MULTIVARIATE COMPETITION PROCESSES[1]

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Summary. A multivariate competition process (M.C.P.) is a stationary, continuous time Markov process whose state space is the lattice points of the positive orthant in N-dimensional space and whose transition probability matrix only allows jumps to certain nearest neighbors. As such it is the natural generalization of birth and death processes. In this paper we extend the results of Reuter [15] to obtain sufficient conditions for a M.C.P. to be regular, positive recurrent, absorbed with certainty, and to have finite mean absorption time. Some explicit examples are given and references to various applications indicated.

1. Introduction. A M.C.P. is a standard, stationary, continuous time Markov process[2] with state space $E = \{(i_1, \cdots, i_N): i_1, \cdots, i_N = 0, 1, \cdots\}$, where $N \geqq 2$, and transition probability functions[3] $\{p_{ij}(t): i, j \varepsilon E, t \geqq 0\}$ for which $p'_{ij}(0) = q_{ij}$. Furthermore, for $i = (i_1, \cdots, i_N)$

$$\begin{aligned} q_{ij} &= s(i) & j &= (i_1, \cdots, i_N) \\ &= \lambda_k(i) & j &= (i_1, \cdots, i_{k-1}, i_k + 1, i_{k+1}, \cdots, i_N) \\ &= \mu_k(i) & j &= (i_1, \cdots, i_{k-1}, i_k - 1, i_{k+1}, \cdots, i_N) \\ &= \gamma_{kl}(i) & j &= (i_1, \cdots, i_{k-1}, i_k - 1, i_{k+1}, \cdots, i_{l-1}, i_l + 1, i_{l+1}, \cdots, i_N) \\ &= 0 & & \text{other } j, \end{aligned}$$

where $k, l = 1, \cdots, N$; $k \neq l$; and $s(i) = -\{\sum_{k=1}^N [\lambda_k(i) + \mu_k(i)] + \sum_{k,l=1; k \neq l}^N \gamma_{kl}(i)\}$. We shall assume that $\mu_k(i) = \gamma_{kl}(i) = 0$ when $i_k = 0$ and that $\lambda_k(i)$, $\mu_k(i)$, and $\gamma_{kl}(i)$ are nonnegative. Also we assume that $\sum_{j=1}^N \lambda_j(i)$ and $\sum_{j=1}^n \mu_j(i)$ are positive when the state i is nonabsorbing with the exception that $\sum_{j=1}^N \mu_j(i) = 0$ when i is the origin. The elements $\lambda_k(i)$ allow a birth of one in the kth coordinate, $\mu_k(i)$ a death of one in the kth coordinate, and $\gamma_{kl}(i)$ a mutation of one from the kth to the lth coordinate. Clearly, a M.C.P. is conservative and its states are stable. Following Reuter [14] we call $\{p_{ij}(t)\}$ a **Q**-process and $\{q_{ij}\}$ the **Q**-matrix.

Received 16 August 1963.

[1] Research sponsored by Office of Naval Research contract Nonr-401(48).

[2] See Chung [1] for a definition of a standard Markov process and other terminology associated with Markov processes.

[3] This definition extends Reuter's [15] definition of a competition process in two dimensions to one in N dimensions. For the case $N = 2$, see [2] for an integral representation of $\{p_{ij}(t)\}$.

In general, there will be many processes possessing the same **Q**-matrix. If the **Q**-matrix uniquely determines the process, we call the **Q**-matrix and the associated process regular. Probabilistically the regularity condition is equivalent to either the process taking only a finite number of jumps in finite time or that almost all sample functions are step functions (cf., Chung [1], p. 237). In many practical applications the only data given for the problem will be a **Q**-matrix. Hence it is important to have a condition for regularity of the **Q**-matrix stated in terms of the infinitesimal parameters q_{ij}.

