THE MOMENTS OF A VARIATE RELATED TO THE NON-CENTRAL t[1]

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1. Introduction. Suppose W is a random normal variate with mean θ and variance 1 and X^2 is independently distributed as chi-squared with n degrees of freedom. Then, the random variate Q with non-centrality θ and n degrees of freedom is defined by

$$Q = W/(W^2 + X^2)^{\frac{1}{2}}, \qquad \text{(all roots positive)}.$$

It is apparent that $n^{\frac{1}{2}}Q/(1 - Q^2)^{\frac{1}{2}}$ is distributed as non-central t with non-centrality θ and n degrees of freedom. Further, Q^2 is distributed as non-central beta, see for example Hodges (1955), with non-centrality θ and shape parameters $\frac{1}{2}$ and $\frac{1}{2}n$ (in Hodges' notation $\lambda = \frac{1}{2}\theta^2$, $a = \frac{1}{2}n$, $b = \frac{1}{2}$).

This paper gives closed form analytic expressions and recurrence relations for the moments about zero of Q. As well, Table 1 gives numerical values of the mean, variance, third and fourth central moments. The present interest in the variate Q derived from another study, Hogben et al. (1962b) in the course of which a knowledge of the moments of Q was desired. Other uses of the table are suggested in Section 6. The present authors know of no previous published work on the moments of Q. Studies of the probability integral of the general non-central beta have been published by Tang (1938), Thompson (1941), Lehmer (1944), Nicholson (1954) and Hodges (1955).

2. Analytic expressions for the moments of Q. By transforming the joint density of W and X to polar coordinates and integrating, the probability density function of Q, $n > 0$, is directly found to be

$$(1) \qquad \begin{aligned} f(q)\,dq &= K_{n,\theta}^{-1}(1 - q^2)^{\frac{1}{2}(n-2)}P_n(\theta q)\,dq, \qquad -1 \leqq q \leqq 1 \\ &= 0, \quad \text{elsewhere,} \end{aligned}$$

where

$$K_{n,\theta} = \pi^{\frac{1}{2}}2^{\frac{1}{2}(n-1)}\Gamma(\tfrac{1}{2}n)e^{\frac{1}{2}\theta^2}$$

and

$$P_n(z) = \int_0^\infty y^n e^{zy - \frac{1}{2}y^2}\,dy.$$

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The function $P_n(z)$ is related to the well known Hh (Hermite hyperbolique) functions, see for example Fisher (1931), by the relation $P_n(z) = n!e^{-\frac{1}{2}z^2}Hh_n(-z)$. The lemmas below may be interpreted in terms of the Hh functions.

When $n = 0$ the variate Q is a transformed two point binomial variate with

$$\Pr\{Q = +1\} = p, \qquad \Pr\{Q = -1\} = q,$$

where $p + q = 1$ and

$$p = (1/2\pi)^{\frac{1}{2}} \int_{-\theta}^{\infty} e^{-\frac{1}{2}u^2}\, du.$$

The rth raw moment, or moment about zero, of Q, for $r = 0, 1, 2, \cdots$, may therefore be written as

$$EQ^r = m_r = K_{n,\theta}^{-1} \int_{-1}^{1} q^r(1 - q^2)^{\frac{1}{2}n-1} P_n(\theta q)\, dq$$

and setting $u = \theta q$ gives

$$m_r = \theta^{1-n-r} K_{n,\theta}^{-1} \int_{-\theta}^{\theta} u^r(\theta^2 - u^2)^{\frac{1}{2}(n-2)} P_n(u)\, du. \tag{2}$$

The evaluation of the last integral is facilitated by the following two lemmas:

LEMMA 1.

$$\int_{-\theta}^{\theta} (\theta^2 - u^2)^{r+\frac{1}{2}n} P_n(u)\, du = \pi^{\frac{1}{2}} 2^{\frac{1}{2}(n+1)} \frac{\Gamma(r + 1 + \frac{1}{2}n)}{\Gamma(r + 1)} \int_0^{\theta} u^n(\theta^2 - u^2)^r e^{\frac{1}{2}u^2}\, du.$$

PROOF. Since $f(q)$ is a density function, the equation

$$t^{n-1}K_{n,t} = \int_{-t}^{t} (t^2 - u^2)^{\frac{1}{2}(n-2)} P_n(u)\, du$$

is an identity in n and t. Multiplying by $t(\theta^2 - t^2)^r$ and integrating on t between 0 and θ gives

$$\int_0^{\theta} t^n K_{n,t}(\theta^2 - t^2)^r\, dt = \int_0^{\theta} t(\theta^2 - t^2)^r \int_{-t}^{t} (t^2 - u^2)^{\frac{1}{2}(n-2)} P_n(u)\, du\, dt.$$

The order of integration on the right hand side may be interchanged to give

$$\int_{-\theta}^{\theta} P_n(u) S(|u|, \theta, r)\, du,$$

where

$$S(x, \theta, r) = \int_x^{\theta} (\theta^2 - t^2)^r(t^2 - x^2)^{\frac{1}{2}(n-2)} t\, dt, \qquad 0 \leqq x \leqq \theta.$$

Putting $z = (t^2 - x^2)/(\theta^2 - x^2)$ yields

$$\begin{aligned} S(x, \theta, r) &= \tfrac{1}{2} (\theta^2 - x^2)^{r+\frac{1}{2}n} \int_0^1 z^{\frac{1}{2}(n-2)}(1 - z)^r\, dz \\ &= [\Gamma(r + 1)\Gamma(\tfrac{1}{2} n)/2\Gamma(r + 1 + \tfrac{1}{2} n)]\, (\theta^2 - x^2)^{r+\frac{1}{2}n}. \end{aligned}$$

Thus,

$$\int_{-\theta}^{\theta} (\theta^2 - u^2)^{r+\frac{1}{2}n} P_n(u)\, du = \frac{2\Gamma(r + 1 + \frac{1}{2}n)}{\Gamma(r + 1)\Gamma(\frac{1}{2}n)} \int_0^{\theta} t^n K_{n,t}(\theta^2 - t^2)^r\, dt,$$

and substituting for $K_{n,t}$ completes the proof.

LEMMA 2.

$$\int_{-\theta}^{\theta} P_{n-1}(u)(\theta^2 - u^2)^{r-1+\frac{1}{2}n} u\, du = \pi^{\frac{1}{2}} 2^{\frac{1}{2}(n-1)} \frac{\Gamma(r + \frac{1}{2}n)}{\Gamma(r + 1)} \int_0^{\theta} u^n (\theta^2 - u^2)^r e^{\frac{1}{2}u^2}\, du.$$

PROOF. The derivative of $P_n(z)$, with respect to z, is

$$P_n'(z) = \int_0^{\infty} y^{n+1} e^{zy - \frac{1}{2}y^2}\, dy = P_{n+1}(z)$$

and therefore integration by parts, as suggested, of

$$\int_{-\theta}^{\theta} (\theta^2 - u^2)^{r+\frac{1}{2}n} P_n(u)\, du = -\frac{1}{2}\left(r + \frac{1}{2}n\right) \int_{-\theta}^{\theta} P_n(u) \left[\frac{d}{du} (\theta^2 - u^2)^{r+\frac{1}{2}n}\right] du$$

yields

$$\int_{-\theta}^{\theta} u(\theta^2 - u^2)^{r+\frac{1}{2}n-1} P_{n-1}(u)\, du = \frac{1}{2(r + \frac{1}{2}n)} \int_{-\theta}^{\theta} (\theta^2 - u^2)^{r+\frac{1}{2}n} P_n(u)\, du.$$

Lemma 1 applied to the right hand side completes the proof.

Alternative expressions for the moments may now be obtained. For the even moments, let $r = 2h$ and write

$$u^r = [\theta^2 - (\theta^2 - u^2)]^h = \sum_{i=0}^{h} (-1)^i \binom{h}{i} \theta^{2h-2i} (\theta^2 - u^2)^i.$$

Substituting in (2) and rearranging gives

$$m_{2h} = \sum_{i=0}^{h} (-1)^i \binom{h}{i} \theta^{-n-2i+1} K_{n,\theta}^{-1} \int_{-\theta}^{\theta} (\theta^2 - u^2)^{\frac{1}{2}n+i-1} P_n(u)\, du.$$

Applying Lemma 1 one obtains

$$(3)\quad m_{2h} = 1 + \sum_{i=1}^{h} (-1)^i \binom{h}{i} \theta^{-n-2i+1} 2e^{-\frac{1}{2}\theta^2} \frac{\Gamma(\frac{1}{2}n + i)}{\Gamma(i)\Gamma(\frac{1}{2}n)} \int_0^{\theta} u^n (\theta^2 - u^2)^{i-1} e^{\frac{1}{2}u^2}\, du,$$

where the first term, ($i = 0$), equals 1, since the kernel of the integral is then the density of Q.

