usefulness of the Jiřina procedure for such sequences of independent identically distributed random variables unless there is some saving in either the expected sample size or a gain in coverage. It is known [2], [3] that for $\eta = 1$, (i.e., either an upper or lower tolerance limit using either the maximum or minimum observation) if the sample size required of the fixed sample is equal to the expected number of observations for the sequential procedure, then the coverage of the fixed sample is always greater. On the other hand for $\eta \geqq 2$, if β is sufficiently near 1 and the fixed sample size equals the expected sample size for the sequential procedure, then the probability that the coverage of the sequential procedure exceeds β is greater than the probability that the coverage of the fixed sample exceeds β. The question as to whether the β required by the theorem is in the range of β's used in application can be answered in each specific instance by the results of this paper.

Let us relax the assumption of identical distribution and assume there is a possible trend in the independent observations. Suppose we establish a tolerance region using the Jiřina procedure and, if no trend exists, we have a specified distribution of coverage and of sample size but, if there is a trend in the observations, the sequential procedure should stop the sampling earlier and increase the coverage in a stochastic sense. Under what conditions this is true is the subject of the following paper.

This area of application makes it of interest to study in detail the distribution of the sample size and the coverage of the Jiřina procedure and to provide tables for their use.

3. The distribution of coverage. In order to be able to determine the parameters necessary to obtain a specified coverage, i.e., to be able to determine η, k such that $\Pr[Q > \beta] \geqq 1 - \alpha$, where α is small, positive and is specified in advance, one must have available, in some form, values of (1.2). Thus one needs values of the function

$$\Lambda_k(\beta) = -\sum_{j=1}^{k} \beta^j/j - \ln(1-\beta) \qquad 0 < \beta < 1 \tag{3.1}$$

for the range of β and k of interest. While a short table of $\sum_{j=1}^{k} \beta^j/j$, at $\beta = .8$, .9, .95, .99 in suitable increments of k from 1 to 50, was included in Jiřina's paper, there does not appear any method for determining anything about the random sample size, its expectation say.

For a given tolerance estimation situation, i.e., for fixed η, β and α, the problem of determining the least k such that (3.1) is satisfied can be done by finding the least integer k such that

$$\Lambda_k(\beta) \leqq (-1/\eta) \ln(1-\alpha). \tag{3.2}$$

That this can be done with an error of at most one from existing tables [6] we now show by the

THEOREM 3.1. *For* $0 < \beta < 1$, $k = 1, 2, \cdots$ *we have*

$$\mathrm{Ei}\,[k \ln(1/\beta)] \leqq \Lambda_k(\beta) \leqq \mathrm{Ei}\,[(k+1)\ln(1/\beta)] \tag{3.2.1}$$

where Ei *is the exponential integral defined by*

$$\mathrm{Ei}\,(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \qquad \text{for} \quad x > 0. \tag{3.2.2}$$

PROOF. Since β^x/x is decreasing for $x > 1$ we have

$$\Lambda_{k-1}(\beta) = \sum_{j=k}^{\infty} \beta^j/j \leqq \int_k^\infty \frac{\beta^x}{x}\,dx \leqq \Lambda_k(\beta)$$

and by making the change of variable $t = x \ln(1/\beta)$ we obtain a double inequality equivalent with (3.2.1).

However, it is convenient to have $\Lambda_k(\beta)$ tabled in the range of interest and Table 1 gives values of $\beta = .8, .85, .9, .95, .99, .999$ and $k = 1(1)30(2)50(5)90$.

