ON CONTINUOUS SINGULAR INFINITELY DIVISIBLE DISTRIBUTION FUNCTIONS[1]

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1. Introduction and summary. A probability distribution function F is said to be infinitely divisible if, for every integer n, there is a distribution function F_n such that F is the n-fold convolution of F_n. If F is infinitely divisible its characteristic function can be written in the Khinchin-Lévy canonical form:

$$(1) \qquad f(u) = \exp\left\{i\gamma u + \int_{-\infty}^{\infty}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\,dG(x)\right\},$$

where γ is a constant and G is a bounded, non-decreasing function. Hartman and Wintner [2] proved that if G is discrete, then the distribution function F is *pure*, i.e., it is either absolutely continuous or discrete or continuous singular, and an example was given of each of these types of pure distributions which was determined by a discrete G. In the example of a discrete G producing a continuous singular F, G was given jumps of size $(1/N)^{2j}$ at $\pm 1/N^j$, $j = 1, 2, \cdots$, where N is a positive integer $\geqq 2$. It was first proved that F must be continuous; then it was proved that $f(u)$ does not converge to zero as $|u| \to \infty$, thus violating the conclusion of the Riemann-Lebesgue lemma.

The main purpose of this paper is to give sufficient conditions that a continuous singular F be obtained from a discrete G. These conditions are not too broad; for example, they do not include the example cited above. However, these conditions should be of considerable interest in that they are obtained by purely probabilistic methods, there being no use made of the Riemann-Lebesgue lemma, and thus supply more insight into the structure of continuous singular infinitely divisible distributions. In Section 2 a lemma is proved which might be of independent interest. This lemma is used to prove Theorem 1 in Section 3, which gives sufficient conditions that F be continuous singular. In Section 4 a theorem is proved, giving sufficient conditions that every m-fold convolution of F, F^{*m}, be continuous singular.

2. A lemma. The following lemma turns out to be a very useful device in Section 3.

Lemma. *Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of discrete random variables which are tail equivalent*, i.e., $\sum_{n=1}^{\infty} P[X_n \neq Y_n] < \infty$, *and assume that $\sum_{n=1}^{\infty} X_n$ converges almost surely (which implies that $\sum_{n=1}^{\infty} Y_n$ converges almost surely). If F and G are the distribution functions of $X = \sum_{n=1}^{\infty} X_n$ and $Y = \sum_{n=1}^{\infty} Y_n$ re-*

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spectively, if F_d, F_s *and* F_{ac} *are the discrete, continuous singular, and absolutely continuous components of* F, *and if* G_d, G_s *and* G_{ac} *denote the corresponding components of* G, *then* (i) $\operatorname{Var} G_d = \operatorname{Var} F_d$, (ii) $\operatorname{Var} G_s = \operatorname{Var} F_s$, *and* (iii) $\operatorname{Var} G_{ac} = \operatorname{Var} F_{ac}$, *where* Var *denotes "total variation of."*

(It should be noted that no assumptions of independence are made here.)

Proof. Let M be the module generated by the values that the X_n's and Y_n's take with positive probabilities. One easily observes that M is countable. Tail equivalence of the two sequences, discreteness of the X_n's and the Y_n's, and the Borel-Cantelli lemma imply that if B is any Borel subset of $(-\infty, +\infty)$, and if $B + M$ denotes $\{b + m \mid b \varepsilon B \text{ and } m \varepsilon M\}$, then the two events $[\sum_{n=1}^{\infty} X_n \varepsilon B + M]$ and $[\sum_{n=1}^{\infty} Y_n \varepsilon B + M]$ differ only by a set of probability zero, and therefore their probabilities are equal. In particular, let B denote the set of points at which F is not continuous, i.e., $\operatorname{Var} F_d = P[X \varepsilon B]$. Then $B \subset B + M$, and the countability of B implies that $B + M$ is denumerable. It is easy to verify that

$$\operatorname{Var} F_d = P[X \varepsilon B] = P[X \varepsilon B + M] = P[Y \varepsilon B + M] \leqq \operatorname{Var} G_d .$$

By a symmetrical argument, one may obtain $\operatorname{Var} G_d \leqq \operatorname{Var} F_d$, which establishes (i) in the theorem. The proof of (ii) is accomplished by proving that $\operatorname{Var}(F_s + F_d) = \operatorname{Var}(G_s + G_d)$. The proof of this equality is the same as that of (i) except for the following changes: the set B is a set of Lebesgue measure zero which carries the discrete and continuous singular part of the measure determined by F, and $M + B$ in this case turns out to be a set of Lebesgue measure zero. The assertion (iii) follows from (i) and (ii).

