THE RELATION BETWEEN PITMAN'S ASYMPTOTIC RELATIVE EFFICIENCY OF TWO TESTS AND THE CORRELATION COEFFICIENT BETWEEN THEIR TEST STATISTICS[1]

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1. Introduction. It is well known (cf. e.g., Cramér [1], pp. 477–483) that, under certain regularity conditions, the efficiency of an unbiased estimate of a parameter relative to an efficient unbiased estimate is equal to the square of the correlation coefficient between the two estimates.

In this paper the relation between Pitman's asymptotic relative efficiency $e(T', T)$, of a test T' with respect to a test T, and the correlation coefficient between their test statistics, t' and t, will be considered.

If the test statistic of T is an efficient estimate of the underlying parameter and $g(t')$ is a consistent estimate of this parameter, then Noether [10] proved (cf. also Kendall and Stuart [6], Section 25.13 and Stuart [13]) that, under certain regularity conditions, the limiting correlation coefficient between t and $g(t')$ equals $[e(T', T]^{\frac{1}{2}}$. This result was used by Stuart ([14] and [15]) to find the asymptotic relative efficiency of several nonparametric tests for normal alternatives.

In this paper it will be shown that, under certain regularity conditions, the statistic t' itself has limiting correlation coefficient $[e(T', T)]^{\frac{1}{2}}$ with the test statistic t of a (in a later to be defined sense) best test T.

For the case of the two sample location problem Hájek [4] proved this relation between $[e(T', T)]^{\frac{1}{2}}$ and $\rho(t', t)$ for rank-orders tests. In his case T' (respectively T) is the locally most powerful rank-order test for testing $\theta = 0$ if the two samples are from distributions with distribution functions $F(x)$ and $F(x + \theta)$ (respectively $G(x)$ and $G(x + \theta)$).

The theorem will be stated and proved in Section 2; Section 3 contains some examples.

2. The theorem. Let T and T' be two tests for the hypothesis H_0: $\theta = \theta_0$ against the alternative $\theta > \theta_0$. Then the relative efficiency of T' with respect to T is the ratio n/n', where n and n' are the number of observations necessary to give T and T' the same power β for a given level of significance α. The concept of asymptotic relative efficiency is due to Pitman [11]. He considers the limit of n/n' for a sequence of alternatives depending on the sample size and converging to H_0 in such a way that the power of both tests converges to a limit <1. Pitman proved the following theorem (cf. e.g., Noether [9] and [10]).

Received March 19, 1963.

[1] This research was supported in part by the National Science Foundation under Grant Number G-19126 and by the National Institutes of Health under Grant Number 2G-43 (7 + 8).

Let T_n be a test for the hypothesis $H_0: \theta = \theta_0$ against the alternative $\theta > \theta_0$ based on n observations, let t_n be the test statistic and let

$$\begin{aligned} \psi_n(\theta) &\equiv E(t_n \mid \theta) \\ \sigma_n^2(\theta) &\equiv \sigma^2(t_n \mid \theta). \end{aligned} \tag{2.1}$$

Let further θ_n be a sequence of alternatives such that

$$\theta_n = \theta_0 + k/n^{\frac{1}{2}}, \tag{2.2}$$

where k is a positive, finite constant independent of n and let the following conditions be satisfied:

A. an ϵ exists such that, for $\theta_0 \leqq \theta \leqq \theta_0 + \epsilon$, $\psi_n'(\theta)$ exists,
B. $\lim_{n\to\infty} \psi_n'(\theta_n)/\psi_n'(\theta_0) = 1$,
C. $\lim_{n\to\infty} \sigma_n(\theta_n)/\sigma_n(\theta_0) = 1$,
D. $c \equiv \lim_{n\to\infty} \psi_n'(\theta_0)/n^{\frac{1}{2}}\sigma_n(\theta_0)$ exists,
E. the distribution of $[t_n - \psi_n(\theta)]/\sigma_n(\theta)$ tends to the normal distribution uniform in θ.

Condition E can be replaced by
E′. the distribution of $[t_n - \psi_n(\theta_n)]/\sigma_n(\theta_n)$ tends to the normal distribution.