Recently J. C. Gower [1] has provided some asymptotic formulae for the

TABLE 1

Table of $\Lambda_k(\beta)$*

k	β					
	.80	.85	.90	.95	.99	.999
1	.809	1.047	1.403	2.046	3.615	5.909
2	.489	.686	.998	1.594	3.125	5.410
3	.319	.481	.755	1.309	2.802	5.077
4	.216	.351	.591	1.105	2.562	4.828
5	.151	.262	.472	.950	2.371	4.629
6	.107	.199	.384	.828	2.214	4.464
7	.772 (−1)	.153	.316	.728	2.081	4.322
8	.562 (−1)	.119	.262	.645	1.966	4.198
9	.413 (−1)	.935 (−1)	.219	.575	1.864	4.088
10	.306 (−1)	.738 (−1)	.184	.515	1.774	3.989
11	.228 (−1)	.586 (−1)	.155	.463	1.693	3.899
12	.170 (−1)	.467 (−1)	.132	.418	1.619	3.817
13	.128 (−1)	.374 (−1)	.112	.379	1.551	3.741
14	.966 (−2)	.301 (−1)	.959 (−1)	.344	1.489	3.670
15	.731 (−2)	.242 (−1)	.822 (−1)	.313	1.432	3.604
16	.555 (−2)	.196 (−1)	.706 (−1)	.286	1.379	3.543
17	.423 (−2)	.159 (−1)	.608 (−1)	.261	1.329	3.485
18	.323 (−2)	.129 (−1)	.524 (−1)	.239	1.283	3.431
19	.247 (−2)	.105 (−1)	.453 (−1)	.219	1.239	3.379
20	.189 (−2)	.858 (−2)	.392 (−1)	.201	1.198	3.330
21	.145 (−2)	.701 (−2)	.340 (−1)	.185	1.160	3.283
22	.112 (−2)	.574 (−2)	.295 (−1)	.170	1.123	3.239
23	.861 (−3)	.470 (−2)	.257 (−1)	.157	1.089	3.196
24	.664 (−3)	.386 (−2)	.224 (−1)	.145	1.056	3.156
25	.513 (−3)	.317 (−2)	.195 (−1)	.134	1.025	3.117
26	.397 (−3)	.261 (−2)	.170 (−1)	.124	.995	3.079
27	.307 (−3)	.215 (−2)	.149 (−1)	.114	.967	3.043
28	.238 (−3)	.177 (−2)	.130 (−1)	.106	.940	3.008
29	.185 (−3)	.146 (−2)	.114 (−1)	.981 (−1)	.914	2.975
30	.143 (−3)	.121 (−2)	.996 (−2)	.909 (−1)	.890	2.943
32	.867 (−4)	.825 (−3)	.765 (−2)	.783 (−1)	.843	2.881
34	.526 (−4)	.565 (−3)	.590 (−2)	.676 (−1)	.801	2.823
36	.320 (−4)	.389 (−3)	.456 (−2)	.584 (−1)	.761	2.769
38	.195 (−4)	.268 (−3)	.353 (−2)	.506 (−1)	.725	2.718
40	.119 (−4)	.185 (−3)	.274 (−2)	.439 (−1)	.691	2.669
42	.730 (−5)	.128 (−3)	.213 (−2)	.382 (−1)	.659	2.623
44	.448 (−5)	.888 (−4)	.166 (−2)	.333 (−1)	.629	2.579
46	.275 (−5)	.617 (−4)	.129 (−2)	.290 (−1)	.601	2.537
48	.169 (−5)	.429 (−4)	.101 (−2)	.253 (−1)	.575	2.496
50	.104 (−5)	.299 (−4)	.789 (−3)	.221 (−1)	.551	2.458
55	.314 (−6)	.122 (−4)	.429 (−3)	.159 (−1)	.495	2.368
60	.958 (−7)	.499 (−5)	.235 (−3)	.114 (−1)	.447	2.287
65	.299 (−7)	.206 (−5)	.129 (−3)	.830 (−2)	.405	2.212
70	.100 (−7)	.854 (−6)	.713 (−4)	.604 (−2)	.368	2.144
75	.396 (−8)	.356 (−6)	.396 (−4)	.441 (−2)	.335	2.080
80	.214 (−8)	.150 (−6)	.221 (−4)	.323 (−2)	.306	2.021
85	.161 (−8)	.638 (−7)	.123 (−4)	.238 (−2)	.279	1.965
90	.149 (−8)	.279 (−7)	.691 (−5)	.175 (−2)	.256	1.913

* Integers in parentheses are powers of 10 by which entry is to be multiplied.

evaluation of $\Lambda_k(\beta)$, the remainder term of the logarithmic series, which are useful when $k(1 - \beta)$ is small. In [1] he quotes an unpublished result of P. M. Grundy which is, in our notation,

$$\Lambda_{k-1}(\beta) = -\text{Ei}\,[k \ln (1/\beta)] + \beta^k/2k$$

with an error less than $(1 - k \ln \beta)\beta^k/12k^2$. This result coupled with Theorem (3.1) should enable one on most practical problems to solve (3.2) readily in cases where the table does not apply.

