

A LIMIT LAW CONCERNING MOVING AVERAGES¹

BY L. A. SHEPP

University of California, Berkeley and Bell Telephone Laboratories

0. Introduction. This note deals with the limiting values of the averages, $T_n = (S_{n+f(n)} - S_n)/f(n)$ where $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \cdots$, is a sum of mutually independent and identically distributed random variables. The function f takes positive integer values and nondecreasingly tends to infinity. We shall determine the almost everywhere constant $\limsup T_n$ in terms of the moment generating function of X_1 and the radius of convergence of $\sum x^{f(n)}$, denoted r .

If we define N_n as the number of *consecutive* successes beginning at trial $n + 1$ in a sequence of Bernoulli trials with success probability p , we let $U_n = N_n/f(n)$. It is shown that, almost surely, $\limsup U_n = \log_p r$. This result will be compared with $\limsup T_n$ for the case of Bernoulli trials. The author is indebted to Professor D. J. Newman for drawing his attention to this problem.

1. Statement of results. The events $\{T_n > c$ infinitely often (i.o.) $\}$ are tail events, $-\infty < c < \infty$, and so $T = \limsup T_n$ is constant with probability one ([3], p. 229).

We form the series $\sum x^{f(n)} = \sum A_k x^k$ and determine its radius of convergence r , $0 \leq r \leq 1$. The constant $T = T(f)$ will depend only on r , and when this is to be emphasized, will be denoted T_r .

Let $M(t) = Ee^{X_1 t} \leq +\infty$ be defined for all t , and set [1], for any a , $m(a) = \min M(t)e^{-at} \leq M(0) = 1$. If $M(t) < \infty$ for some $t > 0$, then $EX_1^+ < \infty$, and so EX_1 is well defined, $-\infty \leq EX_1 < \infty$. Let $\text{ess sup } X_1 = A \leq +\infty$. We assume X_1 is nonconstant a.s., so $EX_1 < A$.

THEOREM 1. *If $M(t) < \infty$ for some $t > 0$, then*

$$(1.1) \quad T_r = A \quad \text{for } 0 \leq r \leq P\{X_1 = A\}.$$

$$(1.2) \quad T_r = a_r \quad \text{for } P\{X_1 = A\} < r < 1$$

where $a = a_r$ is the unique solution (inverse) of $m(a) = r$, $EX_1 < a < A$.

$$(1.3) \quad T_r = EX_1 \quad \text{for } r = 1.$$

Moreover, T_r decreases strictly and continuously for $P\{X_1 = A\} \leq r \leq 1$.

The consideration of the sequence $-X_i$ leads to a similar theorem for $\liminf T_n$. If f is strictly increasing then $r = 1$. The existence of $\lim T_n$ is assured if and only if $r = 1$, whence $\lim T_n = EX_1$ a.s., $-\infty < EX_1 < \infty$. Proofs of these assertions are in Section 3. Our methods break down in case $M(t) = \infty$ for all $t > 0$, and this case is not discussed.

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An immediate corollary of this theorem is that if f_1 and f_2 are two functions, as above, and $f_1(n) \leq f_2(n)$, $n = 1, 2, \dots$, then $T(f_2) \leq T(f_1)$. A class of f in central position are $f_c(n) = 1 + [c \log n]$, $c > 0$, $n = 1, 2, \dots$, whence $r = \exp(-1/c)$.

2. Chernoff's theorem. It is easily seen that $\log P\{S_n \geq na\}$ is subadditive in n for fixed a , and hence $\lim - (1/n) \log P\{S_n \geq na\}$ exists. Chernoff [1], has found this limit in terms of $m(a)$. We suppress the subscript of X_1 in this discussion.