Then Pitman proved

1. if T_n is a test satisfying the above conditions, the asymptotic power of T_n is $\phi(u_\alpha - kc)$, where

$$\phi(u) = (2\pi)^{-\frac{1}{2}} \int_u^\infty e^{-\frac{1}{2}x^2}\, dx, \tag{2.3}$$

2. if T_n and T_n' are two tests satisfying the above conditions with $c > 0$ and $c' > 0$ then the asymptotic relative efficiency $e(T', T)$ of T_n' with respect to T_n is

$$e(T', T) = (c'/c)^2. \tag{2.4}$$

Now let C be the class of all tests of H_0 satisfying the conditions A–E (or A-E') and suppose C contains a test, T_{n0} say, such that

(2.5)
1. for every given α and k no other test in C has a larger asymptotic power than T_{n0}; then $c_0 \geqq c$ for all $T_n \varepsilon C$
2. $c_0 > 0$.

A test T_{n0} satisfying (2.5) will be called a best test in C and the following theorem will be proved.

THEOREM. *If C contains a best test T_{n0} and $T_n \varepsilon C$ with*

(2.6)
1. *the simultaneous distribution of $[t_n - \psi_n(\theta)]/\sigma_n(\theta)$ and $[t_{n0} - \psi_{n0}(\theta)]/\sigma_{n0}(\theta)$ tends to a bivariate normal distribution uniform in θ or the simultaneous distribution of $[t_n - \psi_n(\theta_n)]/\sigma_n(\theta_n)$ and $[t_{n0} - \psi_{n0}(\theta_n)]/\sigma_n(\theta_n)$ tends to a bivariate normal distribution,*
2. $\lim_{n\to\infty} \rho(t_n, t_{n0} \mid \theta_0) = \lim_{n\to\infty} \rho(t_n, t_{n0} \mid \theta_n)$ $(= \rho$, say$)$,
3. $c > 0$

then

$$e(T, T_0) = \rho^2. \tag{2.7}$$

PROOF. Consider the tests $T_n(\lambda)$ based on the test statistic

$$(2.8) \qquad t_n(\lambda) = \lambda\,[t_{n0}/\sigma_{n0}(\theta_0)] + (1-\lambda)\,[t_n/\sigma_n(\theta_0)],$$

where λ is a constant independent of n. It is easily verified that $T_n \varepsilon C$ for all λ. Further

$$(2.9) \qquad c(\lambda) = [\lambda c_0 + (1-\lambda)c]/[\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\rho]^{\frac{1}{2}}.$$

From the fact that $T_n(\lambda) \varepsilon C$ for every λ and the fact that T_{n0} is a best test in C it follows that

$$(2.10) \qquad c(\lambda) \leqq c_0 \quad \text{for every } \lambda$$

and (2.10) is identical with

$$(2.11) \qquad \begin{aligned} &\lambda^2[c^2 - c_0^2 - 2c_0(c-\rho c_0)] - 2\lambda[c^2 - c_0^2 - c_0(c-\rho c_0)] \\ &\qquad\qquad + c^2 - c_0^2 \leqq 0 \quad \text{for every } \lambda. \end{aligned}$$

From (2.11) it follows that

$$(2.12) \qquad [c^2 - c_0^2 - c_0(c-\rho c_0)]^2 - (c^2 - c_0^2)\,[c^2 - c_0^2 - 2c_0(c-\rho c_0)] \leqq 0,$$

which is identical with $c_0^2(c-\rho c_0)^2 \leqq 0$ or, c_0 being positive,

$$(2.13) \qquad \rho = c/c_0$$

and from $c > 0$ and $c_0 > 0$ it then follows that

$$(2.14) \qquad \rho = c/c_0 = [e(T, T_0)]^{\frac{1}{2}}.$$

3. Examples.

3.1 *Tests for the hypothesis that the mean of a symmetric distribution is zero.* Let X be a random variable with a continuous symmetric distribution $F(x \mid \theta)$ with mean θ and let $x_1, \cdots, x_n$ be a sample from this distribution. Let $u_1, \cdots, u_n$ be the ordered absolute values of the observations; let, for $i = 1, \cdots, n$, v_i be defined by

$$(3.1.1) \qquad \begin{aligned} v_i &= 1 \text{ if } u_i \text{ corresponds to a positive observation,} \\ &= -1 \text{ if } u_i \text{ corresponds to a negative observation} \end{aligned}$$

and consider tests for $H_0 : \theta = 0$ based on test statistics of the form

$$(3.1.2) \qquad t_n = \sum_{i=1}^{n} a_i v_i,$$

where, for $i = 1, \cdots, n$, the weights a_i are given functions of i and n. Examples of test statistics of the form (3.1.2) are e.g.,