4. The first and second factorial moments of the sample size. A general result for any sequential tolerance procedure given in [5] allows the determination of the factorial moment generating function of the sample size from the distribution of coverage. When this result is applied to the Jiřina procedure, it yields

$$\mu_{\eta,k}(\beta) = \eta \int_0^\beta (\beta - t)^{\eta-1} t^k \exp\left\{\eta \sum_1^k t^j/j\right\} dt \tag{4.1}$$

where $\mu_{\eta,k}$ is the generating function for the procedure with parameters η and k.

Let us define $N_{\eta,k}$ and $M_{\eta,k}$ as the first and second factorial moments. We then obtain through successive differentiation:

$$N_{1,k} = \exp\left\{\sum_1^k 1/j\right\} \tag{4.1.1}$$

$$M_{1,k} = 2kN_{1,k}\,. \tag{4.1.2}$$

If $\eta \geqq 2$,

$$N_{\eta,k} = \eta(\eta - 1) \int_0^1 (1 - t)^{\eta-2} t^k \exp\left\{\eta \sum_1^k t^j/j\right\} dt. \tag{4.1.3}$$

If $\eta = 2$,

$$M_{2,k} = 2N_{1,k}^2\,. \tag{4.1.4}$$

If $\eta \geqq 3$,

$$M_{\eta,k} = 3!\binom{\eta}{3} \int_0^1 (1 - t)^{\eta-3} t^k \exp\left\{\eta \sum_1^k t^j/j\right\} dt. \tag{4.1.5}$$

Numerical integration of (4.1.3) using both a Gauss seven-point formula (and as a check Simpson's rule) for $\eta = 2, 3$ and $k = 1, \cdots, 25$ indicated (to the authors amazement) that $N_{\eta,k}$ was for fixed η nearly a linear function of k. Tabulation of $N_{1,k}$ indicated the same linear behavior. (See Figure 1.) The points computed are indicated by dots and a slight convexity for k small is not apparent in the drawing.

But, if $N_{\eta,k}$ is linear in k, then $M_{\eta,k}$ should be a quadratic in k. This is true for $M_{1,k}$ and $M_{2,k}$ by the expressions given. This suggests that one might get sufficiently accurate approximations for practical use by expanding in a Maclaurin's

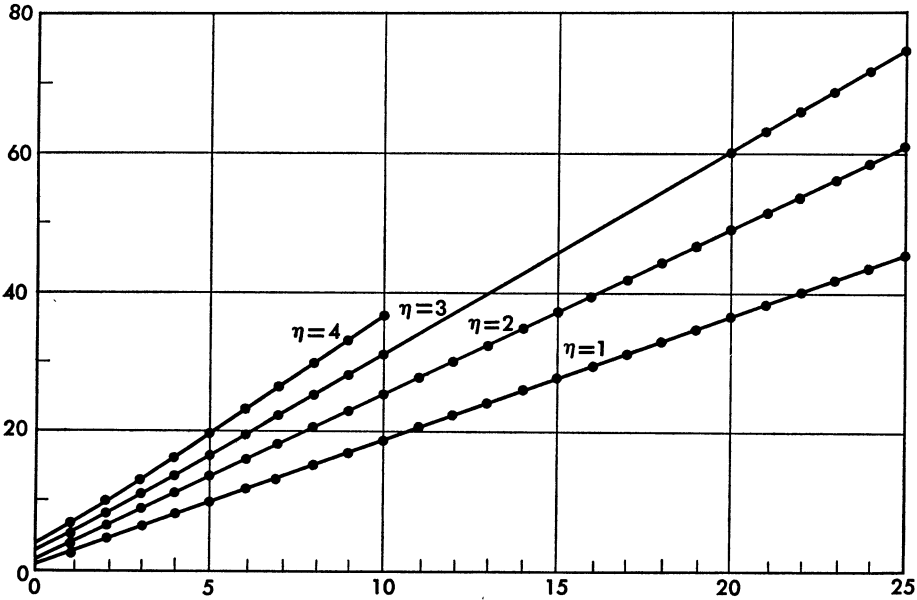

FIG. 1. Graph of the expected sample size of $N_{\eta,k}$.