For the odd moments, let $r = 2h + 1$ and write

$$u^r = u[\theta^2 - (\theta^2 - u^2)]^h = \sum_{i=0}^{h} (-1)^i \binom{h}{i} \theta^{2h-2i} (\theta^2 - u^2)^i u.$$

Substitution in (2), rearranging as before and now applying Lemma 2 gives

$$m_{2h+1} = \sum_{i=0}^{h} (-1)^i \binom{h}{i} \theta^{-n-2i} e^{-\frac{1}{2}\theta^2} \left[\frac{2^{\frac{1}{2}}\Gamma(\frac{1}{2}n + i)}{\Gamma(i + \frac{1}{2})\Gamma(\frac{1}{2}n)}\right] \int_0^{\theta} u^{n+1} (\theta^2 - u^2)^{i-\frac{1}{2}} e^{\frac{1}{2}u^2}\, du.$$

Setting $t^2 = \theta^2 - u^2$ gives

$$(4)\quad m_{2h+1} = \sum_{i=0}^{h} (-1)^i \binom{h}{i} \theta^{-n-2i} 2^{\frac{1}{2}} \left[\frac{\Gamma(\frac{1}{2}n + i)}{\Gamma(i + \frac{1}{2})\Gamma(\frac{1}{2}n)}\right] \int_0^\theta t^{2i}(\theta^2 - t^2)^{\frac{1}{2}n} e^{-\frac{1}{2}t^2}\, dt.$$

Making appropriate substitutions for h in (3) and (4), we have for the first four raw moments

$$(5)\qquad m_1 = (2/\pi)^{\frac{1}{2}}\theta^{-n} \int_0^\theta (\theta^2 - u^2)^{\frac{1}{2}n} e^{-\frac{1}{2}u^2}\, du,$$

$$(6)\qquad m_2 = 1 - n\theta^{-(n+1)} e^{-\frac{1}{2}\theta^2} \int_0^\theta u^n e^{\frac{1}{2}u^2}\, du,$$

$$(7)\qquad m_3 = m_1 - (2/\pi)^{\frac{1}{2}} n\theta^{-(n+2)} \int_0^\theta u^2(\theta^2 - u^2)^{\frac{1}{2}n} e^{-\frac{1}{2}u^2}\, du,$$

$$(8)\qquad m_4 = 2m_2 - 1 + n\,(\tfrac{1}{2}\,n + 1)\,\theta^{-(n+3)} e^{-\frac{1}{2}\theta^2} \int_0^\theta u^n(\theta^2 - u^2) e^{\frac{1}{2}u^2} du.$$

These latter forms for the raw moments of Q lend themselves conveniently for developing both recurrence relations and direct numerical computation procedures.

3. Recurrence relations for raw moments. The numerical evaluation of the raw moments as a function of n is sometimes made more efficient by the use of recurrence relations. Following are forward recurrence relations for the first four moments:

$$(9)\quad m_1(n) = [1 - (n - 1)\theta^{-2}]m_1(n - 2) + (n - 2)\theta^{-2}m_1(n - 4);$$

$$(10)\quad m_2(n) = 1 + n(n - 2)^{-1}\theta^{-2}[1 - (n - 1)m_2(n - 2)];$$

$$(11)\quad m_3(n) = m_1(n) - n[m_1(n) - m_1(n + 2)];$$

$$(12)\quad m_4(n) = m_2(n) - \tfrac{1}{2}n[m_2(n) - m_2(n + 2)].$$

Also, for the first two moments we have the following backward recurrence relations:

$$(13)\quad m_1(n - 4) = (n - 2)^{-1}\{(n - 1)m_1(n - 2) - \theta^2[m_1(n - 2) - m_1(n)]\};$$

$$(14)\quad m_2(n - 2) = (n - 1)^{-1}\{n^{-1}(n - 2)\theta^2[1 - m_2(n)] + 1\}.$$

The procedure by which the forward relations may be derived is illustrated for $m_1(n)$:

$$(15)\quad \begin{aligned} m_1(n) &= (2/\pi)^{\frac{1}{2}}\theta^{-n} \int_0^\theta (\theta^2 - u^2)^{\frac{1}{2}(n-2)}(\theta^2 - u^2)e^{-\frac{1}{2}u^2}\, du \\ &= m_1(n - 2) + (2/\pi)^{\frac{1}{2}}\theta^{-n} \int_0^\theta (\theta^2 - u^2)^{\frac{1}{2}(n-2)} u \left(\frac{d}{du}\, e^{-\frac{1}{2}u^2}\right) du. \end{aligned}$$

Integration by parts, as suggested in the notation, and simple manipulation gives for the second term

$$-\theta^{-2}m_1(n-2) + (n-2)\theta^{-2}\left\{(2/\pi)^{\frac{1}{2}}\theta^{-(n-2)}\int_0^\theta (\theta^2-u^2)^{\frac{1}{2}(n-4)}u^2e^{-\frac{1}{2}u^2}\,du\right\}.$$

From (15), the expression in braces equals $m_1(n-4) - m_1(n-2)$ and then substitution in (15) gives the result.

4. Table of moments. Table 1 gives numerical values of the mean μ, variance σ^2, third central moment μ_3, and fourth central moment μ_4 of Q for values of

$$n = 1(1)25, 30, 35, 40, 50, 60, 80, 100 \quad \text{and}$$
$$\theta = 0.1(0.1)1.0(.2)2.0(0.5)6.0(1.0)10.0.$$