3. Conditions for continuous singularity. In (1) the value of γ has nothing to do with the question under consideration, so we may assume that $\gamma = 0$. In reference to the notation of Section 1, let us denote

$$M(x) = \int_{-\infty}^{x} \frac{1 + \tau^2}{\tau^2} \, dG(\tau) \quad \text{if} \quad x < 0.$$

and

$$N(x) = -\int_{x}^{\infty} \frac{1 + \tau^2}{\tau^2} \, dG(\tau) \quad \text{if} \quad x > 0.$$

Clearly, G is discrete if and only if both M and N are discrete. Let $b_n > 0$ for $n = 0, \pm 1, \pm 2, \cdots$, let $a_n > 0$ if $n = 0, 1, 2, \cdots$, let $a_n < 0$ for $n = -1, -2, \cdots$, and assume $a_n \neq a_m$ if $n \neq m$. We consider a discrete G for which $M(x)$ takes a jump of size b_n at $x = a_n$ when $n < 0$, and $N(x)$ takes a jump of size b_n at $x = a_n$ when $n \geqq 0$, and where $G(+0) - G(-0) = 0$. In order that G be a bounded function (i.e., f be a characteristic function) the condition

$$\sum_{\{n \mid |a_n| \leqq 1\}} a_n^2 b_n < \infty \tag{2}$$

must be satisfied.

THEOREM 1. *If, for each* n, $k_n \geqq 1$ *is an integer such that*

$$\sum_{n=-\infty}^{\infty} (1 - e^{-b_n} - b_n e^{-b_n} - \cdots - b_n^{k_n} e^{-b_n}/k_n!) < \infty, \tag{3}$$

$$\sum_{\{n \mid |a_n| \leqq 1\}} b_n = \infty, \tag{4}$$

and

$$\left(\prod_{j=-n}^{n} (1 + k_j)\right) \sum_{|j| \geqq n+1} k_j |a_j| \to 0 \quad \text{as} \quad n \to \infty, \tag{5}$$

then F *is continuous singular.*

PROOF. We may write the characteristic function, f, of F by

$$f(u) = \prod_{n=-\infty}^{\infty} \exp\{b_n(e^{iua_n} - 1) - iua_n b_n/(1 + a_n^2)\}.$$

Let $\{\cdots, X_{-1}, X_0, X_1, \cdots\}$ be a sequence of independent random variables where the distribution of X_n is Poisson with expectation b_n, $n = 0, \pm 1, \pm 2, \cdots$. Let $\{Y_n\}$ be a sequence of (independent) random variables defined by

$$Y_n = a_n X_n I_{[X_n \leqq k_n]}, \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{6}$$

Hypothesis (3) implies that $\{a_n X_n\}$ and $\{Y_n\}$ are tail-equivalent. One observes that F is the distribution function of