series in k. To do this, substitute

$$\sum_{j=1}^{k} t^j/j = -\ln(1-t) - \int_0^t \frac{x^k}{1-x}\,dx \qquad 0 < t < 1 \tag{4.2}$$

and let this define a function for any real $k \geqq 0$. However, perhaps because of the singular behavior of the right hand side of the expression for $k = -1$, the circle of convergence was too small to allow such an expression in the region of interest.

What (4.2) does yield by substitution into (4.1) and simplifying is that

$$N_{\eta,0} = \eta \qquad \eta = 1, 2, \cdots . \tag{4.3}$$

5. The asymptotic behavior for k large. Let us define $N^*_{\eta,k}$ for $k = 1, 2, \cdots$ and $\eta = 2, 3, \cdots$ by

$$N^*_{\eta,k} = [N_{\eta,k}/\eta(\eta-1)] = \int_0^1 \frac{1}{(1-t)}\, e^{-\eta\Lambda_k(t)}\Lambda_k'(t)\,dt \tag{5.1}$$

where we have made use of the identity (4.2) and the definition (3.2) to set

$$\Lambda_k(t) = \int_0^t \frac{x^k}{1-x}\,dx, \qquad 0 < t < 1. \tag{5.2}$$

Then we also write

$$M^*_{\eta,k} = \int_0^1 \frac{1}{(1-t)^2} \exp\,[-\eta\Lambda_k(t)]\Lambda_k'(t)\,dt. \tag{5.3}$$

Knowing from the results of the numerical integration of (4.1.2), as commented upon in the preceding section, that the integral is approximately linear, one is led to the

THEOREM 5.1. *For any* $\eta = 2, 3, \cdots$ *we have*

$$\lim_{k\to\infty}(N^*_{\eta,k}/k) = \int_C^\infty [\exp\{-\eta\mathrm{Ei}(v) - v\}/v^2]\, dv = S_\eta \tag{5.4}$$

where Ei *is the exponential integral defined by* (3.2.2).

PROOF. Now

$$N^*_{\eta,k-1}/k) = \int_0^1 \frac{\exp[-\eta\Lambda_{k-1}(t)]kt^{k-1}}{[k(1-t)]^2}\, dt \tag{5.4.1}$$

and, if we put $t^k = e^{-v}$, then

$$N^*_{\eta,k-1}/k = \int_0^\infty \frac{\exp[-\eta\Lambda_{k-1}(e^{-v/k})]e^{-v}}{[k(1-e^{-v/k})]^2}\, dv. \tag{5.5}$$

But by the same change of variable we have

$$\Lambda_{k-1}(e^{-v/k}) = \int_v^\infty \frac{e^{-u}}{k(1-e^{-u/k})}\, du. \tag{5.5.1}$$

Since $k(1 - e^{-v/k}) \to v$, as $k \to \infty$ for each $v \geqq 0$, there remains only the justification for the interchange of limits. We will first show that for each $v > 0$

$$\lim_{k\to\infty} \Lambda_{k-1}(e^{-v/k}) = \mathrm{Ei}(v). \tag{5.5.2}$$

Let $h_k(u)$ be the integrand defined in the right hand side of Equation (5.5.1). Now $h_k(u) \to e^{-u}/u$ as $k \to \infty$. To see this, note $k(1 - e^{-u/k}) = \int_0^u e^{-t/k}\, dt$ is an increasing function of k and by the Lebesgue monotone convergence theorem (5.5.2) is proved.