THEOREM 2. (Chernoff) *We suppose $M(t) < \infty$ for some $t > 0$. We have $-\infty \leq EX < \infty$, and $m(a) = \min_{t \geq 0} Ee^{(X-a)t} < 1$, if $a > EX$. Moreover, for $n = 1, 2, \dots$*

$$(2.1) \quad P\{S_n \geq na\} \leq (m(a))^n,$$

and for $0 < \epsilon < m(a)$, and $n > n_0(\epsilon)$,

$$(2.2) \quad P\{S_n \geq na\} > (m(a) - \epsilon)^n.$$

This is a slight variant of [1], and we should complete the proof in [1] by noting that for $0 < t < \delta$, some $\delta > 0$, $r(t) = M(t)e^{-at}$ is convex, and so $r'(t) = E(X - a)e^{(X-a)t}$ decreases to $E(X - a) < 0$ as t decreases to zero (dominated convergence). It follows that $r'(t) < 0$, $0 < t < \delta_1$, some $\delta_1 > 0$, and so $r(t) = r(0^+) + \int_0^t r'(u) du < r(0^+) = 1$, $0 < t < \delta_1$, whence $m(a) < 1$. To prove the validity of the statement $m(a) = \min_{t \geq 0} Ee^{(X-a)t}$ note that this is trivial if $M(t) = \infty$ for all $t < 0$, and if $M(t) < \infty$ for some $t < 0$ then $r'(0) = EX - a < 0$ and so the minimum occurs to the right of zero, by convexity.

If $A = \text{ess sup } X = \infty$, set $m(\infty) = 0$; if $EX = -\infty$, set $m(-\infty) = 1$. The function $m(a)$ is then defined for extended real numbers in the interval $EX \leq a \leq A$.

(2.3) **LEMMA.** *If $M(t) < \infty$ for some $t > 0$, then for $EX \leq a \leq A$, $m(a)$ is continuous and strictly monotonic. Moreover, $m(EX) = 1$.*

PROOF. For $h > 0$ and $EX < a < a + h < A$, we have by the first part of Theorem 2,

$$(2.4) \quad m(a + h) = \min_{t \geq 0} Ee^{(X-a)t} e^{-ht} < m(a).$$

Continuity of $m(a)$ at finite a , $EX \leq a < A$ is easily shown using the fact that $m(a) = Ee^{(X-a)t_1}$ for $0 \leq t_1 < \infty$. If $A < \infty$, $\epsilon > 0$, we set $t = \epsilon^{-\frac{1}{2}}$, obtaining

$$(2.5) \quad m(A - 0) = \lim_{\epsilon \downarrow 0} \min_t Ee^{(X-A+\epsilon)t} \leq \lim_{\epsilon \downarrow 0} E \exp [(X - A)\epsilon^{-\frac{1}{2}} + \epsilon^{\frac{1}{2}}].$$

It follows that $m(A - 0) \leq P\{X = A\} = m(A)$. However, for $\epsilon > 0$,

$$(2.6) \quad m(A - \epsilon) \geq \min_{t \geq 0} Ee^{(X-A)t} \min_{t \geq 0} e^{\epsilon t} = m(A).$$

Thus m is continuous at A , if $A < \infty$. If $A = \infty$, then for some fixed $t > 0$,

$$(2.7) \quad 0 \leq \lim_{a \rightarrow \infty} m(a) \leq \lim_{a \rightarrow \infty} Ee^{(X-a)t} = 0 = m(\infty).$$

If $EX = -\infty$, we have $1 \geq \lim_{a \rightarrow -\infty} m(a) \geq \lim_{a \rightarrow -\infty} P\{X \geq a\} = 1 = m(-\infty)$. This proves the continuity assertions. If $EX > -\infty$, then $M'(0^+) = EX$, and $m(EX) = 1$ follows. This proves (2.3).

3. Proof of Theorem 1. We shall need the following simple lemma for power series. Here f is a function of the type described in the introduction.