If regularity of the **Q**-matrix can be established, one can then ask questions about the recurrence, transience, and absorption properties of the process. This is the type of program we shall attempt to carry out for M.C.P.'s. The results of this paper represent an extension of the work of Reuter [15]. In fact most of the results for these more general processes make use of his methods. In Section 2 a sufficient condition for regularity of a M.C.P. is given. Section 3 is devoted to a sufficient condition for irreducible M.C.P.'s to be positive recurrent. M.C.P.'s with absorbing states are introduced in Section 4 and a sufficient condition for absorption with probability one is given. Section 5 gives a sufficient condition for the mean absorption time to be finite. Actually the conditions obtained in Sections 2–5 apply to a slightly more general class of processes. Finally, in Section 6 we give a number of specific examples of M.C.P.'s, indicate a few qualitative properties of the processes, and give some references to applications of M.C.P.'s.

2. Regularity. For conservative **Q**-matrices Reuter ([14], Theorem 7) has obtained the following necessary and sufficient condition for regularity:

LEMMA 1 (Reuter). *Let* **Q** *be conservative, and consider the set of equations*

$$(\lambda + q_i)\xi_i = \sum_{\alpha \varepsilon E, \alpha \neq i} q_{i\alpha}\xi_\alpha . \tag{1}$$

Each of the following conditions is necessary and sufficient in order that there be only one **Q**-*process.*

(a) *For some* $\lambda > 0$, (1) *has no bounded solution other than* $\xi_i \equiv 0$ (*all* $i \varepsilon E$)

(b) *For some* $\lambda > 0$, (1) *has no bounded non-negative solution other than* $\xi_i \equiv 0$ (*all* $i \varepsilon E$).

To verify (a) or (b) for a specific M.C.P. is in general a very difficult matter. However, it is possible to obtain a sufficient condition for regularity by appealing to a simple probabilistic notion. The idea of the proof is to "bound" in an appropriate sense our M.C.P. by one-dimensional birth and death processes. For birth and death processes a simple necessary and sufficient condition for regularity has been obtained by Karlin and McGregor [4]. Since a process can only fail to be regular by "going to infinity in finite time," if we can "bound" our M.C.P. by a regular birth and death process which is moving "toward infinity" at a faster rate than the M.C.P., this will yield a sufficient condition for regularity of the M.C.P.

Before proceeding to the theorem, we shall need to introduce certain definitions and results from the theory of birth and death processes. (For a compre-hensive discussion of birth and death processes see the two papers by Karlin and McGregor [4] and [5].) A birth and death process is a one-dimensional analog of a M.C.P. The state space is the nonnegative integers and the **Q**-matrix is of the form

$$\begin{aligned} q_{ij} &= s_i && (j = i) \\ &= \lambda_i && (j = i + 1) \\ &= \mu_i && (j = i - 1) \\ &= 0 && (\text{other } j), \end{aligned}$$

where $i = 0, 1, \cdots; s_i = -(\lambda_i + \mu_i); \lambda_i, \mu_i > 0$ for $i = 1, 2, \cdots$; and $\mu_0 \geqq 0$. For $s > 0$ let $Q_i(-s)$ be the solution of the set of equations

$$\begin{aligned} &Q_0(-s) \equiv 1 \\ (2)\qquad &sQ_0(-s) = s_0Q_0(-s) + \lambda_0Q_1(-s) \\ &sQ_i(-s) = s_iQ_i(-s) + \lambda_iQ_{i+1}(-s) + \mu_iQ_{i-1}(-s) \qquad (i \geqq 1). \end{aligned}$$

Notice that the set of equations (2) is exactly that of (1) for $\lambda = s$ and $\xi_0 = 1$. We define a new sequence $\pi = \{\pi_i\}$ in terms of the $\{\lambda_i\}$ and $\{\mu_i\}$ by

$$\begin{aligned} (3)\qquad &\pi_0 = 1 \\ &\pi_i = (\lambda_0 \cdots \lambda_{i-1})/(\mu_1 \cdots \mu_i) \qquad (i \geqq 1). \end{aligned}$$

The following lemma attributed to Stieltjes by Karlin and McGregor was proved in [4], p. 504.

LEMMA 2. *A necessary and sufficient condition for the sequence* $\{Q_n(-s)\}$ *to be unbounded as a function of n is that the series*

$$(4)\qquad \sum_{n=0}^{\infty} (1/\lambda_n\pi_n) \sum_{i=0}^{n} \pi_i \quad \textit{is divergent.}$$

Thus combining Lemmas 1 and 2 we see that (4) is necessary and sufficient for a conservative ($\mu_0 = 0$) birth and death process to be regular.