$$X = \sum_{n=-\infty}^{\infty} (a_n X_n - a_n b_n/(1 + a_n^2)). \tag{7}$$

Since the series (7) of independent random variables converges in distribution (to (1)), it converges almost surely. For $n \geqq 0$, since $0 \leqq Y_n \leqq k_n a_n$, then $\sum_{n=0}^{\infty} Y_n$ converges almost surely since it is a monotone non-decreasing series bounded above by $\sum_{n=0}^{\infty} k_n a_n < \infty$ (by (5)). Similarly $\sum_{n=-\infty}^{-1} Y_n$ converges almost surely, and thus $\sum_{n=-\infty}^{\infty} Y_n$ converges almost surely. Since $\{a_n X_n\}$ and $\{Y_n\}$ are tail equivalent, we obtain that the series $\sum_{n=-\infty}^{\infty} a_n X_n$ converges almost surely. Hence $\sum_{n=-\infty}^{\infty} |a_n| b_n/(1 + a_n^2) < \infty$, and the distribution F is continuous singular if and only if the distribution of $\sum_{n=-\infty}^{\infty} a_n X_n$ is. By the lemma in Section 2 we need only prove that the distribution of $Y = \sum_{n=-\infty}^{\infty} Y_n$ is continuous singular. Hypothesis (4) implies that the distribution function of $\sum_{n=-\infty}^{\infty} (a_n X_n - a_n b_n/(1 + a_n^2))$ is continuous; this follows from a result due to Blum and Rosenblatt [1] and also due to Hartman and Wintner [2]. Hence the lemma in Section 2 implies that the distribution function of $Y = \sum_{n=-\infty}^{\infty} Y_n$ is continuous. The distribution of Y can be proved to be continuous singular if one can find a set S of real numbers of Lebesgue measure zero for which $P[Y \in S] = 1$. In order to do this, for each positive integer n let $s_{n,1}, s_{n,2}, \cdots, s_{n,m_n}$ denote the set of all possible distinct sums of numbers of the form $\sum_{j=-n}^{n} v_j a_j$, $0 \leqq v_j \leqq k_j$. It is clear that $m_n \leqq \prod_{j=-n}^{n} (1 + k_j)$. Let $J_{n,k} = [\alpha_{n,k}, \beta_{n,k}]$, where

$$\alpha_{n,k} = s_{n,k} + \sum_{j=-\infty}^{-(n+1)} k_j a_j$$

and

$$\beta_{n,k} = s_{n,k} + \sum_{j=n+1}^{\infty} k_j a_j ,$$

where $1 \leqq k \leqq m_n$. Then let $S_n = \bigcup_{k=1}^{m_n} J_{n,k}$. Each set S_n is a closed set, and it is easy to verify that $P[Y \varepsilon S_n] = 1$ for all n. Further, one can verify that $S_n \supset S_{n+1}$ for all n. Hence if we let $S = \bigcap_{n=1}^{\infty} S_n$, we have that S is closed, non-empty and $P[Y \varepsilon S] = 1$. However, for every n

$$\text{meas } S \leqq \text{meas } S_n \leqq \left(\prod_{j=-n}^{n} (1 + k_j) \right) \sum_{|j| \geqq n+1} k_j |a_j|,$$

and, because of (5), one obtains that meas $S = 0$, thus concluding the proof of the theorem.

4. An application. In the example of Hartman and Winter cited in Section 1, it is easy to see that any m-fold convolution of F with itself, F^{*m}, is also continuous singular. Indeed, if the G-function of F is discrete, then it follows that the G-function of F^{*m} is discrete. However, since F is continuous, then F^{*m} is continuous. The theorem of Hartman and Wintner then implies that the distribution F^{*m} is pure. Since the characteristic function f of F satisfies

$$\limsup_{|u| \to \infty} |f(u)| > 0,$$

then $f^m(u)$ satisfies the same assertion. Hence by the Riemann-Lebesgue lemma, F^{*m} is not absolutely continuous, and, since it is pure and is continuous, it must be continuous singular. The purpose of this section is to obtain conditions on a discrete G which are sufficient for F^{*m} to be continuous singular for all m.

Let $\{a_n, n = 0, \pm 1, \pm 2, \cdots\}$ be a sequence of distinct numbers such that $a_n < 0$ if $n < 0$ and $a_n > 0$ if $n \geqq 0$, let $\{b_n, n = 0, \pm 1, \pm 2, \cdots\}$ be a sequence of positive numbers, and assume $\{a_n\}$ and $\{b_n\}$ satisfy (2) and (4). Again, let F be an infinitely divisible distribution function whose M- and N-functions take jumps of size b_n at a_n.

THEOREM 2. *If, for each n, $k_n \geqq 1$ is an integer such that*

$$(3) \qquad \sum_{n=-\infty}^{\infty} (1 - e^{-b_n} - b_n e^{-b_n} - \cdots - b_n^{k_n} e^{-b_n}/k_n!) < \infty,$$

and

$$(5') \qquad \left(\prod_{j=-n}^{n} (1 + k_j)^{|j|} \right) \sum_{|j| \geqq n+1} k_j |a_j| \to 0 \quad \text{as} \quad n \to \infty,$$

then F^{*m} is continuous singular for every $m \geqq 1$.