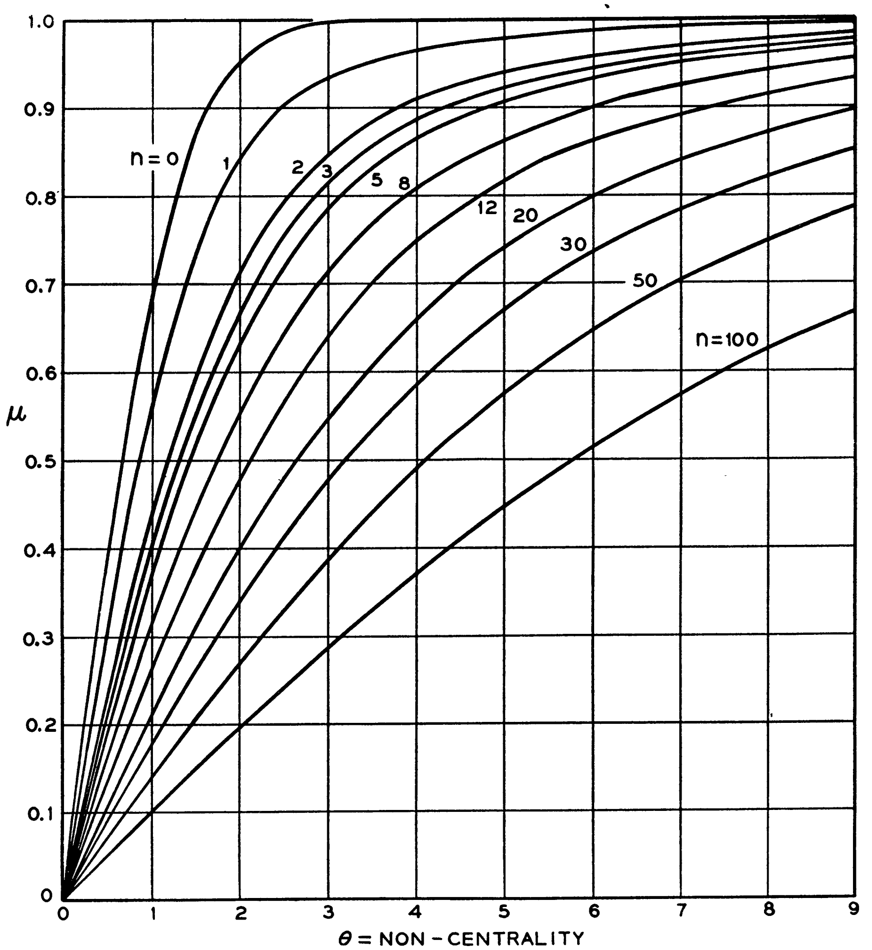

FIG. 1. The Mean of Q

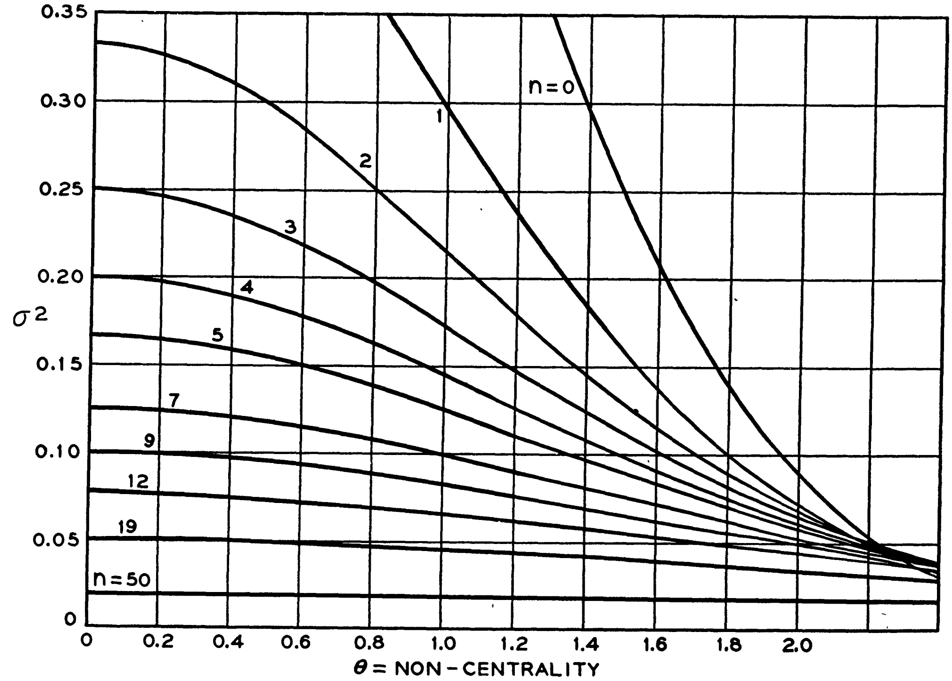

FIG. 2. The Variance of Q

Note that values of θ appear in the center column and refer to each pair of n values. In addition some graphs of the mean and variance of Q are presented in Figures 1 and 2 respectively. More extensive tables, including additional values of n and θ, the raw moments and the fourth cumulant, with seven decimal accuracy, are given in Hogben et al. (1961a, 1962a).

Linear interpolation is usually poor and no simple interpolation procedure has been found which produces good results. However, interpolation on n will not be necessary except for the less frequently used large values and the grid on θ is sufficiently fine so that quadratic interpolation will usually be satisfactory for applied purposes.

When $\theta = 0$, $\mu = \mu_3 = 0$, $\sigma^2 = 1/(n+1)$ and $\mu^4 = 3/(n+1)(n+3)$.

5. Computational procedures. All computations were performed on an I.B.M. "650" using the Wolontis (1956) interpretative system of programming, with a double precision modification for large values of θ. The central moments were computed from the appropriate linear combinations of the raw moments to at least eight significant digits, but for ease in tabulation and to assure a specified accuracy only six decimal places were reproduced.

Several problems arose in trying to find a general computational procedure which was uniformly satisfactory for all values of θ and n. Infinite series worked well for small values of θ, but did not converge rapidly enough for large values of

θ and small values of n. On the other hand, the recurrence relations worked well for large values of θ and n, but failed for small values of θ and n. The issue was resolved by using different procedures for different parts of the grid on θ and n as follows.

(a) For $0 \leqq \theta \leqq 1.8$ power series expansions were used to evaluate the first four raw moments for all given values of n. Expanding the integrand of (5), (6), (7) and (8) and integrating term by term yields:

$$m_1(n) = k_n \theta \sum_{i=0}^{\infty} a_i , \tag{16}$$

$$m_2(n) = (n+1)^{-1} + n(n+1)^{-1} \sum_{i=1}^{\infty} b_i \tag{17}$$

$$m_3(n) = k_n \theta \sum_{i=0}^{\infty} c_i , \tag{18}$$

$$m_4(n) = 3(n+1)^{-1}(n+3)^{-1} + n(n+1)^{-1} \sum_{i=1}^{\infty} d_i , \tag{19}$$

where

$$k_1 = (\pi/8)^{\frac{1}{2}}, \qquad (n+2)k_n k_{n+1} = 1,$$

$$a_0 = 1, \qquad a_{i+1} = -(2i+1)\theta^2 a_i/(2i+2)(2i+n+3),$$

$$b_1 = \theta^2/(n+3), \qquad b_{i+1} = -\theta^2 b_i/(2i+n+3), \qquad c_0 = 3/(n+3),$$

$$c_{i+1} = -(2i+1)[2i+5-2(i+1)n] \cdot \theta^2 c_i/(2i+2)(2i+n+5)(2i+3-2in),$$

$$d_1 = 6\theta^2/(n+3)(n+5),$$

$$d_{i+1} = -(6+2i-in)\theta^2 d_i/(2i+n+5)[6+(i-1)(2-n)], \quad n \neq 3, 4, 5, 8.$$

For $n = 3, 4, 5$ and 8 the recurrence relation breaks down for $i = 7, 4, 3$ and 2 respectively because d_i equals zero. In these cases the series has to be modified by computing new starting values before reusing the recurrence relations.

(b) For $2 \leqq \theta \leqq 10$, m_3 and m_4 were evaluated from m_1 and m_2 using the forward recurrence relations (11) and (12). The moments m_1 and m_2 were evaluated by using the series (16) and (17) to obtain the starting values $m_1(102)$, $m_1(101)$, $m_1(100)$, $m_1(99)$, $m_2(102)$ and $m_2(101)$ and then using the backward relations (13) and (14) to obtain the remaining values. For large values of $\theta = 7, 8, 9, 10$, the backward relations developed large errors for small n and the procedure had to be modified.

(c) When the backward relations failed to give accurate results, the forward relations were used until the two results agreed to at least 8 significant digits. For m_1, the starting values

$$m_1(0) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\theta}^{\theta} e^{-\frac{1}{2}u^2} \, du \quad \text{and}$$

$$m_1(2) = \theta^{-1} \frac{d}{d\theta} m_1(0) + (1 - \theta^{-2}) m_1(0),$$

obtained from Works Project Administration (1942), were used with (9) to obtain the moments for even n. The moments for odd n were obtained from Lagrange 7-point interpolation using Works Projects Administration (1944).

For m_2, the starting values $m_2(1)$ and $m_2(2)$ were used with the forward recurrence relation (10) to obtain the required values. Direct substitution and integration yields $m_2(1) = 1 - \theta^{-2}(1 - e^{-\frac{1}{2}\theta^2})$. Integration by parts and setting $t = u/2^{\frac{1}{2}}$ gives

$$m_2(2) = 1 - \frac{2}{\theta^2} + \frac{2(2)^{\frac{1}{2}}}{\theta^3} F\left(x = \frac{\theta}{2^{\frac{1}{2}}}\right), \quad \text{where} \quad F(x) = e^{-x^2} \int_0^x e^{t^2}\, dt.$$

The function $F(x)$ has been tabled by Karpov (1954) and Lash Miller and Gordon (1931), but greater accuracy was desired and the semi-convergent series (see Lash Miller and Gordon p. 2875)

$$F(x) = 1/2x + 1/4x^3 + 1\cdot 3/8x^5 + 1\cdot 3\cdot 5/16x^7 + 1\cdot 3\cdot 5\cdot 7/32x^9 + \cdots$$

was used to give

$$m_2(2) = 1 - 2/\theta^2 + 2(1/\theta^4 + 1/\theta^6 + 1\cdot 3/\theta^8 + 1\cdot 3\cdot 5/\theta^{10} + \cdots)$$

which gave at least 12 digit accuracy in the range of θ and n for which it was used.