Next we return to the M.C.P. and define a sequence of finite subsets of E and two sequences of constants. Let A denote the set of absorbing states $i \varepsilon E$ $(q_i) = 0)$. For $k = 0, 1, \cdots$ we let

$$\begin{aligned} &E_k = \left\{ i = (i_1, \cdots, i_N) : i \varepsilon E - A \quad \text{and} \quad \sum_{j=1}^{N} i_j = k \right\}; \\ (5)\qquad &\lambda_k = \max_{i \varepsilon E_k} \left\{ \sum_{j=1}^{N} \lambda_j(i) \right\}; \quad \text{and} \\ &\mu_k = \min_{i \varepsilon E_k} \left\{ \sum_{j=1}^{N} \mu_j(i) \right\}. \end{aligned}$$

If E_k is empty, λ_k and μ_k are not defined. With this preparation we can proceed to

THEOREM 1. *Let l_0 denote the smallest integer l for which E_l is nonempty. Then a sufficient condition for regularity of the* M.C.P. *is that*

$$\sum_{k=l_0}^{\infty} (1/\lambda_k \sigma_k) \sum_{i=l_0}^{k} \sigma_i \quad \textit{diverge, where} \tag{6}$$

$\sigma_{l_0} = 1$ *and* $\sigma_{l_0+i} = (\lambda_{l_0} \cdots \lambda_{l_0+i-1})/(\mu_{l_0+1} \cdots \mu_{l_0+i})$ for $i \geqq 1$.

PROOF. Let $s > 0$ and let $\mathbf{P}(-s) = \{P_i(-s)\}$ be any nonnegative, nonvanishing solution of the equation

$$\mathbf{QP}(-s) = s\mathbf{P}(-s), \tag{7}$$

where $\mathbf{Q} = \{q_{ij}\}$. Writing out this equation for $i = (i_1, \cdots, i_N)$ we obtain

$$\begin{aligned} &\left\{ s + \sum_{j=1}^{N} [\lambda_j(i) + \mu_j(i)] + \sum_{\substack{j,l=1\\ j\neq l}}^{N} \gamma_{jl}(i) \right\} P_i(-s) \\ &= \sum_{j=1}^{N} \lambda_j(i) P_{(i_1,\cdots,i_j+1,\cdots,i_N)}(-s) + \sum_{j=1}^{N} \mu_j(i) P_{(i_1,\cdots,i_j-1,\cdots i_N)}(-s) \\ &\qquad + \sum_{\substack{j,l=1\\ j\neq l}}^{N} \gamma_{jl}(i) P_{(i_1,\cdots,i_j-1,\cdots i_l+1,\cdots,i_N)}(-s). \end{aligned} \tag{8}$$

Let $Q_k^*(-s) = \max_{i \varepsilon E_k \cap A} \{P_i(-s)\}$. If E_k is empty, $Q_k^*(-s) = 0$ since $P_i(-s) = 0$ when $i \varepsilon A$ by virtue of (7). If we let $i(k) \varepsilon E_k \cup A$ denote the state for which $Q_k^*(-s) = P_{i(k)}(-s)$, the following inequality follows from (8).

$$\begin{aligned} &\left\{ s + \sum_{j=1}^{N} [\lambda_j(i(k)) + \mu_j(i(k))] + \sum_{\substack{j,l=1\\ j\neq l}}^{N} \gamma_{jl}(i(k)) \right\} Q_k^*(-s) \\ &\leqq \sum_{j=1}^{N} \lambda_j(i(k)) Q_{k+1}^*(-s) + \sum_{j=1}^{N} \mu_j(i(k)) Q_{k-1}^*(-s) + \sum_{\substack{j,l=1\\ j\neq l}}^{N} \gamma_{jl}(i(k)) Q_k^*(-s). \end{aligned} \tag{9}$$

Combining terms and rearranging (9) yields

$$\begin{aligned} &\sum_{j=1}^{N} \lambda_j(i(k)) [Q_{k+1}^*(-s) - Q_k^*(-s)] \\ &\qquad \geqq \sum_{j=1}^{N} \mu_j(i(k)) [Q_k^*(-s) - Q_{k-1}^*(-s)] + sQ_k^*(-s). \end{aligned} \tag{10}$$