We will now show that in Equation (5.5) upon taking the limit as $k \to \infty$, the limit may be passed under the integral sign. We shall use the Lebesgue dominated convergence theorem. One sees that

$$\frac{\exp[-\eta\Lambda_{k-1}(e^{-v/k})]}{[k(1-e^{-v/k})]^2} \leqq \frac{\exp[-\eta\mathrm{Ei}(v)]}{(1-e^{-v})^2}. \tag{5.5.3}$$

To show that the right hand side of (5.5.3) is an integrable function (with respect to e^{-v}), it is sufficient to show it is bounded near zero. To do this it is sufficient, since $\eta \geqq 2$ and $v/(1 - e^{-v}) \to 1$ as $v \to 0$, to show that $v^{-1}e^{-\mathrm{Ei}(v)}$ approaches a limit as $v \to 0$. We examine

$$\ln v + \mathrm{Ei}(v) = \int_v^1 \frac{e^{-x} - 1}{x}\, dx + \mathrm{Ei}(1) \tag{5.5.4}$$

for which the result is obvious.

We now derive a form more useful in calculation.

COROLLARY 5.1. *For* $\eta \geqq 2$

$$S_\eta = \int_0^\infty [\exp\{-\eta \mathrm{Ei}(v)\}/\eta v^2]\, dv. \tag{5.5.5}$$

PROOF. We obtain by (5.4) and the definition (5.2) that

$$S_\eta = \int_0^\infty -\frac{1}{v} \exp\{-\eta \mathrm{Ei}(v)\}\mathrm{Ei}'(v)\, dv \tag{5.5.6}$$

from which the result is obvious by integration by parts.

In essentially the same fashion we obtain the

THEOREM 5.2. *For any* $\eta = 3, 4, \cdots$ *we have*

$$\lim_{k\to\infty} M^*_{\eta,k}/k^2 = \int_0^\infty [\exp\{-\eta \mathrm{Ei}(v) - v\}/v^3]\, dv = T_\eta \tag{5.5.7}$$

where Ei is as defined in (3.2.2), and the

COROLLARY 5.2. *For* $\eta \geqq 3$

$$T_\eta = \frac{2}{\eta}\int_0^\infty [\exp\{-\eta \mathrm{Ei}(v)\}/v^3]\, dv, \tag{5.5.8}$$

which we state without proof.

From the point of view of application the results (5.4) and (5.5.7) suffer from the fact that for each different value of η asymptotic values T_η and S_η must be computed by numerical methods. A short table is given in Table 2.

TABLE 2

η	2	3	4	5	6	7	8
S_η	1.1380	.4775	.2792	.1894	.1401	.1095	.0890
T_η		1.4534	.5168	.2673	.1641	.1120	.0819

Before we proceed we see that the preceding results suggest that we may be able to obtain the asymptotic distribution of the sample size divided by the parameter k upon letting k tend to infinity. We have the

THEOREM 5.3. *If* $W_{\eta,k}$ *is the (random) sample size of the Jiřina procedure with parameter* η, k, *then the limit of the Laplace transform* $W_{\eta,k}/k$ *as* $k \to \infty$, *call it* φ, *is given by*

$$\varphi(t) = \eta \int_t^\infty \left(\frac{u-t}{u}\right)^{\eta-1} \frac{e^{-u}}{u} e^{-\eta \mathrm{Ei}(u)}\, du \qquad \text{for} \quad t > 0. \tag{5.6}$$

PROOF. We use the definition of Λ_k in (5.2) and let $t = \beta s$ in (4.1) to obtain

$$\mu_{\eta,k}(\beta) = \eta\beta^{\eta+k}\int_0^1 \frac{(1-s)^{\eta-1}s^k}{(1-\beta s)^\eta} \exp\{-\eta\Lambda_k(\beta s)\}\, ds.$$

If we replace β by $e^{-t/k}$ we obtain the Laplace transform of $W_{\eta,k}/k$, call it φ_k.