6. Applications. The initial interest of the present authors in the random variable Q derived from its relevance in Hogben et al. (1962b), wherein the distribution of the product of two independent Q variables arose. A natural method of approximating that distribution is by the method due to Cornish and Fisher (1937), which requires the moments of the random variable. One application of the present numerical values of Table 1 is thus in approximating the distribution of Q and of certain functions of one or more Q-type random variables.

Approximate moments of non-linear functions of Q may also be obtained from the moments of Q from a Taylor's series approximation, Hogben et al. (1961b). As an illustration, the mean value of a non-central $t = g(q)$ with degrees of freedom $n = 9$ and non-centrality $\theta = 1$ is 1.0942. This can be approximated by

$$\sum_{i=0}^{4} \frac{\mu_i}{i!} \frac{d^i g(q)}{dq^i}\bigg|_{q=\mu}, \quad \text{where} \quad \begin{aligned} \mu &= E(Q) \\ \mu_i &= E(Q - \mu)^i, i > 0 \end{aligned}$$

$$= 1.0923$$

using the results of Table 1. For

$$\mu = 0.2962357$$

$$g(q) = n^{\frac{1}{2}}q(1 - q^2)^{-\frac{1}{2}}, \qquad g(\mu) = 0.930471$$

$$g'(q) = n^{\frac{1}{2}}(1 - q^2)^{-\frac{3}{2}}$$

$$q''(q) = 3n^{\frac{1}{2}}q(1 - q^2)^{-\frac{5}{2}}, \qquad g''(\mu) = 3.354300$$

$$g'''(q) = 3n^{\frac{1}{2}}(1 + 4q^2)(1 - q^2)^{-\frac{7}{2}}, \qquad g'''(\mu) = 16.769334$$

$$g^{\text{iv}}(q) = 15n^{\frac{1}{2}}q(3 + 4q^2)(1 - q^2)^{-\frac{9}{2}}, \qquad g^{\text{iv}}(\mu) = 67.534730.$$

Another application of the moments of Q is in the selection of an appropriate Beta variable, the distribution of a linear function of which may be used as an approximation to the distribution of Q. This is discussed in Hogben et al. (1964).

As a final illustration of the use of Table 1, the moments may be regarded as numerical values of certain integrals, some of which are of interest in connection with problems in mechanics and heat transfer, see Lash Miller and Gordon (1931).

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TABLE 1

Moments of Q (for negative θ change the sign of μ and μ_3)

$n = 1$					$n = 2$			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.062588	.497331	.046665	.376227	.1	.053139	.331841	.021144	.200006
.124708	.489415	.091355	.379631	.2	.105961	.327409	.041498	.199957
.185906	.476522	.132241	.384432	.3	.158155	.320168	.060324	.199671
.245748	.459085	.167772	.389462	.4	.209422	.310330	.076986	.198870
.303835	.437672	.196776	.393346	.5	.259487	.298175	.090981	.197233
.359808	.412955	.218523	.394704	.6	.308093	.284039	.101970	.194437
.413357	.385676	.232740	.392342	.7	.355017	.268294	.109787	.190211
.464226	.356602	.239592	.385406	.8	.400066	.251336	.114432	.184372
.512214	.326497	.239624	.373473	.9	.443082	.233562	.116059	.176851
.557180	.296082	.233674	.356579	1.0	.483942	.215358	.114950	.167705
.637755	.236846	.208087	.310075	1.2	.558879	.179067	.106073	.145324
.705890	.183000	.171816	.252906	1.4	.624584	.144863	.091300	.119598
.762222	.137001	.132947	.193772	1.6	.681239	.114379	.074077	.093457
.807891	.099751	.097193	.139930	1.8	.729399	.088466	.057126	.069459
.844320	.070957	.067596	.095638	2.0	.769866	.067304	.042172	.049243
.904783	.028398	.022816	.029872	2.5	.843590	.032596	.017034	.017389
.937445	.011320	.006461	.007264	3.0	.889444	.015791	.006058	.005031
.955799	.004995	.001743	.001480	3.5	.918439	.008116	.002122	.001310
.966939	.002550	.000526	.000284	4.0	.937508	.004530	.000808	.000347
.974242	.001472	.000198	.000063	4.5	.950618	.002729	.000349	.000105
.979325	.000922	.000091	.000019	5.0	.960000	.001748	.000170	.000038
.983021	.000612	.000047	.000007	5.5	.966942	.001174	.000091	.000016
.985799	.000423	.000027	.000003	6.0	.972222	.000819	.000052	.000007
.989631	.000222	.000010	.000001	7.0	.979592	.000435	.000019	.000002
.992092	.000128	.000004	.000000	8.0	.984375	.000252	.000008	.000001
.993768	.000079	.000002	.000000	9.0	.987654	.000156	.000004	.000000
.994962	.000052	.000001	.000000	10.0	.990000	.000102	.000002	.000000

$n = 3$					$n = 4$			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.046960	.249043	.011684	.124831	.1	.042524	.199333	.007259	.085544
.093686	.246198	.022981	.124307	.2	.084865	.197349	.014300	.085029
.139949	.241539	.033524	.123373	.3	.126847	.194094	.020921	.084152
.185528	.235186	.042993	.121952	.4	.168296	.189642	.026937	.082891
.230213	.227299	.051128	.119951	.5	.209048	.184095	.032199	.081220
.273813	.218072	.057743	.117286	.6	.248949	.177574	.036592	.079118
.316154	.207722	.062730	.113886	.7	.287862	.170218	.040045	.076573
.357083	.196480	.066063	.109716	.8	.325660	.162176	.042528	.073587
.396472	.184586	.067787	.104780	.9	.362235	.153603	.044054	.070178
.434214	.172275	.068013	.099127	1.0	.397496	.144655	.044668	.066384
.504458	.147281	.064651	.086060	1.2	.463799	.126229	.043495	.057881
.567450	.123059	.057606	.071601	1.4	.524187	.107981	.039854	.048674
.623166	.100766	.048616	.057051	1.6	.578538	.090763	.034715	.039453
.671869	.081114	.039176	.043578	1.8	.626955	.075159	.028970	.030846
.714032	.064408	.030354	.031981	2.0	.669716	.061485	.023315	.023304
.794950	.034906	.014077	.012668	2.5	.754693	.035968	.012011	.010143
.849205	.018769	.005843	.004267	3.0	.814585	.020740	.005570	.003846
.885751	.010444	.002375	.001332	3.5	.856704	.012191	.002506	.001366
.910931	.006134	.001014	.000423	4.0	.886717	.007439	.001158	.000489
.928802	.003810	.000471	.000147	4.5	.908550	.004738	.000566	.000186
.941868	.002485	.000238	.000058	5.0	.924800	.003142	.000296	.000077
.951680	.001689	.000129	.000025	5.5	.937163	.002159	.000164	.000035
.959223	.001187	.000075	.000012	6.0	.946759	.001530	.000096	.000017
.969865	.000637	.000028	.000003	7.0	.960433	.000829	.000037	.000005
.976841	.000372	.000012	.000001	8.0	.969482	.000487	.000016	.000002
.981655	.000231	.000006	.000000	9.0	.975766	.000304	.000008	.000001
.985114	.000152	.000003	.000000	10.0	.980300	.000200	.000004	.000000