Since $\mathbf{P}(-s)$ is not identically zero, there exists an integer k_0 which is the smallest k such that $Q_k^*(-s) > 0$. This implies that $i(k_0) \not\varepsilon A$ and by assumption $\sum_{j=1}^{N} \lambda_j(i(k_0))$ and $\sum_{j=1}^{N} \mu_j(i(k_0))$ are positive. Clearly, $k_0 \geqq l_0$. By assumption $\mathbf{P}(-s)$ [and therefore $Q_k^*(-s)$] is nonnegative. Thus from (10) we can easily show inductively that $Q_k^*(-s) - Q_{k-1}^*(-s) > 0$ and $i(k) \not\varepsilon A$ for $k \geqq k_0$.

Hence using the definition of λ_k and μ_k we obtain from (10) the inequality

$$\lambda_k[Q^*_{k+1}(-s) - Q^*_k(-s)] \geqq \mu_k[Q^*_k(-s) - Q^*_{k-1}(-s)] + sQ^*_k(-s) \tag{11}$$

for $k \geqq k_0$.

Let the polynomials $\{Q_k(-s)\}$ satisfy the equations

$$Q_{k_0-1}(-s) = 0$$

$$Q_{k_0}(-s) = Q^*_{k_0}(-s)$$

$$sQ_k(-s) = -(\lambda_k + \mu_k)Q_k(-s) + \lambda_k Q_{k+1}(-s) + \mu_k Q_{k-1}(-s) \quad (k \geqq k_0).$$

We now wish to show inductively that

(i) $Q^*_k(-s) \geqq Q_k(-s) \qquad (k \geqq k_0)$; and

(ii) $Q^*_k(-s) - Q^*_{k-1}(-s) \geqq Q_k(-s) - Q_{k-1}(-s) \qquad (k \geqq k_0)$.

For $k = k_0$, the inequalities (i) and (ii) are trivially true. Assume (i) and (ii) are true for $k = K$. Then

$$\begin{aligned} \lambda_K Q_{K+1}(-s) &= (s + \lambda_K)Q_K(-s) + \mu_K[Q_K(-s) - Q_{K-1}(-s)] \\ &\leqq (s + \lambda_K)Q^*_K(-s) + \mu_K[Q^*_K(-s) - Q^*_{K-1}(-s)] \\ &\leqq \lambda_K Q^*_{K+1}(-s), \end{aligned}$$

where the last inequality is obtained from (11). This shows (i) is true for $k = K + 1$. On the other hand,

$$\begin{aligned} \lambda_K[Q_{K+1}(-s) - Q_K(-s)] &= \mu_K[Q_K(-s) - Q_{K-1}(-s)] + sQ_K(-s) \\ &\leqq \mu_K[Q^*_K(-s) - Q^*_{K-1}(-s)] + sQ^*_K(-s) \\ &\leqq \lambda_K[Q^*_{K+1}(-s) - Q^*_K(-s)] \end{aligned}$$

which proves (ii).

Using (i) and Lemma 2 it is easy to show that (6) is a sufficient condition for $\{P_i(-s)\}$ to be unbounded as a function of i. Thus appealing to Lemma 1 we see that (6) is a sufficient condition for regularity of the M.C.P.

Remarks. Notice that the parameters $\{\gamma_{jl}(i)\}$ do not appear in the regularity condition, (6). Hence in applications we need not be concerned with the form of these mutation parameters as far as regularity is concerned. Furthermore, if the proof of this theorem is carefully examined, it is clear that we could in fact introduce additional transitions from the state $i \varepsilon E_k$ to any other state $j \varepsilon E_k \cup A$ without changing the result. Thus (6) will serve as a regularity condition for a class of processes much larger than simply M.C.P.'s.

3. Positive recurrence. In this section we shall only consider regular irreducible M.C.P. Our objective will be to obtain a sufficient condition for the process to be positive recurrent (i.e., recurrent with finite mean recurrence time).