1. the sign test with $a_i = 1$ for $i = 1, \cdots, n$,
2. Wilcoxon's signed rank test with $a_i = i$ for $i = 1, \cdots, n$,
3. a test based on the statistic t_n with weights a_i satisfying

$$(3.1.3) \qquad \frac{2}{(2\pi)^{\frac{1}{2}}} \int_0^{a_i} e^{-\frac{1}{2}x^2}\,dx = \frac{i}{n+1} \qquad i = 1, \cdots, n.$$

So if $\psi(\alpha)$ is defined by $\alpha = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\psi(\alpha)} e^{-\frac{1}{2}x^2} dx$, then

$$a_i = \psi[(n+1+i)/(2n+2)] \qquad i = 1, \cdots, n. \tag{3.1.4}$$

This test is analogous to van der Waerden's two sample test; we will call this test van der Waerden's one sample test.

4. a test where a_i is the expected value of the ith order statistic of a sample of size n from a χ-distribution. This test, which is asymptotically identical with van der Waerden's one sample test (cf., e.g., Lehmann [8]), was considered by Fraser [3].

Another test for H_0 that will be considered is the test based on the sample mean

$$\bar{x}_n = n^{-1} \sum_{i=1}^{n} x_i = n^{-1} \sum_{i=1}^{n} u_i v_i . \tag{3.1.5}$$

The correlation coefficient under H_0 of two test statistics of the form (3.1.2) with weights a_i and a_i' respectively is

$$\rho(t_n, t_n' \mid H_0) = \left(\sum_{i=1}^{n} a_i a_i' \Big/ \left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} a_i'^2 \right]^{\frac{1}{2}} \right) \tag{3.1.6}$$

and the correlation coefficient under H_0 between a statistic of the form (3.1.2) and $\bar{x}_n$ is

$$\rho(t_n, \bar{x}_n \mid H_0) = \left(\sum_{i=1}^{n} a_i Eu_i \Big/ \sigma \left[n \sum_{i=1}^{n} a_i^2 \right]^{\frac{1}{2}} \right), \tag{3.1.7}$$

where σ^2 is the variance of X.

Two distributions $F(x \mid \theta)$ will be considered for X.

I. *X has a normal distribution with mean θ and variance* 1. Best tests in this case are the test based on the mean, van der Waerden's one sample test and Fraser's test. So the asymptotic relative efficiency of the sign test with respect to a best test may e.g. be found from (3.1.6) with $a_i = 1$ and $a_i' = \psi[(n+1+i)/(2n+2)]$ or from (3.1.7) with $a_i = 1$, $\sigma = 1$ and u_i is the ith order statistic of the absolute values of $x_1, \cdots, x_n$. From (3.1.6) we obtain (cf. also van Eeden and Benard [2])

$$e = \rho^2 = \lim_{n\to\infty} \frac{\left(\sum_{i=1}^{n} \psi\left(\frac{n+1+i}{2n+2} \right) \right)^2}{n \sum_{i=1}^{n} \left\{ \psi\left(\frac{n+1+i}{2n+2} \right) \right\}^2} = \frac{\left(\int_0^1 \psi\left(\frac{y+1}{2} \right) dy \right)^2}{\int_0^1 \left\{ \psi\left(\frac{y+1}{2} \right) \right\}^2 dy} = \frac{2}{\pi} \tag{3.1.8}$$

and from (3.1.7)

$$e = \rho^2 = \lim_{n\to\infty} \left(n^{-1} \sum_{i=1}^{n} Eu_i \right)^2 = \lim_{n\to\infty} \left(n^{-1} \sum_{i=1}^{n} E\,|x_i| \right)^2 = (E\,|X|)^2 = \frac{2}{\pi}. \tag{3.1.9}$$