TABLE 1—*Continued*

$n = 5$					$n = 6$			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.039142	.166175	.004874	.062358	.1	.036455	.142480	.003459	.047505
.078137	.164711	.009615	.061930	.2	.072788	.141355	.006833	.047163
.116842	.162306	.014100	.061213	.3	.108880	.139504	.010040	.046593
.155116	.159009	.018215	.060200	.4	.144616	.136963	.013005	.045796
.192827	.154888	.021864	.058889	.5	.179883	.133778	.015666	.044774
.229848	.150026	.024975	.057278	.6	.214576	.130008	.017971	.043533
.266065	.144516	.027496	.055370	.7	.248598	.125720	.019885	.042081
.301371	.138460	.029404	.053179	.8	.281858	.120987	.021387	.040430
.335676	.131965	.030696	.050723	.9	.314276	.115886	.022471	.038600
.368898	.125141	.031393	.048032	1.0	.345781	.110497	.023147	.036611
.431843	.110926	.031176	.042100	1.2	.405826	.099165	.023365	.032269
.489833	.096602	.029225	.035756	1.4	.461636	.087583	.022318	.027655
.542698	.082818	.026116	.029413	1.6	.513018	.076258	.020368	.023040
.590448	.070049	.022416	.023447	1.8	.559924	.065579	.017890	.018669
.633240	.058592	.018596	.018142	2.0	.602437	.055812	.015213	.014733
.720478	.036270	.010403	.008519	2.5	.690836	.036110	.009097	.007359
.784262	.022031	.005251	.003541	3.0	.757346	.022854	.004914	.003292
.830599	.013508	.002551	.001385	3.5	.806928	.014501	.002539	.001389
.864499	.008504	.001253	.000541	4.0	.843995	.009373	.001311	.000581
.889667	.005535	.000640	.000220	4.5	.871993	.006220	.000695	.000250
.908688	.003726	.000344	.000096	5.0	.893440	.004246	.000383	.000113
.923328	.002588	.000194	.000044	5.5	.910120	.002979	.000220	.000054
.934793	.001849	.000115	.000022	6.0	.923290	.002144	.000132	.000027
.951282	.001012	.000045	.000006	7.0	.942397	.001186	.000053	.000008
.962293	.000598	.000020	.000002	8.0	.955265	.000705	.000024	.000003
.969984	.000375	.000010	.000001	9.0	.964307	.000444	.000012	.000001
.975558	.000247	.000005	.000000	10.0	.970885	.000293	.000006	.000001
$n = 7$					$n = 8$			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.034253	.124701	.002561	.037409	.1	.032407	.110868	.001958	.030230
.068404	.123809	.005063	.037137	.2	.064726	.110144	.003875	.030012
.102351	.122341	.007451	.036684	.3	.096870	.108950	.005711	.029650
.135995	.120321	.009674	.036055	.4	.128753	.107306	.007429	.029148
.169243	.117784	.011687	.035252	.5	.160294	.105237	.008998	.028510
.202002	.114774	.013455	.034284	.6	.191413	.102776	.010391	.027743
.234190	.111339	.014951	.033158	.7	.222038	.099961	.011588	.026854
.265727	.107534	.016158	.031886	.8	.252099	.096833	.012575	.025854
.296544	.103416	.017070	.030483	.9	.281535	.093436	.013347	.024754
.326578	.099045	.017689	.028966	1.0	.310288	.089816	.013902	.023568
.384087	.089783	.018102	.025672	1.2	.365556	.082094	.014393	.021000
.437919	.080204	.017566	.022183	1.4	.417589	.074029	.014156	.018282
.487870	.070712	.016317	.018687	1.6	.466184	.065946	.013348	.015550
.533861	.061630	.014610	.015354	1.8	.511239	.058115	.012149	.012930
.575920	.053189	.012683	.012318	2.0	.552747	.050740	.010731	.010521
.664779	.035667	.008015	.006476	2.5	.641605	.035052	.007106	.005775
.733201	.023347	.004579	.003077	3.0	.711356	.023605	.004257	.002885
.785306	.015247	.002491	.001381	3.5	.765436	.015802	.002420	.001363
.824978	.010083	.001342	.000610	4.0	.807266	.010663	.001354	.000632
.855396	.006808	.000735	.000275	4.5	.839764	.007314	.000764	.000296
.878977	.004708	.000415	.000129	5.0	.865229	.005120	.000441	.000143
.897491	.003336	.000243	.000063	5.5	.885397	.003660	.000262	.000072
.912219	.002418	.000147	.000032	6.0	.901554	.002672	.000161	.000038
.933764	.001351	.000060	.000010	7.0	.925372	.001507	.000066	.000012
.948393	.000808	.000027	.000003	8.0	.941672	.000907	.000030	.000004
.958730	.000511	.000013	.000001	9.0	.953250	.000576	.000015	.000002
.966280	.000338	.000007	.000001	10.0	.961741	.000383	.000008	.000001

TABLE 1—*Continued*

$n = 9$					$n = 10$			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.030830	.099799	.001537	.024941	.1	.029463	.090740	.001233	.020931
.061584	.099199	.003044	.024764	.2	.058859	.090235	.002443	.020786
.092185	.098209	.004492	.024472	.3	.088119	.089400	.003609	.020548
.122558	.096844	.005853	.024067	.4	.117178	.088249	.004710	.020217
.152631	.095124	.007105	.023553	.5	.145973	.086796	.005728	.019798
.182336	.093075	.008227	.022936	.6	.174442	.085062	.006648	.019295
.211608	.090724	.009203	.022224	.7	.202526	.083070	.007457	.018715
.240385	.088106	.010023	.021423	.8	.230172	.080845	.008146	.018064
.268611	.085254	.010680	.020544	.9	.257328	.078416	.008711	.017351
.296236	.082204	.011174	.019598	1.0	.283949	.075811	.009148	.016582
.349505	.075663	.011686	.017550	1.2	.335422	.070194	.009652	.014921
.399900	.068772	.011626	.015383	1.4	.384316	.064234	.009700	.013159
.447217	.061798	.011105	.013197	1.6	.430436	.058151	.009369	.011377
.491345	.054970	.010250	.011089	1.8	.473660	.052141	.008754	.009648
.532251	.048466	.009191	.009133	2.0	.513940	.046358	.007953	.008031
.620797	.034336	.006335	.005201	2.5	.601965	.033564	.005677	.004722
.691448	.023693	.003954	.002712	3.0	.673196	.023656	.003672	.002554
.747081	.016207	.002335	.001338	3.5	.730048	.016494	.002243	.001310
.790707	.011135	.001350	.000646	4.0	.775176	.011518	.001336	.000655
.825001	.007749	.000782	.000314	4.5	.811027	.008121	.000793	.000328
.852137	.005485	.000461	.000156	5.0	.839647	.005809	.000476	.000167
.873801	.003956	.000278	.000080	5.5	.862666	.004225	.000292	.000087
.891269	.002907	.000173	.000042	6.0	.881341	.003125	.000183	.000047
.917210	.001655	.000072	.000013	7.0	.909267	.001796	.000078	.000015
.935095	.001002	.000033	.000005	8.0	.928657	.001094	.000036	.000006
.947866	.000639	.000017	.000002	9.0	.942574	.000700	.000018	.000002
.957267	.000426	.000009	.000001	10.0	.952855	.000468	.000010	.000001

$n = 11$					$n = 12$			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.028263	.083189	.001007	.017817	.1	.027198	.076798	.000835	.015351
.056465	.082758	.001996	.017698	.2	.054342	.076426	.001656	.015252
.084548	.082045	.002952	.017501	.3	.081378	.075810	.002451	.015087
.112451	.081061	.003857	.017228	.4	.108253	.074959	.003206	.014859
.140118	.079817	.004699	.016882	.5	.134915	.073882	.003912	.014571
.167493	.078331	.005464	.016467	.6	.161315	.072594	.004557	.014225
.194525	.076620	.006144	.015989	.7	.187406	.071109	.005135	.013827
.221165	.074706	.006730	.015453	.8	.213142	.069444	.005638	.013380
.247365	.072611	.007219	.014865	.9	.238482	.067618	.006064	.012890
.273085	.070358	.007607	.014232	1.0	.263387	.065651	.006409	.012362
.322932	.065482	.008087	.012863	1.2	.311752	.061375	.006861	.011219
.370448	.060272	.008200	.011409	1.4	.357997	.056782	.007011	.010002
.415443	.054918	.007999	.009932	1.6	.401939	.052029	.006899	.008762
.457795	.049583	.007555	.008492	1.8	.443454	.047261	.006578	.007547
.497443	.044406	.006942	.007136	2.0	.482473	.042597	.006106	.006395
.584804	.032766	.005110	.004314	2.5	.569075	.031961	.004619	.003964
.656373	.023529	.003412	.002410	3.0	.640796	.023335	.003173	.002279
.714179	.016688	.002148	.001278	3.5	.699341	.016806	.002052	.001244
.760565	.011826	.001313	.000660	4.0	.746784	.012071	.001285	.000661
.797769	.008441	.000797	.000340	4.5	.785167	.008713	.000797	.000349
.827714	.006097	.000488	.000177	5.0	.816294	.006353	.000496	.000186
.851963	.004469	.000303	.000094	5.5	.841663	.004691	.000312	.000101
.871749	.003326	.000192	.000052	6.0	.862474	.003513	.000200	.000056
.901532	.001929	.000083	.000017	7.0	.893997	.002055	.000088	.000019
.922354	.001182	.000039	.000006	8.0	.916181	.001266	.000042	.000007
.937371	.000760	.000020	.000003	9.0	.932255	.000817	.000021	.000003
.948505	.000509	.000011	.000001	10.0	.944215	.000549	.000011	.000001