PROOF. Because of (1), it is easy to see that F^{*m} has discrete M- and N-functions which take jumps of sizes $\{mb_n\}$ at points $\{a_n\}$. In order to prove the theorem we need only show that the sequences $\{a_n\}$, $\{mb_n\}$ and $\{mk_n\}$ of this theorem satisfy requirements (2), (3), (4) and (5) of Theorem 1. We first note that (2) and (4) are trivially satisfied. In order to prove that these three sequences satisfy (3), it is sufficient to prove that, for every n,

$$1 - e^{-mb_n} \sum_{j=0}^{mk_n} (mb_n)^j/j! \leqq m\{1 - e^{-b_n} \sum_{j=0}^{k_n} b_n^j/j!\}.$$

Toward this end, let $U_1, U_2, \cdots, U_m$ be m independent, indentically distributed random variables, each of whose distributions is Poisson with expectation b_n. If $V = U_1 + \cdots + U_m$, then the distribution of V is Poisson with expectation mb_n. Since

$$[V > mk_n] \subset [U_1 > k_n] \cup \cdots \cup [U_m > k_n],$$

we obtain, by Boole's inequality, $P[V > mk_n] \leqq mP[U_1 > k_n]$, which is precisely what the desired inequality states. We now verify that the three sequences satisfy (5). Let us denote the expression in (5′) by $B(n)$. Then, for $n > m$, we have

$$0 < \left(\prod_{j=-n}^{n} (1 + mk_j)\right) \sum_{|j| \geqq n+1} mk_j\, |a_j| < \left(\prod_{j=-n}^{n} (1 + k_j)^m\right) \sum_{|j| \geqq n+1} mk_j\, |a_j|$$
$$< m \left(\prod_{i=-m+1}^{m-1} (1 + k_j)^{m-|i|}\right) B(n).$$

Since by (5′) $B(n) \to 0$ as $n \to \infty$, and since the coefficient of $B(n)$ in this last expression does not depend on n, we may conclude that (5) in Theorem 1 is satisfied by the three sequences. Thus, Theorem 1 implies that F^{*m} is continuous singular.

5. Remarks. Three remarks are now in order. It should first be noted (as remarked in Section 1) that the conditions of Theorem 1 (and therefore of Theorem 2) are not broad enough to include the example due to Hartman and Wintner, and one easily shows that hypothesis (5) (and therefore (5′)) is violated when $N = 2$. Indeed, in this case it is easy to see that for all n,

$$\left(\prod_{j=-n}^{n} (1 + k_j)\right) \sum_{|j| \geqq n+1} k_j\, 2^{-j} \geqq 2.$$

The second remark is that it is possible to have sequences $\{a_n\}$, $\{b_n\}$, $\{k_n\}$ which satisfy Theorem 1 and such that $k_n = 1$ for all n. One can verify that this is the case when $a_n = (2/3)^n$ and $b_n = 1/n$, $n = 1, 2, \ldots$.

The third remark is that the sequence $\{b_n\}$ *only determines* whether F is continuous or discrete. According to the result of Hartman and Wintner stated in Section 1, the distribution function F is either continuous singular or absolutely continuous when (4) is satisfied. If the jumps of the M- and N-functions are $\{b_n\}$ at $\{a_n\}$, and if F is absolutely continuous, then by pressing the points at which these jumps occur closer to the origin one can obtain an F which is continuous singular, and by pressing them even still closer one can obtain an F such that F^{*m} is continuous singular for all m. Indeed, given $\{b_n\}$ which satisfy (4), one can always select positive integers $\{k_n\}$ such that (3) is true. Now let

$$a_n = \min\left\{\frac{1}{2^n(b_n)^{\frac{1}{2}}}, \left(\prod_{j=-n}^{n} (1 + k_j)^{|j|}\right)^{-1} k_n^{-1} |n|^{-3}\right\}.$$

It is easy to see that $\{a_n\}$, $\{b_n\}$, $\{k_n\}$ now satisfy (2) and (5′) (and therefore (5)). Thus we obtain from this remark that in order that a discrete G produce an absolutely continuous F, it must of course satisfy (4), but the discontinuities of G cannot pile up at too rapid a rate about zero.

REFERENCES

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