The proof of the theorem giving a sufficient condition for positive recurrence follows the method of Reuter ([15], p. 424). In obtaining his result Reuter first proves the following general lemma.

LEMMA 3 (Reuter). *Suppose the* **Q**-*matrix is regular and irreducible. If there exists a state I and a finite sequence* $\{u_i : i \varepsilon E\}$ *such that*

(a) $u_i \geqq 0$,

(b) $\sum_{j\varepsilon E} q_{ij}u_j + 1 \leqq 0 \quad (i \neq I)$, *and*

(c) $\sum_{j\varepsilon E} q_{ij}u_j \quad (i = I)$, *is finite,*

then the process is positive recurrent.

Using this result we obtain

THEOREM 2. *A sufficient condition for a regular, irreducible* M.C.P. *to be positive recurrence is that*

$$\sum_{j=0}^{\infty} \sigma_j \quad \textit{converge, where} \tag{12}$$

the sequence $\{\sigma_j\}$ *is defined in Theorem* 1 *in terms of the* $\{\lambda_k\}$ *and* $\{\mu_k\}$ *of* (5).

PROOF. We shall apply Lemma 3 for $I = (0, \cdots, 0)$. Condition (c) will clearly be satisfied for a finite sequence $\{u_i\}$, since only a finite number of the $\{q_{Ij}\}$ are non-zero. Let $u_i = U_k$ for $i \varepsilon E_k$. If $\{u_i\}$ is to satisfy condition (6), we must have

$$\sum_{j=1}^{N} \lambda_j(i)[U_{k+1} - U_k] + 1 \leqq \sum_{j=1}^{N} \mu_j(i)[U_k - U_{k-1}] \qquad (i \neq I). \tag{13}$$

Furthermore, if we can choose the U_k increasing, inequality (13) will be implied by

$$\lambda_k[U_{k+1} - U_k] + 1 \leqq \mu_k[U_k - U_{k-1}] \qquad (k \geqq 1).$$

Following Reuter we let $U_{k+1} - U_k = V_k$, let $V_0 \geqq 0$ be arbitrary, and define $V_k (k \geqq 1)$ recursively by $\lambda_k V_k + 1 = \mu_k V_{k-1}$ $(k \geqq 1)$, so that

$$V_k = (\lambda_0/\sigma_k)\{V_0 - (1/\lambda_0)(\sigma_1 + \sigma_2 + \cdots + \sigma_k)\} \qquad (k \geqq 1).$$

Since $\sum_{k=1}^{\infty} \sigma_k < \infty$ by hypothesis, V_0 can be chosen sufficiently large so that $V_k \geqq 0$ $(k \geqq 1)$. Now if we choose $U_0 \geqq 0$, the sequence $\{V_k\}$ will determine a sequence $\{U_k\}$ satisfying the conditions of Lemma 3.

REMARKS. Again as in Theorem 2 the mutation parameters $\{\gamma_{jl}(i)\}$ do not play a role in the condition for positive recurrence. Hence the same remarks that follow Theorem 1 could also be made in this case. It is also interesting to observe that the sufficient condition (12) is also a necessary and sufficient condition for the regular birth and death process with parameters $\{\lambda_k\}$ and $\{\mu_k\}$ to be positive recurrent (cf. [5]).

4. Absorption with certainty. We now consider regular M.C.P.'s possessing a nonempty set of absorbing states A. We denote the set of nonabsorbing states $E - A$ by the letter D. The number ρ_{ij} for $i \varepsilon D$ and $j \varepsilon A$ is the probability that the process starting in state i will ultimately be absorbed in state j. Thus $\alpha_i = \sum_{j\varepsilon A} \rho_{ij}$ for $i \varepsilon D$ is the probability of the process being absorbed in set A having started in state i. We know that either $\alpha_i < 1$ for all $i \varepsilon D$ or $\alpha_i = 1$ for all $i \varepsilon D$

(cf., Kendall and Reuter, [11]). Our objective is to find a sufficient condition for $\alpha_i = 1$ for all $i \varepsilon D$.