TABLE 1—*Continued*

n = 13					n = 14			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.026245	.071320	.000701	.013365	.1	.025386	.066571	.000596	.011741
.052442	.070995	.001392	.013281	.2	.050727	.066285	.001183	.011670
.078540	.070457	.002061	.013142	.3	.075979	.065812	.001754	.011552
.104492	.069714	.002700	.012950	.4	.101099	.065157	.002299	.011389
.130253	.068773	.003298	.012708	.5	.126043	.064327	.002811	.011182
.155775	.067645	.003848	.012417	.6	.150770	.063332	.003285	.010935
.181017	.066343	.004344	.012081	.7	.175241	.062181	.003715	.010650
.205937	.064882	.004781	.011704	.8	.199418	.060888	.004096	.010329
.230496	.063276	.005154	.011291	.9	.223265	.059465	.004425	.009978
.254659	.061543	.005462	.010846	1.0	.246749	.057926	.004701	.009599
.301668	.057763	.005883	.009881	1.2	.292509	.054559	.005091	.008777
.346736	.053681	.006053	.008852	1.4	.336485	.050906	.005272	.007897
.389689	.049433	.006003	.007799	1.6	.378509	.047085	.005265	.006994
.430404	.045143	.005773	.006762	1.8	.418458	.043205	.005102	.006101
.468804	.040919	.005408	.005773	2.0	.456255	.039361	.004818	.005244
.554583	.031162	.004192	.003658	2.5	.541171	.030377	.003818	.003390
.626312	.023093	.002953	.002157	3.0	.612797	.022816	.002752	.002045
.685423	.016865	.001957	.001210	3.5	.672332	.016875	.001865	.001175
.733753	.012263	.001254	.000659	4.0	.721404	.012411	.001220	.000655
.773166	.008946	.000792	.000356	4.5	.761718	.009142	.000784	.000362
.805351	.006579	.000501	.000193	5.0	.794852	.006780	.000503	.000200
.831740	.004893	.000320	.000107	5.5	.822171	.005076	.000325	.000112
.853498	.003685	.000207	.000060	6.0	.844806	.003845	.000213	.000064
.886653	.002174	.000092	.000021	7.0	.879492	.002287	.000096	.000022
.910132	.001348	.000044	.000008	8.0	.904205	.001426	.000046	.000009
.927224	.000873	.000023	.000003	9.0	.922275	.000927	.000024	.000004
.939984	.000588	.000012	.000001	10.0	.935810	.000626	.000013	.000002

n = 15					n = 16			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.024606	.062415	.000512	.010396	.1	.023893	.058748	.000443	.009270
.049171	.062161	.001016	.010335	.2	.047749	.058521	.000880	.009218
.073654	.061742	.001507	.010234	.3	.071530	.058146	.001306	.009131
.098016	.061160	.001977	.010095	.4	.095199	.057627	.001714	.009010
.122217	.060423	.002420	.009918	.5	.118719	.056967	.002101	.008858
.146219	.059538	.002831	.009706	.6	.142057	.056175	.002461	.008675
.169986	.058514	.003207	.009461	.7	.165178	.055258	.002791	.008463
.193483	.057361	.003542	.009187	.8	.188049	.054224	.003088	.008226
.216677	.056091	.003834	.008885	.9	.210641	.053083	.003348	.007965
.239537	.054715	.004082	.008560	1.0	.232924	.051845	.003572	.007684
.284143	.051696	.004442	.007853	1.2	.276460	.049123	.003904	.007071
.327100	.048408	.004626	.007094	1.4	.318466	.046147	.004087	.006412
.368249	.044953	.004649	.006313	1.6	.358789	.043006	.004131	.005732
.407469	.041425	.004536	.005538	1.8	.397312	.039785	.004056	.005053
.444679	.037911	.004316	.004790	2.0	.433954	.036559	.003885	.004396
.528709	.029611	.003489	.003153	2.5	.517087	.028869	.003199	.002942
.600144	.022515	.002568	.001942	3.0	.588262	.022196	.002400	.001847
.659986	.016847	.001777	.001140	3.5	.648315	.016787	.001692	.001106
.709679	.012521	.001184	.000650	4.0	.698524	.012599	.001147	.000642
.750780	.009308	.000774	.000365	4.5	.740314	.009447	.000762	.000368
.784766	.006956	.000504	.000206	5.0	.775065	.007112	.000503	.000210
.812935	.005243	.000330	.000117	5.5	.804013	.005394	.000333	.000121
.836381	.003992	.000218	.000068	6.0	.828210	.004128	.000222	.000071
.872506	.002394	.000099	.000024	7.0	.865687	.002495	.000103	.000026
.898394	.001501	.000048	.000009	8.0	.892696	.001572	.000050	.000010
.917405	.000980	.000025	.000004	9.0	.912613	.001031	.000026	.000004
.931692	.000663	.000014	.000002	10.0	.927629	.000700	.000015	.000002

TABLE 1—*Continued*

$n = 17$					$n = 18$			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.023240	.055488	.000387	.008318	.1	.022636	.052570	.000340	.007505
.046444	.055284	.000768	.008272	.2	.045241	.052386	.000676	.007466
.069579	.054947	.001140	.008197	.3	.067780	.052082	.001003	.007400
.092611	.054480	.001498	.008092	.4	.090224	.051659	.001319	.007308
.115506	.053887	.001838	.007960	.5	.112541	.051123	.001619	.007192
.138232	.053174	.002155	.007801	.6	.134700	.050478	.001900	.007053
.160757	.052347	.002447	.007617	.7	.156673	.049729	.002160	.006892
.183050	.051414	.002711	.007410	.8	.178431	.048883	.002396	.006712
.205085	.050384	.002945	.007184	.9	.199948	.047948	.002607	.006513
.226833	.049264	.003147	.006938	1.0	.221198	.046930	.002790	.006298
.269371	.046797	.003454	.006404	1.2	.262804	.044683	.003074	.005829
.310485	.044090	.003633	.005827	1.4	.303080	.042211	.003247	.005321
.350029	.041222	.003691	.005230	1.6	.341886	.039581	.003315	.004794
.387888	.038269	.003645	.004632	1.8	.379110	.036863	.003291	.004265
.423979	.035297	.003513	.004051	2.0	.414671	.034116	.003190	.003747
.506213	.028150	.002941	.002752	2.5	.496009	.027458	.002712	.002582
.577076	.021867	.002246	.001758	3.0	.566517	.021530	.002105	.001675
.637258	.016703	.001611	.001072	3.5	.626761	.016597	.001534	.001039
.687893	.012650	.001110	.000634	4.0	.677746	.012678	.001074	.000625
.730286	.009562	.000749	.000369	4.5	.720667	.009656	.000734	.000370
.765726	.007249	.000500	.000214	5.0	.756726	.007369	.000496	.000218
.795386	.005530	.000334	.000125	5.5	.787038	.005654	.000335	.000129
.820280	.004254	.000225	.000074	6.0	.812580	.004370	.000228	.000077
.859030	.002590	.000106	.000027	7.0	.852528	.002681	.000108	.000029
.887108	.001641	.000052	.000011	8.0	.881625	.001708	.000054	.000012
.907897	.001080	.000027	.000005	9.0	.903255	.001128	.000029	.000005
.923619	.000735	.000015	.000002	10.0	.919661	.000770	.000016	.000002