For the Wilcoxon test the asymptotic relative efficiency with respect to a best test may e.g. be found from (3.1.6) with $a_i = i$, $a_i' = \psi[(n+1+i)/(2n+2)]$

$$(3.1.10)\quad e = \rho^2 = \lim_{n\to\infty} \frac{\left(\sum_{i=1}^{n} i\psi\left(\frac{n+1+i}{2n+2}\right)\right)^2}{\sum_{i=1}^{n} i^2 \sum_{i=1}^{n}\left\{\psi\left(\frac{n+1+i}{2n+2}\right)\right\}^2} = 3\frac{\left(\int_0^1 y\psi\left(\frac{y+1}{2}\right)dy\right)^2}{\int_0^1\left\{\psi\left(\frac{y+1}{2}\right)\right\}^2 dy} = \frac{3}{\pi}.$$

The asymptotic relative efficiency of the sign test with respect to the Wilcoxon test follows from (3.1.8) and (3.1.10): $e = \frac{2}{3}$. This efficiency is not equal to the square of the correlation coefficient between the test statistics. For the correlation coefficient we find from (3.1.6) (cf. also van Eeden and Bernard [2])

$$(3.1.11)\qquad \rho = \lim_{n\to\infty}\left[\sum_{i=1}^{n} i \Big/ \left(n \sum_{i=1}^{n} i^2\right)^{\frac{1}{2}}\right] = \tfrac{1}{2}3^{\frac{1}{2}}.$$

II. *X has a double exponential distribution with density* $\frac{1}{2}e^{-|x-\theta|}$. A best test in this case is the sign test (cf. Ruist [12] or Hoeffding and Rosenblatt [5]). So the asymptotic relative efficiency of van der Waerden's one sample test with respect to a best test follows from the correlation coefficient between the test statistics. This correlation coefficient is independent of the distribution of X, the two tests being simultaneously nonparametric. So the asymptotic relative efficiency of van der Waerden's one sample test with respect to the sign test is $2/\pi$ for a sample from a double exponential distribution, the same as the asymptotic relative efficiency of the sign test with respect to van der Waerden's one sample test for a sample from a normal distribution.

For the Wilcoxon test we find (cf. (3.1.11)) $e = \frac{3}{4}$ for the asymptotic relative efficiency with respect to a best test.

For the test based on the mean the asymptotic relative efficiency with respect to a best test follows from (3.1.7) with $a_i = 1$, σ^2 is the variance of a double exponential distribution with density $\frac{1}{2}e^{-|x-\theta|}$ and u_i is the ith order statistic of the absolute value of $x_1, \cdots, x_n$, so

$$(3.1.12)\qquad e = \rho^2 = \lim_{n\to\infty}\left(\sum_{i=1}^{n} Eu_i/n\sigma\right)^2 = (E\,|x|/\sigma)^2 = \tfrac{1}{2}.$$

Finally a test analogous to van der Waerden's one sample test will be considered, i.e., we choose the a_i such that (cf. (3.1.3))

$$(3.1.13)\qquad \int_0^{a_i} e^{-x}\,dx = \frac{i}{n+1} \quad\text{or}\quad a_i = ln\,\frac{n+1}{n+1-i} \qquad i = 1, \cdots, n.$$

The asymptotic relative efficiency of this test with respect to a best test follows from (3.1.6) with $a_i = 1$ and $a_i' = \ln[(n+1)/(n+1-i)]$

$$(3.1.14)\quad e = \rho^2 = \lim_{n\to\infty}\frac{\left(\sum_{i=1}^{n}\ln\frac{n+1}{n+1-i}\right)^2}{n\sum_{i=1}^{n}\left\{\ln\frac{n+1}{n+1-i}\right\}^2} = \frac{\left(\int_0^1 \ln(1-x)\,dx\right)^2}{\int_0^1\{\ln(1-x)\}^2\,dx} = \tfrac{1}{2},$$

so the test with weights (3.1.13) has the same efficiency as the test based on the mean. The same holds for the normal distribution: van der Waerden's one sample test has the same efficiency as the test based on the mean.