(3.1) LEMMA. *If $0 < x < y < 1$, $b > 0$ and $\sum x^{f(n)} < \infty$, then $\sum y^{f(n_k)} = \infty$ where $[bf(n_k)] \geq 1$ and for $k = 1, 2, \dots$,*

$$(3.2) \quad n_{k+1} = n_k + [bf(n_k)].$$

PROOF. Choose K so that for $n \geq n_K$,

$$(3.3) \quad (x/y)^{f(n)} f(n) < 1.$$

Since f is *nondecreasing*, we have for $k = 1, 2, \dots$,

$$(3.4) \quad \sum_{j=0}^{n_{k+1}-n_k-1} y^{f(n_k+j)}/f(n_k+j) \leq y^{f(n_k)}(n_{k+1}-n_k)/f(n_k) \leq by^{f(n_k)}.$$

Using (3.3) and the fact that $n_k \rightarrow \infty$ (since $[bf(n_k)] \geq 1$ in (3.2)), we get

$$(3.5) \quad \sum_{k=K}^{\infty} y^{f(n_k)} \geq b^{-1} \sum_{k=K}^{\infty} \sum_{j=n_k}^{n_{k+1}-1} y^{f(j)}/f(j) \geq b^{-1} \sum_{n \geq n_K} x^{f(n)} = \infty,$$

proving (3.1).

To prove Theorem 1, let $A_n(a)$ be the event $\{T_n \geq a\}$. Since we are dealing with identically distributed random variables, we have

$$(3.6) \quad P\{A_n(a)\} = P\{S_{f(n)} \geq af(n)\}.$$

In proving (1.1), we assume first $A < \infty$. If $r \leq m(A)$, then for $h > 0$, sufficiently small, $m(A-h) > r$ and we choose $\epsilon > 0$ so that $\alpha = m(A-h) - 2\epsilon > r$. Since $\sum \alpha^{f(n)} = \infty$, we may apply (3.1) with $b = 1$, $x = \alpha$, $y = \alpha + \epsilon$. For the sequence n_k of (3.1), we have $\sum y^{f(n_k)} = \infty$. By (2.2) and (3.6), $P\{A_n(A-h)\} \geq y^{f(n)}$, for n sufficiently large. It follows that $\sum P\{A_{n_k}(A-h)\} = \infty$. Since A_{n_k} depends only on that segment of $\{X_n\}$ for which $n_k < n \leq n_k + f(n_k) = n_{k+1}$, and these segments are disjoint, the events A_{n_k} are independent, $k = 1, 2, \dots$. By Borel's lemma ([3], p. 228), $P\{A_{n_k}(A-h) \text{ i.o.}\} = 1$ and so $T \geq A-h$. Since $T_n \leq A$, a.s. for each n , we obtain $T = A$. If now $A = \infty$ and $r \leq m(A)$, then $r = 0$. Since $m(a) > 0$ for finite $a > EX_1$ we have for $2\epsilon < m(a)$, $\sum P\{A_n(a)\} \geq \sum (m(a) - \epsilon)^{f(n)} = \infty$ and arguing as above, $P\{A_{n_k}(a) \text{ i.o.}\} = 1$. Hence $T \geq a$ for every a , and $T = \infty = A$. This proves (1.1).

If $m(A) < r < 1$, then there is a unique a , $EX_1 < a < A$, for which $m(a) = r$ by (2.3). We shall show $T_r = a$. Let $h > 0$ be sufficiently small in the following. Using (2.1) we have, since $m(a+h) < m(a) = r$,

$$(3.7) \quad \sum P\{A_n(a+h)\} \leq \sum (m(a+h))^{f(n)} < \infty.$$

By Cantelli's lemma ([3], p. 228) we have $T_r \leq a + h$ so $T_r \leq a$. On the other hand, by (2.2), $P\{A_n(a - h)\} \geq (m(a - h) - \epsilon)^{f(n)}$ for $0 < \epsilon < m(a - h)$. If ϵ is so small that $m(a - h) - 2\epsilon > m(a) = r$, then $\sum (m(a - h) - 2\epsilon)^{f(n)} = \infty$. By (3.1), again with $b = 1$, we obtain a sequence n_k for which $\sum P\{A_{n_k}(A - h)\} \geq \sum (m(a - h) - \epsilon)^{f(n_k)} = \infty$. As before we conclude that $T_r \geq a - h$ and so $T_r = a$ and (1.2) is proved. To prove (1.3) assume that $y > EX_1$. By Theorem 2, $m(y) < 1$ and so

