EXCHANGEABLE PROCESSES WHICH ARE FUNCTIONS OF STATIONARY MARKOV CHAINS[1]

By S. W. Dharmadhikari

University of California, Berkeley

Let $\{V_n, -\infty < n < \infty\}$ be a stochastic process and let $W_n = \{V_k, -\infty < k \leqq n\}$. Then the W-process is a Markov process and the V-process is a function of the W-process. The W-process is stationary if, and only if, the V-process is stationary. Thus every stationary process is a function of a stationary Markov process. The W-process has a uncountable state-space even when the V-process has only two states. It is thus of some interest to isolate stationary processes which are functions of stationary Markov chains with a countable number of states. The present note solves this problem for exchangeable processes.

Let $\{Y_n, n \geqq 1\}$ be an exchangeable process (see [2], p. 365) with a countable state-space J. States of J will be denoted by δ and finite sequences of states of J will be denoted by s. For a sequence s of length n let $p(s) = P[(Y_1, \cdots, Y_n) = s]$. Let Q denote the space of all probability distributions on J. Each $q \varepsilon Q$ is then defined by a sequence $\{q(\delta), \delta \varepsilon J\}$ of non-negative real numbers which add up to 1.

From de Finetti's work [1] we know that $\{Y_n\}$ is a mixture of sequences of independent and identically distributed random variables with values in J. In our notation, this means that, if $s = \delta_1 \cdots \delta_n$, then

$$p(s) = \int_Q q(\delta_1) \cdots q(\delta_n)\, d\mu(q), \tag{1}$$

when μ is a probability measure on the Borel sets in Q. The measure μ is uniquely determined by the probability function p.

Theorem. *An exchangeable process $\{Y_n\}$ with a countable state-space J is a function of a stationary countable-state Markov chain if, and only if, it is a countable mixture of sequences of independent and identically distributed random variables with values in J.*

Proof. Let the measure μ be discrete that is, concentrated on a countable subset $\{q_\nu, \nu \geqq 1\}$ of Q. Let $a_\nu = \mu(\{q_\nu\})$. Then (1) can be written as

$$p(s) = \sum_\nu a_\nu q_\nu(\delta_1) \cdots q_\nu(\delta_n). \tag{2}$$

The state-space I of the underlying Markov chain is then defined as $I = \{(\nu, \delta) \mid \delta \varepsilon J, \nu \geqq 1\}$. Let M_ν be the square matrix with all its rows equal to q_ν and let M be the direct sum of the M_ν's. The probability measure $\mathbf{m}$ on I which gives a mass $a_\nu q_\nu(\delta)$ to the state (ν, δ) is a stationary initial distribution for M. Finally, let f be the function on I into J defined by $f[(\nu, \delta)] = \delta$.

Received 26 March 1963.

[1] Prepared with the partial support of the National Science Foundation, Grant GP-10.

Now (2) shows easily that if $\{X_n\}$ is a stationary Markov chain with state-space I, initial distribution $\mathbf{m}$ and transition matrix M, then $\{f(X_n)\}$ and $\{Y_n\}$ have the same distribution. This proves the "if" part.

Conversely, suppose that $\{Y_n\}$ is a function f of a stationary Markov chain $\{X_n\}$ with transition matrix M and initial distribution $\mathbf{m}$. Since $\{X_n\}$ is stationary, all its states can be assumed to be recurrent. Since $\{Y_n\}$ is exchangeable, we would get the same function process if, instead of M, we take as the transition matrix any power M^k or any average $(1/n)\sum_{k=1}^{n} M^k$ or even the limit $M^* = \lim_{n\to\infty} (1/n)\sum_{k=1}^{n} M^k$. But M^* must be a direct sum of matrices M_ν, $\nu \geqq 1$, such that each M_ν has identical rows. By lumping together states which have the same functional value under f we can assume, without loss of generality, that M_ν has all its rows equal to some $q_\nu \varepsilon Q$. The states corresponding to M_ν can be numbered (ν, δ), $\delta \varepsilon J$ and each (ν, δ) must receive a probability $a_\nu q_\nu(\delta)$ in $\mathbf{m}$.

Now (2) follows easily. The uniqueness of μ implies that μ is concentrated on a countable subset of Q. This proves the "only if" part and completes the proof of the theorem.

It should be remarked that if, throughout the statement of the theorem, we replace the word "countable" by the word "finite", we still get a true theorem.

REFERENCES

[1] DE FINETTI, BRUNO (1937). La prévision: ses lois logiques, ses sources subjectives. *Ann. Inst. Henri Poincaré* **7** 1–68.

[2] LOÈVE, MICHEL (1960). *Probability Theory*, 2nd ed. Van Nostrand, Princeton.