REMARK. As already remarked earlier in this section van der Waerden's one sample test and Fraser's test are asymptotically identical; their asymptotic relative efficiency is 1 for a sample from any distribution and their correlation coefficient is asymptotically 1. So these tests are, for a sample from, e.g. a double exponential distribution, an example of a case where the relation $e = \rho^2$ holds and none of the two tests is a best test.

An example will now be given of this situation with $e \neq 1$. Let $T_{n,1}$ and $T_{n,2}$ be two tests in C with $0 < c_2 \leqq c_1$,

$$\rho \equiv \lim_{n\to\infty} \rho(t_{n,1}, t_{n,2} \mid \theta_0) = \lim_{n\to\infty} \rho(t_{n,1}, t_{n,2} \mid \theta_n) \tag{3.1.15}$$

and $\rho \neq c_2/c_1$. Consider the test based on the statistic

$$t_n(\lambda) = \lambda\,[t_{n1}/\sigma_{n1}(\theta_0)] + (1-\lambda)\,[t_{n2}/\sigma_{n2}(\theta_0)] \tag{3.1.16}$$

and choose $\lambda = \lambda_0$ in such a way that $c(\lambda)$ is a maximum. Then

$$\begin{aligned} \lambda_0 &= (c_1 - \rho c_2)/(1-\rho)\,(c_1 + c_2) \\ c(\lambda_0) &> 0 \\ [c(\lambda_0)]^2 &= c_1^2 + (c_2 - \rho c_1)^2/(1-\rho^2) > c_1^2. \end{aligned} \tag{3.1.17}$$

So, if $T_n(\lambda_0) \,\varepsilon\, C$, $T_n(\lambda_0)$ has higher efficiency than T_{n1} and T_{n2}. Further

$$\begin{aligned} \lim_{n\to\infty} &\rho^2(t_n(\lambda_0), t_{n1} \mid \theta_0) \\ &= [\lambda_0 + (1-\lambda_0)\rho]^2/[\lambda_0^2 + (1-\lambda_0)^2 + 2\lambda_0(1-\lambda_0)\rho] \\ &= (c_1/c(\lambda_0))^2 = e(T_1, T(\lambda_0)). \end{aligned} \tag{3.1.18}$$

So $T_{n,1}$ and $T_n(\lambda_0)$ are two tests for which the relation $e = \rho^2$ holds; that $T_n(\lambda_0)$ is not necessarily a best test follows from the following example; let $x_1, \cdots, x_n$ be a sample from a normal distribution with known variance and let $T_{n,1}$ and $T_{n,2}$ respectively be the Wilcoxon and the sign test. Then $c_1 = (3/\pi)^{\frac{1}{2}}$, $c_2 = (2/\pi)^{\frac{1}{2}}$ and $\rho = \frac{1}{2}3^{\frac{1}{2}}$ (cf. (3.1.11)). So $\rho \neq c_2/c_1$ and

$$[c(\lambda_0)]^2 = 3/\pi + 4\,[(2/\pi)^{\frac{1}{2}} - 3/2\pi^{\frac{1}{2}}]^2 = [20 - 12\,(2)^{\frac{1}{2}}]/\pi < 1. \tag{3.1.19}$$

A best test in this case has $c_0 = 1$, so $T_n(\lambda_0)$ is not a best test.

3.2 *Tests for the hypothesis that two distributions are identical.* Let X and Y be two independent random variables with distribution functions $F(x)$ and $G(y)$ respectively. Let $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ be two samples from F and G respectively, let (with $N = m + n$) $z_1, \cdots, z_N$ be the ordered observations in the pooled samples and let

$$\begin{aligned} v_i &= 1 \quad \text{if } z_i \text{ is an observation of } X \\ &= 0 \quad \text{if } z_i \text{ is an observation of } Y. \end{aligned} \tag{3.2.1}$$

The tests to be considered for the hypothesis H_0 that F and G are identical have statistics of the form

$$t_N = \sum_{i=1}^{N} a_i(v_i - m/N), \tag{3.2.2}$$

where, for $i = 1, \cdots, N$, the weights a_i are given functions of i and N. Examples of tests of the form (3.2.2) are:

1. Wilcoxon's two sample test with $a_i = i$
2. van der Waerden's two sample test with $a_i = \psi[i/(N+1)]$,
3. the test of Ansari and Bradley with $a_i = |i/(N+1) - \frac{1}{2}|$,
4. Mood's test with $a_i = (i/(N+1) - \frac{1}{2})^2$
5. a test with $a_i = \{\psi[i/(N+1)]\}^2$. In this case the a_i correspond to applying the technique of van der Waerden to Mood's test.