$$(3.8) \quad \sum P\{A_n(y)\} \leq m(y)^{f(n)} < \infty,$$

since $r = 1$. It follows that $T_1 \leq EX_1$. The weak law of large numbers, ([2], p. 228) shows that for $E|X_1| < \infty$ and $\epsilon > 0$, $P\{S_n > (EX_1 - \epsilon)n\} \rightarrow 1$. Thus $P\{A_n(EX_1 - \epsilon)\} \rightarrow 1$ and so for $0 \leq r \leq 1$, $T_r \geq EX_1$. We have $T_1 = EX_1$ in any case and (1.3) is proved. In this last case we have $\lim T_n$ existing, since

$$(3.9) \quad \liminf T_n = -\limsup -T_n = -E(-X_1) = EX_1 = \limsup T_n$$

almost surely. In (3.9) we have tacitly assumed that $M(t) < \infty$ also for some $t < 0$ and so $E|X_1| < \infty$ here. Thus Theorem 1 and the assertions following are proved.

4. An example. As an illustration of Theorem 1 we consider the Bernoulli case, $P\{X_1 = 1\} = p = 1 - P\{X_1 = 0\}$ for $0 < p < 1$. We have for $p \leq a \leq 1$,

$$(4.1) \quad m(a) = (p/a)^a((1 - p)/(1 - a))^{1-a}$$

and thus, for $p \leq r \leq 1$, $T_r = a$ where $m(a) = r$, and for $0 \leq r \leq p$, $T_r = 1$.

5. A related problem on success runs. If we set $N_n = k$ if $X_{n+1} = \dots = X_{n+k-1} = 1$ and $X_{n+k} = 0$; $k = 1, 2, \dots$, in the example, then N_n counts the waiting time until the first failure following the n th Bernoulli trial (success runs [2] p. 197, Example 5, D. J. Newman). If f is a function of the type considered above, we set $U_n = N_n/f(n)$, $n = 1, 2, \dots$ and $U = \limsup U_n$. It will be shown that U is a constant a.s., depending on f only through r , the radius of convergence of $\sum x^{f(n)}$.

(5.1) COROLLARY. For $0 \leq r \leq 1$, $U = \limsup U_n = \log_p r$.

PROOF. Let $B_n(b)$ denote the event $\{U_n > b\}$, $0 < b$ and $n = 1, 2, \dots$. We have $P\{B_n(b)\} = p^{\lfloor bf(n) \rfloor + 1}$, and if $p^b < r$,

$$(5.2) \quad \sum P\{B_n(b)\} \leq \sum p^{bf(n)} < \infty.$$

It follows that $P\{B_n(b) \text{ i.o.}\} = 0$ and that $U \leq \log_p r$. If $p^b > r$, then finding an $\epsilon > 0$ for which $p^{b+\epsilon} > r$, $\sum p^{(b+\epsilon)f(n)} = \infty$. Using (3.1) with $x = p^{b+\epsilon}$, $y = p^b$ we obtain a sequence n_k for which

$$(5.3) \quad \sum p^{bf(n_k)} = \infty$$

$$(5.4) \quad n_{k+1} = n_k + \lfloor bf(n_k) \rfloor.$$

The event $B_{n_k}(b)$ refers to the section of $\{X_n\}$ for $n_k < n \leq n_k + \lfloor bf(n_k) \rfloor = n_{k+1}$

and thus $B_{n_k}(b)$ are independent, $k = 1, 2, \dots$. Moreover, $\sum P\{B_{n_k}(b)\} \geq p \sum p^{b/(n_k)} = \infty$ by (5.3). It follows that $U \geq b$ and so $U \geq \log_p r$. This proves (5.1).

Comparing T and U in the Bernoulli case, it is easily seen that whenever $U < 1$, we have $U_n \leq T_n$ eventually and so $U \leq T \leq 1$. If $T = a$ and $U = b$ we have the relationship, from (4.1) and (5.1)

$$(5.5) \quad (p/a)^a ((1-p)/(1-a))^{1-a} = p^b = r$$

for $p \leq a \leq 1$ and $0 \leq b \leq 1$.

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