Again we follow Reuter's method. We shall apply

LEMMA 4 (Reuter, [15]). *Suppose the Q-matrix is regular. If there exists a sequence* $\{u_i : i \varepsilon E\}$ *such that*

(a) $u_i \geqq 0 \quad (i \varepsilon E)$,

(b) *the* $\{u_i\}$ *are unbounded as a function of* i, *and*

(c) $\sum_{j \varepsilon E} q_{ij} u_j \leqq 0$ *for all* $i \varepsilon E$, then

$$\sum_{j \varepsilon E} p_{ij} = 1 \quad \textit{for all } i \varepsilon E.$$

Using Lemma 4 we obtain

THEOREM 3. *For a regular* M.C.P. *assume there exists a nonnegative integer* l_0 *such that* E_l *is empty for* $l < l_0$ *and* E_l *is nonempty for* $l \geqq l_0$. *Then a sufficient condition for the process to be nondissipative* ($\alpha_i = 1$ *for* $i \varepsilon D$) *is that*

$$\sum_{j=l_0}^{\infty} 1/\lambda_j \sigma_j \quad \textit{diverge,} \tag{14}$$

where $\{\lambda_j\}$ *and* $\{\mu_j\}$ *are defined in* (5).

PROOF. We shall construct a sequence $\{u_i\}$ satisfying (a)-(c) of Lemma 4. Let $u_i = U_k$ for $i \varepsilon E_k \cup A$. Observe that (c) will be satisfied for any sequence $\{u_j\}$ when $i \varepsilon A$. Hence we need only be concerned about (c) for $i \varepsilon D$. Condition (c) for $i \varepsilon E_k$ requires that

$$\left[\sum_{j=1}^{N} \lambda_j(i) + \sum_{j=1}^{N} \mu_j(i) + \sum_{\substack{j,l=1 \\ j \neq l}}^{N} \gamma_{jl}(i)\right] U_k$$
$$\geqq \sum_{j=1}^{N} \lambda_j(i) U_{k+1} + \sum_{j=1}^{N} \mu_j(i) U_{k-1} + \sum_{\substack{j,l=1 \\ j \neq l}}^{N} \gamma_{jl}(i) U_k \qquad (k \geqq l_0).$$

If we let $U_{l_0-1} = 0$, $U_{l_0} = 1$, and define U_k recursively by

$$\mu_k(U_k - U_{k-1}) = \lambda_k(U_{k+1} - U_k) \qquad (k > l_0),$$

then

$$U_k = 1 + \mu_{l_0} \sum_{j=l_0}^{k-1} 1/\lambda_i \sigma_j \qquad (k > l_0).$$

Thus if (14) holds, then the sequence $\{u_i\}$ satisfies Lemma 4.

REMARK. Again the mutation parameters $\{\gamma_{jl}(i)\}$ play no role. Observe also that condition (14) is a necessary and sufficient condition for the regular birth and death process with parameters $\{\lambda_j\}$ and $\{\mu_i\}$ $(j \geqq l_0)$ to be absorbed with certainty in the state l_0, cf. ([5], p. 380).

5. Finite mean absorption time. In this section we consider regular M.C.P.'s possessing a nonempty set of absorbing states A and being nondissipative in the sense that $\alpha_i = 1$ for all $i \varepsilon D$. Let τ_i be the mean time to absorption in A, the process having started at $i \varepsilon D$. Our objective in this section is to find a sufficient condition for $\tau_i < \infty$ for all $i \varepsilon D$.

We shall apply the criterion

LEMMA 5 (Reuter, [15]). *Suppose the Q-matrix is regular. If there exists a finite sequence* $\{u_j\}$ *such that*

(a) $u_j \geqq 0$, and

(b) $\sum_{j \varepsilon E} q_{ij} u_j + 1 \leqq 0 \qquad (i \varepsilon D)$,

then $\alpha_i = 1$ *and* $\tau_i \leqq u_i < \infty$.

Lemma 4 yields directly

THEOREM 4. *For a regular, nondissipative* M.C.P. *assume there exists a nonnegative integer* l_0 *such that* E_l *is empty for* $l < l_0$ *and* E_l *is nonempty for* $l \geqq l_0$. *Then a sufficient condition for the process to possess* $\tau_i < \infty$ *for all* $i \varepsilon D$ *is that*

$$\sum_{j=l_0}^{\infty} \sigma_j \quad \textit{converge.} \tag{15}$$

PROOF. The construction of a sequence $\{u_i\}$ satisfying Lemma 5 is carried out as in Theorem 2 and will not be repeated here.