The Examples 1 and 2 are both tests for location; 3, 4, and 5 are, if it is assumed that X and Y have the same median, tests for scale. Another test for location is the test based on the difference between the sample means

$$\bar{x} - \bar{y} = m^{-1} \sum_{i=1}^{m} x_i - n^{-1} \sum_{j=1}^{n} y_j = (N/mn) \sum_{i=1}^{N} z_i(v_i - m/N). \tag{3.2.3}$$

The correlation coefficient under H_0 between two statistics of the form (3.2.2) is

$$\rho(t_N, t_N' \mid H_0) = \frac{\sum_{i=1}^{N} a_i a_i' - N^{-1} \sum_{i=1}^{N} a_i \sum_{i=1}^{N} a_i'}{\left[\left\{\sum_{i=1}^{N} a_i^2 - N^{-1}\left(\sum_{i=1}^{N} a_i\right)^2\right\}\left\{\sum_{i=1}^{N} a_i'^2 - N^{-1}\left(\sum_{i=1}^{N} a_i'\right)^2\right\}\right]^{\frac{1}{2}}} \tag{3.2.4}$$

and the correlation coefficient under H_0 between a statistic of the form (3.2.2) and $\bar{x} - \bar{y}$ is

$$\rho(t_N, \bar{x} - \bar{y} \mid H_0) = \frac{\sum_{i=1}^{N} a_i Ez_i - N^{-1} \sum_{i=1}^{N} a_i \sum_{i=1}^{N} a_i'}{(N-1)^{\frac{1}{2}}\sigma\left[\sum_{i=1}^{N} a_i^2 - N^{-1}\left(\sum_{i=1}^{N} a_i\right)^2\right]^{\frac{1}{2}}}, \tag{3.2.5}$$

where σ^2 is the (common) variance of X and Y.

For the tests for location we will consider the case where X and Y both have a logistic distribution with equal known variance and difference between means θ; for the tests for scale we will consider the case where X and Y both have normal distributions with equal known means and ratio of variance θ.

I. *X and Y have a logistic distribution.* Let X and Y have distribution functions

$$F(x) = 1/(1 + e^{-(x-\theta_1)}) \qquad G(y) = 1/(1 + e^{-(y-\theta_2)}) \tag{3.2.6}$$

respectively. A best test for testing $H_0 : \theta_1 - \theta_2 = 0$ is the Wilcoxon two sample test (cf. Lehmann [8]).

So the asymptotic relative efficiency of the van der Waerden test with respect to the Wilcoxon test for samples from logistic distributions is $3/\pi$, the same as the asymptotic relative efficiency of the Wilcoxon test with respect to the van der Waerden test for samples from normal distributions, the two tests being simultaneously nonparametric and the van der Waerden test being a best test in the case of a normal distribution.

For the test based on the difference between the sample means the asymptotic relative efficiency with respect to a best test follows from (3.2.5) with $a_i = i$

$$(3.2.7)\quad e = \rho^2 = \lim_{N\to\infty} \frac{\left[\sum_{i=1}^{N} iEz_i - EX \sum_{i=1}^{N} i\right]^2}{(N-1)\sigma^2\left(\sum_{i=1}^{N} i^2 - \frac{1}{N}\left(\sum_{i=1}^{N} i\right)^2\right)} = \frac{36}{\pi^2}\lim_{N\to\infty}\left(\frac{1}{N^2}\sum_{i=1}^{N} iEz_i\right)^2,$$

where

$$(3.2.8)\quad \begin{aligned}\sum_{i=1}^{N} i\, Ez_i &= N\, EX + N(N-1)\, EXF(X)\\ &= N(N-1)\, EXF(X) = \tfrac{1}{2}N(N-1).\end{aligned}$$

So from (3.2.7) and (3.2.8) we obtain $e = \rho^2 = 9/\pi^2$.