In the integrand make the transformation $s^k = e^{-v}$, and we obtain

$$\varphi_k(t) = \eta e^{-t(1+\eta/k)} \int_0^\infty \frac{(1 - e^{-v/k})^{\eta-1}}{[1 - \exp\{-(t + v)/k\}]^\eta} \frac{e^{-v/k}}{k} e^{-v-\eta\Lambda_k[\exp\{-(t+v)/k\}]} dv.$$

Now take the limit as $k \to \infty$. The interchange of limits is easily justified and we obtain

$$\varphi(t) = \eta e^{-t} \int_0^\infty \left(\frac{v}{t + v}\right)^{\eta-1} \frac{e^{-v}}{v + t} e^{-\eta \mathrm{Ei}(t+v)} dv.$$

Now we let $u = t + v$ to obtain the form given.

REMARK. Unfortunately the task of obtaining an explicit form for the inverse transform of (5.6) does not appear to be easy.

6. Some numerical comparisons. It is known that for $\eta = 1$, if any sequential tolerance procedure has an expected sample size equal to the sample size of the Wilks' fixed sample procedure, then the coverage of the Wilks' procedure, say Q_W, is stochastically greater than the coverage of the sequential procedure, say Q_S.

This strong a result is not true for $\eta \geqq 2$. In fact, we know that

$$P[Q_W > \beta] < P[Q_S > \beta] \tag{6.1.1}$$

for β in some interval $(\beta_0, 1)$ and the reverse inequality holds in (6.1.1) for β in the interval $(0, \beta_0)$. See [2], [3].

We now make some numerical comparisons between the Jiřina procedure with coverage Q_J and the Wilks' fixed sample procedure in the one-sided and two-sided cases, with $\eta = 1, 2$ respectively.

Now from (5.4) and (4.3) we make the approximation

$$N_{\eta,k} \cong \eta + \eta(\eta - 1)kS_\eta \qquad \eta \geqq 2 \tag{6.2}$$

and in addition from (4.1.1) we have by (4.3) that

$$N_{1,k} \cong 1 + ke^\gamma \tag{6.3}$$

where $e^\gamma = 1.78107 \cdots$ and γ is Euler's constant.

If we let $V_{\eta,k}$ be the variance of the sample size of the procedure with parameters η, k we have

$$V_{\eta,k} = M_{\eta,k} + N_{\eta,k} - N_{\eta,k}^2. \tag{6.4}$$

In particular

$$V_{1,k} \cong (2 - e^\gamma)k(1 + ke^\gamma). \tag{6.5}$$

Now applying (6.2) for $\eta = 2$, we have

$$N_{2,k} \cong 2 + 2kS_2. \tag{6.6}$$

From (4.1.4) we have

$$V_{2,k} = N_{1,k}^2 + N_{1,k} \cong 2 + 3ke^{\gamma} + k^2e^{2\gamma}. \tag{6.7}$$

Let us set $\eta = 1$. Then

$$P[Q_W > \beta] = 1 - \beta^n, \qquad P[Q_J > \beta] = e^{-\Lambda_k(\beta)}.$$

Now for fixed α we want the least integers k, and n such that

$$n \geqq \ln \alpha/\ln \beta, \qquad \Lambda_k(\beta) \leqq -\ln(1 - \alpha).$$

The first equation is easily solved and the second can be handled by Table 1. Let us pick $\alpha = .05$, $\beta = .9$, and we obtain $n = 29$ $k = 19$. Thus from (6.3) and (6.5) we find $N_{1,19} \cong 34.8$ $(V_{1,19})^{\frac{1}{2}} \cong 12.3$. Let us set $\eta = 2$. Then $P[Q_W > \beta] = 1 - \beta^n - n(1 - \beta)\beta^{n-1}$, $P[Q_J > \beta] = \exp[-2\Lambda_k(\beta)]$. Now for fixed α the problem of finding the least n such that $\beta^n + n(1 - \beta)\beta^{n-1} \leqq \alpha$ is probably best handled by the Birnbaum-Zuckerman method of [7]. Using this method and Table 1 for $\alpha = .02$, $\beta = .9$, we obtain $n = 56$ $k = 30$ and thus by (6.6) and (6.7) $N_{2,30} \cong 73.1$ and $(V_{2,30})^{\frac{1}{2}} \cong 54.9$. Thus in the last instance the sample size for the Jiřina procedure might well be two or three times the fixed sample required.

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