REMARK. Again the $\{\gamma_{jl}(i)\}$ play no role. Also the condition (15) is a necessary and sufficient condition for the regular nondissipative birth and death process with parameters $\{\lambda_j\}$ and $\{\mu_j\}$ $(j \geqq l_0)$ to possess a finite mean absorption time, cf. ([5], p. 380).

6. Examples. We conclude this paper by giving some examples of M.C.P.'s. These examples include a number of processes which occur in physical and biological models. In Table 1 are listed irreducible M.C.P.'s. Table 2 deals with processes having a nonempty set of absorbing states. For all of the processes listed the mutation parameters $\{\gamma_{jl}(i)\}$ are arbitrary, as they play no role in the sufficient conditions of our theorems. All processes in Tables 1 and 2 are regular.

TABLE 1

Irreducible multivariate competition processes

Process Label	Birth Rates	Death Rates	λ_k, $k \geqq 0$	μ_k, $k \geqq 0$	σ_k, $k > 0$	Recurrence
A	$\lambda_j(i) = \lambda^{(j)} > 0$	$\mu_j(i) = \mu^{(j)}$ $(i_j > 0)$ $= 0$ $(i_j = 0)$	λ	$\mu'(k \geqq 1)$ $0\ (k = 0)$	$(\lambda/\mu')^k$	positive recurrent for $\lambda < \mu'$; inconclusive for $\lambda \geqq \mu'$
B	$\lambda_j(i) = \lambda^{(j)} > 0$	$\mu_j(i) = i_j\mu^{(j)}$ $(\mu^{(j)} > 0)$	λ	$k\mu'$	$\frac{(\lambda/\mu')^k}{k!}$	positive recurrent
C	$\lambda_j(i) = (i_j + \beta)\lambda^{(j)}$ $(\beta, \lambda^{(j)} > 0)$	$\mu_j(i) = i_j\mu^{(j)}$ $(\mu^{(j)} > 0)$	$(k + \beta)\lambda'$	$k\mu'$	$\left(\frac{\lambda'}{\mu'}\right)^k \frac{(k + \beta - 1)_k}{k!}$	positive recurrent for $\lambda' < \mu'$; inconclusive for $\lambda \geqq \mu'$

TABLE 2

Multivariate competition processes with absorbing states

Process Label	Absorbing States (A)	Birth Rates ($i \notin A$)	Death Rates ($i \notin A$)	l_0
D	i such that $i_j = 0$ for some j $(1 \leqq j \leqq N)$	$\lambda_j(i) = \lambda^{(j)} > 0$	$\mu_j(i) = \mu^{(j)} > 0$	N
E	i such that $i_j = 0$ for some j $(1 \leqq j \leqq N)$	$\lambda_j(i) = \lambda^{(j)} > 0$	$\mu_j(i) = i_j\mu^{(j)}$ $(\mu^{(j)} > 0)$	N
F	i such that $i_j = 0$ for some j $(1 \leqq j \leqq N)$	$\lambda_j(i) = (i_j + \beta)\lambda^{(j)}$ $(\beta, \lambda^{(j)} > 0)$	$\mu_j(i) = i_j\mu^{(j)}$ $(\mu^{(j)} > 0)$	N

Process Label	$\lambda_k,\ k \geqq l_0$	$\mu_k,\ k \geqq l_0$	$\sigma_{l_0+k},\ k > 0$	Absorption	Mean Absorption Time
D	λ	$\mu + \mu'$	$[\lambda/(\mu + \mu')]^k$	certain for $\lambda < \mu + \mu'$; inconclusive for $\lambda \geqq \mu + \mu'$	finite for $\lambda < \mu + \mu'$; inconclusive for $\lambda \geqq \mu + \mu'$
E	λ	$(k - l_0 + \gamma)\mu'$ $(\gamma \equiv \mu/\mu')$	$\dfrac{(\lambda/\mu')^k}{(k - l_0 + \gamma)_k}$	certain	finite
F	$(k - l_0 + \alpha)\lambda'$ $(\alpha \equiv (1 + \beta)\lambda/\lambda')$	$(k - l_0 + \gamma)\mu'$ $(\gamma \equiv \mu/\mu')$	$\dfrac{(\lambda'/\mu')^k(k - 1 - l_0 + \alpha)_k}{(k - l_0 + \gamma)_k}$	certain for $\lambda' < \mu'$; inconclusive for $\lambda' \geqq \mu'$	finite for $\lambda' < \mu'$; inconclusive for $\lambda' \geqq \mu'$