We now consider a test analogous to van der Waerden's test, i.e., we choose a_i such that

$$(3.2.9)\quad 1/(1 + e^{-a_i}) = i/(N+1) \quad \text{or} \quad a_i = \ln[i/(N+1-i)].$$

This test has (analogous to the case of the normal and the exponential distribution) the same efficiency as the test based on the difference between the sample means. Its efficiency with respect to a best test follows from (3.2.4) with $a_i = i$ and $a_i' = \ln[i/(N+1-i)]$

$$(3.2.10)\quad \begin{aligned} e = \rho^2 &= \lim_{N\to\infty} \frac{\left[\sum_{i=1}^{N} i \ln\frac{i}{N+1-i} - N^{-1}\sum_{i=1}^{N} i \sum_{i=1}^{N} \ln\frac{i}{N+1-i}\right]^2}{\left\{\sum_{i=1}^{N} i^2 - N^{-1}\left(\sum_{i=1}^{N} i\right)^2\right\} \cdot \left\{\sum_{i=1}^{N}\left\{\ln\frac{i}{N+1-i}\right\}^2 - N^{-1}\left\{\sum_{i=1}^{N}\ln\frac{i}{N+1-i}\right\}^2\right\}} \\ &= 12\,\frac{\left(\int_0^1 x\ln[x/(1-x)]\,dx\right)^2}{\int_0^1 (\ln[x/(1-x)])^2\,dx} = 9/\pi^2. \end{aligned}$$

II. *X and Y have a normal distribution.* Let X and Y have normal distributions with mean zero and variances θ_1 and θ_2 respectively. Best tests for testing $H_0 : \theta_1/\theta_2 = 1$ are the F-test, based on the ratio of the sample variances and the test of the form (3.2.2) with $a_i = (\psi[i/(N+1)])^2$ (cf. Klotz [7]).

So for the test of Ansari and Bradley the asymptotic relative efficiency with respect to a best test is found from (3.2.4) with $a_i = (\psi[i/(N+1)])^2$ and $a_i' = |i/(N+1) - \frac{1}{2}|$

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(3.2.11)\qquad e = \rho^2 = \lim_{N\to\infty} \frac{\left[\sum_{i=1}^{N}\left|\frac{i}{N+1} - \frac{1}{2}\right|\left\{\psi\left(\frac{i}{N+1}\right)\right\}^2 - N^{-1}\sum_{i=1}^{N}\left|\frac{i}{N+1} - \frac{1}{2}\right|\sum_{i=1}^{N}\left\{\psi\left(\frac{i}{N+1}\right)\right\}^2\right]^2}{\left[\sum_{i=1}^{N}\left\{\psi\left(\frac{i}{N+1}\right)\right\}^4 - N^{-1}\left\{\sum_{i=1}^{N}\left(\psi\left(\frac{i}{N+1}\right)\right)^2\right\}^2\right]\cdot\left[\sum_{i=1}^{N}\left(\frac{i}{N+1} - \frac{1}{2}\right)^2 - N^{-1}\left(\sum_{i=1}^{N}\left|\frac{i}{N+1} - \frac{1}{2}\right|\right)^2\right]}
$$

$$
= \frac{\left[\int_0^1 |x - \frac{1}{2}|\{\psi(x)\}^2\,dx - \int_0^1 |x - \frac{1}{2}|\,dx \int_0^1 \{\psi(x)\}^2\,dx\right]^2}{\left[\int_0^1 \{\psi(x)\}^4\,dx - \left\{\int_0^1 \{\psi(x)\}^2\,dx\right\}^2\right]\cdot\left[\int_0^1 (x - \frac{1}{2})^2\,dx - \left\{\int_0^1 |x - \frac{1}{2}|\,dx\right\}^2\right]} = \frac{6}{\pi^2}.
$$

For Mood's test we find from (3.2.4) with $a_i = \{\psi[i/(N+1)]\}^2$ and $a_i' = ([i/(N+1)] - \frac{1}{2})^2$ $e = \rho^2 = 15/2\pi^2$.

4. Acknowledgment. The author wants to thank the referee for his helpful suggestions and references.

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