$n = 19$					$n = 20$			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.022078	.049944	.000301	.006806	.1	.021559	.047568	.000267	.006201
.044126	.049777	.000598	.006771	.2	.043090	.047416	.000532	.006170
.066114	.049501	.000888	.006713	.3	.064565	.047164	.000790	.006119
.088013	.049117	.001168	.006633	.4	.085956	.046814	.001040	.006048
.109793	.048630	.001435	.006531	.5	.107237	.046369	.001279	.005958
.131426	.048043	.001686	.006409	.6	.128380	.045833	.001504	.005849
.152886	.047361	.001918	.006267	.7	.149361	.045210	.001713	.005724
.174145	.046591	.002131	.006108	.8	.170154	.044505	.001904	.005584
.195179	.045738	.002321	.005933	.9	.190736	.043724	.002077	.005428
.215963	.044809	.002488	.005744	1.0	.211085	.042873	.002230	.005261
.256697	.042754	.002750	.005329	1.2	.250998	.040986	.002473	.004893
.296184	.040486	.002916	.004880	1.4	.289741	.038898	.002632	.004493
.334291	.038067	.002991	.004413	1.6	.327184	.036665	.002710	.004076
.370909	.035556	.002983	.003941	1.8	.363222	.034339	.002715	.003654
.405957	.033010	.002907	.003478	2.0	.397776	.031971	.002658	.003239
.486408	.026791	.002506	.002428	2.5	.477352	.026149	.002322	.002287
.556529	.021191	.001975	.001599	3.0	.547061	.020852	.001856	.001527
.616778	.016476	.001461	.001007	3.5	.607268	.016341	.001393	.000976
.668047	.012685	.001037	.000615	4.0	.658762	.012677	.001002	.000605
.711427	.009732	.000718	.000369	4.5	.702542	.009791	.000702	.000368
.748044	.007474	.000491	.000220	5.0	.739661	.007565	.000486	.000222
.778955	.005765	.000335	.000132	5.5	.771121	.005866	.000335	.000135
.805098	.004477	.000230	.000080	6.0	.797823	.004575	.000231	.000082
.846174	.002766	.000110	.000030	7.0	.839963	.002847	.000113	.000032
.876245	.001771	.000056	.000012	8.0	.870963	.001832	.000057	.000013
.898684	.001174	.000030	.000005	9.0	.894183	.001219	.000031	.000006
.915755	.000804	.000017	.000003	10.0	.911899	.000837	.000017	.000003

TABLE 1—*Continued*

n = 21					n = 22			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.021075	.045408	.000239	.005673	.1	.020622	.043436	.000215	.005209
.042123	.045269	.000476	.005645	.2	.041219	.043307	.000428	.005185
.063120	.045038	.000707	.005600	.3	.061767	.043095	.000636	.005144
.084037	.044717	.000931	.005537	.4	.082242	.042800	.000838	.005088
.104851	.044309	.001145	.005457	.5	.102618	.042425	.001031	.005017
.125537	.043818	.001348	.005361	.6	.122874	.041973	.001214	.004931
.146070	.043246	.001537	.005249	.7	.142987	.041446	.001385	.004832
.166426	.042599	.001711	.005124	.8	.162934	.040850	.001543	.004720
.186585	.041881	.001868	.004986	.9	.182694	.040188	.001687	.004596
.206524	.041098	.002008	.004837	1.0	.202247	.039466	.001816	.004463
.245665	.039360	.002233	.004509	1.2	.240659	.037858	.002025	.004169
.283703	.037431	.002384	.004152	1.4	.278029	.036072	.002169	.003849
.320515	.035362	.002465	.003778	1.6	.314240	.034150	.002250	.003512
.355998	.033201	.002480	.003399	1.8	.349192	.032137	.002272	.003170
.390075	.030994	.002439	.003024	2.0	.382808	.030074	.002244	.002831
.468790	.025532	.002157	.002160	2.5	.460679	.024940	.002007	.002043
.538068	.020514	.001746	.001460	3.0	.529512	.020179	.001645	.001397
.598193	.016197	.001328	.000946	3.5	.589521	.016044	.001266	.000916
.649862	.012653	.000967	.000594	4.0	.641320	.012617	.000934	.000583
.693990	.009836	.000686	.000366	4.5	.685750	.009869	.000670	.000364
.731561	.007643	.000480	.000224	5.0	.723727	.007711	.000473	.000225
.763525	.005956	.000333	.000137	5.5	.756153	.006038	.000331	.000139
.790747	.004666	.000232	.000084	6.0	.783860	.004749	.000232	.000086
.833890	.002923	.000114	.000033	7.0	.827950	.002995	.000116	.000034
.865778	.001890	.000059	.000014	8.0	.860686	.001946	.000060	.000014
.889751	.001262	.000032	.000006	9.0	.885384	.001304	.000032	.000006
.908092	.000869	.000018	.000003	10.0	.904332	.000900	.000018	.000003

n = 23					n = 24			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.020197	.041627	.000194	.004800	.1	.019797	.039964	.000176	.004438
.040371	.041509	.000386	.004778	.2	.039573	.039854	.000350	.004418
.060498	.041313	.000574	.004742	.3	.059304	.039673	.000520	.004386
.080556	.041041	.000757	.004692	.4	.078971	.039421	.000686	.004340
.100522	.040695	.000932	.004628	.5	.098550	.039101	.000846	.004283
.120375	.040277	.001098	.004551	.6	.118022	.038714	.000997	.004214
.140091	.039791	.001254	.004462	.7	.137365	.038263	.001139	.004134
.159652	.039240	.001398	.004362	.8	.156562	.037752	.001272	.004043
.179037	.038627	.001530	.004251	.9	.175591	.037184	.001393	.003944
.198226	.037958	.001648	.004131	1.0	.194436	.036563	.001502	.003835
.235947	.036468	.001843	.003867	1.2	.231503	.035177	.001683	.003597
.272684	.034808	.001980	.003578	1.4	.267637	.033630	.001813	.003336
.308322	.033018	.002060	.003274	1.6	.302728	.031959	.001893	.003060
.342765	.031138	.002088	.002964	1.8	.336682	.030199	.001925	.002779
.375937	.029206	.002070	.002656	2.0	.369426	.028386	.001915	.002498
.452980	.024371	.001872	.001935	2.5	.445659	.023824	.001749	.001837
.521358	.019848	.001551	.001339	3.0	.513574	.019523	.001465	.001284
.581221	.015885	.001208	.000888	3.5	.573269	.015722	.001154	.000861
.633113	.012571	.000901	.000572	4.0	.625220	.012516	.000870	.000561
.677802	.009890	.000653	.000361	4.5	.670129	.009901	.000637	.000358
.716144	.007768	.000465	.000226	5.0	.708800	.007815	.000458	.000226
.748996	.006110	.000329	.000141	5.5	.742042	.006175	.000326	.000142
.777153	.004826	.000232	.000088	6.0	.770619	.004896	.000232	.000090
.822137	.003063	.000117	.000036	7.0	.816447	.003128	.000118	.000037
.855685	.002000	.000061	.000015	8.0	.850771	.002052	.000062	.000016
.881082	.001345	.000033	.000007	9.0	.876843	.001384	.000034	.000007
.900620	.000930	.000019	.000003	10.0	.896954	.000960	.000019	.000003

TABLE 1—*Continued*

n = 25					n = 30			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.019421	.038428	.000160	.004115	.1	.017814	.032234	.000104	.002929
.038820	.038326	.000318	.004097	.2	.035611	.032162	.000208	.002918
.058179	.038158	.000474	.004068	.3	.053376	.032042	.000310	.002899
.077475	.037925	.000625	.004027	.4	.071093	.031875	.000409	.002874
.096689	.037627	.000770	.003975	.5	.088746	.031662	.000505	.002841
.115801	.037268	.000909	.003913	.6	.106319	.031404	.000598	.002802
.134793	.036849	.001039	.003840	.7	.123798	.031103	.000685	.002757
.153644	.036373	.001160	.003759	.8	.141167	.030761	.000768	.002705
.172338	.035845	.001272	.003669	.9	.158412	.030379	.000845	.002648
.190856	.035266	.001373	.003571	1.0	.175520	.029961	.000916	.002586
.227302	.033975	.001542	.003355	1.2	.209270	.029021	.001038	.002448
.262862	.032530	.001666	.003118	1.4	.242319	.027961	.001133	.002296
.297429	.030966	.001744	.002867	1.6	.274577	.026804	.001200	.002133
.330914	.029315	.001779	.002610	1.8	.305969	.025571	.001241	.001965
.363244	.027610	.001776	.002353	2.0	.336428	.024284	.001256	.001794
.438685	.023299	.001637	.001746	2.5	.408180	.020962	.001204	.001380
.506134	.019203	.001385	.001232	3.0	.473291	.017705	.001064	.001015
.565639	.015555	.001102	.000835	3.5	.531644	.014703	.000885	.000720
.617619	.012454	.000839	.000550	4.0	.583436	.012062	.000704	.000496
.662717	.009904	.000620	.000355	4.5	.629076	.009816	.000542	.000335
.701682	.007855	.000450	.000226	5.0	.669100	.007952	.000409	.000223
.735282	.006233	.000323	.000144	5.5	.704096	.006433	.000304	.000147
.764250	.004961	.000231	.000092	6.0	.734656	.005208	.000225	.000097
.810875	.003188	.000119	.000038	7.0	.784661	.003443	.000122	.000043
.845942	.002101	.000063	.000016	8.0	.822992	.002318	.000067	.000019
.872666	.001422	.000035	.000008	9.0	.852651	.001595	.000038	.000009
.893332	.000989	.000020	.000004	10.0	.875869	.001123	.000022	.000005