Process Label	Absorbing States (A)	Birth Rates ($i \notin A$)	Death Rates ($i \notin A$)	l_0
G	i such that $i_j = 0$ for $j = j_0$	$\lambda_j(i) = \lambda^{(j)} > 0$	$\mu_j(i) = \mu^{(j)} \quad (i_j > 0)$ $= 0 \quad (i_j = 0)$	1
H	i such that $i_j = 0$ for $j = j_0$	$\lambda_j(i) = \lambda^{(j)} > 0$	$\mu_j(i) = i_j\mu^{(j)}$ $(\mu^{(j)} > 0)$	1
I	i such that $i_j = 0$ for $j = j_0$	$\lambda_j(i) = (i_j + \beta)\lambda^{(j)}$ $(\beta, \lambda^{(j)} > 0)$	$\mu_j(i) = i_j\mu^{(j)}$ $(\mu^{(j)} > 0)$	1

Process Label	$\lambda_k,\ k \geqq l_0$	$\mu_k,\ k \geqq l_0$	$\sigma_{l_0+k},\ k > 0$	Absorption	Mean Absorption Time
G	λ	$\mu^{(j_0)}$	$[\lambda/\mu^{(j_0)}]^k$	certain for $\lambda < \mu^{(j_0)}$; inconclusive for $\lambda \geqq \mu^{(j_0)}$	finite for $\lambda < \mu^{(j_0)}$; inconclusive for $\lambda \geqq \mu^{(j_0)}$
H	λ	$k\mu^{(j_0)}$	$\dfrac{[\lambda/\mu^{(j_0)}]^k}{k!}$	certain	finite
I	$(k - l_0 + \alpha)\lambda'$ $(\alpha \equiv (1+\beta)\lambda^{(j_0)}/\lambda')$	$(k - l_0 + \gamma)\mu'$ $(\gamma \equiv \mu^{(j_0)}/\mu')$	$\dfrac{(\lambda'/\mu')^k(k-1-l_0+\alpha)_k}{(k-l_0+\gamma)_k}$	certain for $\lambda' < \mu'$; inconclusive for $\lambda' \geqq \mu'$	finite for $\lambda' < \mu'$; inconclusive for $\lambda' \geqq \mu'$

Throughout Tables 1 and 2 we let

$$\lambda = \sum_{j=1}^{N} \lambda^{(j)}, \qquad \mu = \sum_{j=1}^{N} \mu^{(j)}$$

$$\lambda' = \max_{1 \leq j \leq N} \{\lambda^{(j)}\}, \quad \text{and} \quad \mu' = \min_{1 \leq j \leq N} \{\mu^{(j)}\}.$$

Processes A, D, and G represent extensions of the birth and death processes which arise in single server queues (cf., Karlin and McGregor [7]). Processes B, E, and H are generalizations of the telephone trunking problem (cf. [7]). Finally, processes C, F, and I are extensions of linear growth, birth and death processes (cf., Karlin and McGregor [6]).

The number of applications of M.C.P.'s seems large. We shall mention a few of these. Karlin and McGregor have applied M.C.P.'s to problems in genetics in which the coordinates of the process represent numbers of various alleles (cf., [8]). Jackson has used M.C.P.'s to model a network of queue; (jobshop) in which the coordinates of the process represent the length of the queues at the various work centers (cf., [3]). Neyman, Park, and Scott have studied competition among various species (cf., [13]). Kendall has applied M.C.P.'s to problems in epidemics in which the coordinates represent suspectible, infected, and removed persons (cf., [9]) and to the study of the growth of populations subject to mutation (cf. [10]).

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