n = 35					n = 40			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.016549	.027760	.000073	.002191	.1	.015521	.024376	.000053	.001700
.033085	.027706	.000145	.002183	.2	.031031	.024334	.000105	.001695
.049595	.027616	.000216	.002171	.3	.046519	.024264	.000157	.001686
.066066	.027490	.000285	.002154	.4	.061975	.024167	.000208	.001674
.082486	.027330	.000353	.002132	.5	.077388	.024042	.000257	.001659
.098841	.027137	.000418	.002106	.6	.092747	.023891	.000305	.001641
.115119	.026910	.000480	.002075	.7	.108043	.023715	.000352	.001619
.131309	.026652	.000539	.002041	.8	.123265	.023513	.000396	.001595
.147398	.026364	.000595	.002003	.9	.138404	.023287	.000437	.001568
.163376	.026046	.000647	.001961	1.0	.153449	.023039	.000477	.001538
.194952	.025332	.000738	.001867	1.2	.183223	.022477	.000547	.001472
.225954	.024521	.000812	.001764	1.4	.212518	.021837	.000606	.001398
.256309	.023630	.000868	.001652	1.6	.241268	.021130	.000652	.001318
.285948	.022674	.000906	.001535	1.8	.269415	.020366	.000686	.001234
.314815	.021668	.000928	.001416	2.0	.296908	.019558	.000708	.001147
.383303	.019030	.000917	.001120	2.5	.362504	.017412	.000717	.000928
.446160	.016378	.000837	.000851	3.0	.423245	.015212	.000672	.000724
.503181	.013869	.000721	.000625	3.5	.478884	.013086	.000595	.000546
.554426	.011601	.000594	.000446	4.0	.529393	.011121	.000505	.000402
.600149	.009619	.000473	.000312	4.5	.574918	.009365	.000414	.000289
.640730	.007931	.000368	.000215	5.0	.615728	.007837	.000331	.000205
.676622	.006521	.000282	.000147	5.5	.652169	.006534	.000260	.000144
.708299	.005357	.000214	.000100	6.0	.684624	.005437	.000202	.000101
.760869	.003631	.000122	.000046	7.0	.739145	.003768	.000120	.000049
.801837	.002493	.000069	.000022	8.0	.782254	.002634	.000070	.000024
.833970	.001742	.000040	.000011	9.0	.816479	.001866	.000042	.000012
.859403	.001241	.000024	.000005	10.0	.843841	.001344	.000026	.000006

TABLE 1—*Continued*

$n = 50$					$n = 60$			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.013933	.019599	.000031	.001109	.1	.012750	.016387	.000020	.000780
.027858	.019571	.000062	.001106	.2	.025495	.016368	.000040	.000778
.041768	.019525	.000092	.001102	.3	.038227	.016335	.000059	.000776
.055653	.019462	.000122	.001095	.4	.050941	.016290	.000079	.000772
.069508	.019380	.000151	.001087	.5	.063631	.016233	.000098	.000767
.083323	.019281	.000180	.001077	.6	.076290	.016163	.000116	.000761
.097092	.019165	.000208	.001065	.7	.088914	.016081	.000135	.000754
.110807	.019032	.000234	.001052	.8	.101496	.015987	.000152	.000745
.124461	.018883	.000260	.001037	.9	.114031	.015881	.000169	.000736
.138046	.018719	.000284	.001020	1.0	.126513	.015764	.000185	.000726
.164984	.018345	.000329	.000984	1.2	.151296	.015498	.000216	.000704
.191567	.017917	.000367	.000942	1.4	.175803	.015191	.000242	.000678
.217746	.017440	.000400	.000897	1.6	.199995	.014848	.000266	.000650
.243475	.016921	.000426	.000849	1.8	.223834	.014472	.000285	.000620
.268712	.016366	.000445	.000798	2.0	.247286	.014068	.000301	.000588
.329423	.014865	.000467	.000668	2.5	.304029	.012960	.000324	.000505
.386387	.013282	.000455	.000542	3.0	.357768	.011766	.000325	.000421
.439328	.011702	.000421	.000427	3.5	.408231	.010547	.000310	.000342
.488128	.010191	.000373	.000329	4.0	.455263	.009351	.000284	.000272
.532804	.008793	.000320	.000248	4.5	.498819	.008217	.000251	.000212
.573481	.007535	.000267	.000184	5.0	.538939	.007169	.000217	.000163
.610359	.006424	.000219	.000135	5.5	.575734	.006220	.000183	.000124
.643686	.005460	.000176	.000098	6.0	.609364	.005376	.000153	.000093
.700804	.003928	.000112	.000051	7.0	.667918	.003988	.000102	.000052
.747080	.002832	.000070	.000027	8.0	.716298	.002952	.000067	.000029
.784593	.002059	.000043	.000014	9.0	.756203	.002193	.000043	.000016
.815119	.001513	.000027	.000008	10.0	.789165	.001642	.000028	.000009

$n = 80$					$n = 100$			
μ	σ^2	μ_3	μ_4	θ	μ	σ^2	μ_3	μ_4
.011076	.012342	.000010	.000446	.1	.009925	.009899	.000006	.000288
.022148	.012331	.000020	.000445	.2	.019848	.009891	.000011	.000288
.033213	.012312	.000030	.000444	.3	.029764	.009880	.000017	.000287
.044265	.012287	.000039	.000442	.4	.039672	.009863	.000023	.000286
.055301	.012254	.000049	.000440	.5	.049569	.009841	.000028	.000285
.066318	.012213	.000058	.000437	.6	.059451	.009815	.000034	.000284
.077310	.012166	.000067	.000434	.7	.069316	.009785	.000039	.000282
.088275	.012112	.000076	.000431	.8	.079161	.009749	.000045	.000280
.099209	.012051	.000085	.000426	.9	.088983	.009710	.000050	.000278
.110107	.011983	.000094	.000422	1.0	.098779	.009666	.000055	.000276
.131784	.011828	.000110	.000412	1.2	.118285	.009565	.000064	.000270
.153276	.011649	.000124	.000400	1.4	.137657	.009447	.000074	.000264
.174557	.011447	.000138	.000387	1.6	.156874	.009314	.000082	.000257
.195600	.011224	.000149	.000373	1.8	.175917	.009166	.000089	.000249
.216381	.010982	.000159	.000358	2.0	.194768	.009005	.000096	.000241
.267038	.010307	.000177	.000317	2.5	.240936	.008551	.000109	.000218
.315607	.009559	.000185	.000275	3.0	.285545	.008039	.000117	.000194
.361849	.008771	.000184	.000233	3.5	.328394	.007489	.000119	.000169
.405599	.007971	.000176	.000194	4.0	.369329	.006918	.000118	.000145
.446760	.007186	.000164	.000159	4.5	.408242	.006343	.000113	.000123
.485295	.006433	.000148	.000128	5.0	.445070	.005779	.000105	.000102
.521221	.005725	.000131	.000102	5.5	.479789	.005235	.000097	.000084
.554596	.005072	.000114	.000081	6.0	.512408	.004720	.000087	.000069
.614083	.003942	.000083	.000049	7.0	.571520	.003796	.000068	.000045
.664726	.003042	.000059	.000030	8.0	.622946	.003023	.000051	.000029
.707646	.002343	.000041	.000018	9.0	.667426	.002396	.000037	.000018
.743965	.001809	.000028	.000011	10.0	.705773	.001897	.000027	.000011