BALANCED DESIGNS WITH UNEQUAL NUMBERS OF REPLICATES

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1. Introduction. Rao [5] proved as a theorem that a necessary and sufficient condition for a design to be balanced, i.e., for var$(\hat{\tau}_i - \hat{\tau}_{i'})$ to be the same for all pairs (i, i'), $i \neq i'$, is that the matrix $\mathbf{C}$ of the adjusted intrablock normal equations shall have all its diagonal elements equal and all its off-diagonal elements equal. If r_i denotes the number of times that the ith treatment is replicated, k_j denotes the number of plots in the jth block and $\mathbf{n} = (n_{ij})$ is the incidence matrix, the elements of $\mathbf{C}$ are

$$c_{ii} = r_i - \sum_j (n_{ij}^2/k_j); \qquad c_{ii'} = -\sum_j (n_{ij}n_{i'j}/k_j), \qquad i \neq i'.$$

The design is said to be proper if $k_j = k$ for all k, equireplicate if all the r_i are equal and binary if n_{ij} takes only the values 0 or 1.

Rao also proved as a corollary that if a binary balanced design is proper, then it must be equireplicate, and gave an example of an equireplicate binary balanced design with unequal block sizes. Results published later by Atiqullah [1], Graybill [3], and Hanani [4], enable us to give a simpler proof of Rao's theorem and to derive examples showing that the corollary does not hold when either the binary requirement or the requirement of equal block sizes is relaxed.

2. The theorem. $\mathbf{1}_v$ will denote the column vector of v elements each of which is unity; $\mathbf{J}_v$ will denote the matrix $\mathbf{1}_v\mathbf{1}_v'$. $\mathbf{C}$ is a real symmetric matrix of rank $(v - 1)$ such that $\mathbf{C1} = 0$. The normal equations are solved under the side condition $\mathbf{1}'\boldsymbol{\tau} = \mathbf{1}'\hat{\boldsymbol{\tau}} = 0$.

Graybill's method of solution ([3], p. 292) is to consider the augmented matrix $\mathbf{C}^*$ and its inverse

$$\mathbf{C}^* = \begin{pmatrix} \mathbf{C} & \mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix}, \mathbf{C}^{*-1} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & B_{22} \end{pmatrix}.$$

Then it is readily shown that

$$\hat{\boldsymbol{\tau}} = \mathbf{B}_{11}\mathbf{Q} \tag{1}$$

$$\mathbf{1}'\mathbf{B}_{11} = 0 = \mathbf{B}_{11}\mathbf{1} \tag{2}$$

$$\text{cov}\,(\hat{\boldsymbol{\tau}}) = \mathbf{B}_{11}\mathbf{C}\mathbf{B}_{11}\sigma^2 = \mathbf{B}_{11}\sigma^2 \tag{3}$$

$$\mathbf{C}\mathbf{B}_{11} = \mathbf{I}_v - v^{-1}\mathbf{J}_v = \mathbf{B}_{11}\mathbf{C}. \tag{4}$$

Atiqullah [1] has shown that necessary and sufficient conditions for a design to be balanced are that var $\hat{\tau}_i$ shall be the same for all i, and that cov $(\hat{\tau}_i, \hat{\tau}_{i'})$ shall be the same for all pairs i, i', $i \neq i'$. From (2) and (3) these conditions are equivalent to $\mathbf{B}_{11} = a(\mathbf{I}_v - v^{-1}\mathbf{J}_v)$ where a is some positive constant.

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Suppose the design is balanced; then (4) becomes

$$a\mathbf{C}(\mathbf{I}_v - v^{-1}\mathbf{J}_v) = \mathbf{I}_v - v^{-1}\mathbf{J}_v$$

but $\mathbf{CJ} = 0$, and so $a\mathbf{C} = \mathbf{I}_v - v^{-1}\mathbf{J}_v$ and $\mathbf{C} = (\mathbf{I}_v - v^{-1}\mathbf{J}_v)/a$.

Conversely, suppose that $\mathbf{C} = b\mathbf{I}_v + c\mathbf{J}_v$ where b and c are constants.

Since $\mathbf{CJ} = 0$, $b + vc = 0$, and so $\mathbf{C} = b(\mathbf{I}_v - v^{-1}\mathbf{J}_v)$, $b > 0$. Then (4) becomes

$$b\mathbf{B}_{11}(\mathbf{I}_v - v^{-1}\mathbf{J}_v) = \mathbf{I}_v - v^{-1}\mathbf{J}_v$$

whence by (2) $\mathbf{B}_{11} = (\mathbf{I}_v - v^{-1}\mathbf{J}_v)/b$ and the theorem is proved.

3. The examples. The designs involve $(v + 1)$ treatments $t_0, t_1, \cdots, t_v$, and each design consists of two portions. In the first portion the jth block, $j = 1, \cdots, v$, consists of k^* plots with t_0 and one plot with t_j; the second portion is a BIBD in the treatments $t_1, \cdots, t_v$, with parameters (b, v, r, k, λ). Then $r_0 = k^*v$ and $r_i = r + 1, i > 0$.

Proper designs. Let $k = 3$, $k^* = 2$, $r_0 = 2v$. Then

$$c_{00} = 2v/3, \qquad c_{ii} = 2(r + 1)/3,$$
$$c_{0i} = -\tfrac{2}{3}, \qquad c_{ii'} = -\lambda/3 \qquad (i, i' > 0, i \neq i').$$

For balance $c_{0i} = c_{ii'}$ or $\lambda = 2$ and $c_{00} = c_{ii}$ or $r = (v - 1)$, $b = v(v - 1)/3$, so that $r_i = v \neq r_0$.

Such a design exists whenever the BIBD portion exists. For $k = 3$ it has been shown by Bose [2] and Hanani [4] that necessary and sufficient conditions for the existence of a BIBD are that

$$\text{(i)}\quad \lambda(v - 1) \equiv 0 \pmod 2 \qquad \text{and} \qquad \text{(ii)}\quad \lambda v(v - 1) \equiv 0 \pmod 6.$$

Condition (i) is satisfied by $\lambda = 2$. Condition (ii) is equivalent to the condition that b be an integer and is satisfied for $\lambda = 2$ if $v \equiv 0$ or $1 \pmod 3$.

Hence designs of the above type which are balanced, proper but not equireplicate exist for all values of v such that $v \equiv 0$ or $1 \pmod 3$.

The design for $v = 4$ may be written with i denoting t_i as 001, 002, 003, 004, 123, 124, 134, 234.

Binary designs with unequal block sizes. Let $k^* = 1$, $r_0 = v$. Then

$$c_{00} = v/2, \qquad c_{ii} = r(k - 1)/k + \tfrac{1}{2},$$
$$c_{0i} = -\tfrac{1}{2}, \qquad c_{ii'} = -\lambda/k.$$

The design is balanced if $\lambda = k/2$, in which case

$$r = k(v - 1)/2(k - 1) \qquad \text{and} \qquad r_i = 1 + r \neq r_0, \qquad k > 2.$$

In particular, if $k = 4$, $\lambda = 2$, Hanani [4] showed that necessary and sufficient conditions for the existence of a BIBD with $k = 4$ are

$$\text{(i)}\quad \lambda(v - 1) \equiv 0 \pmod 3 \qquad \text{and} \qquad \text{(ii)}\quad \lambda v(v - 1) \equiv 0 \pmod{12}.$$

With $\lambda = 2$, both conditions are satisfied when $v \equiv 1 \pmod 3$.

Hence there exist designs of the above type which are balanced, binary but not equireplicate for all values of v such that $v \equiv 1 \pmod 3$.

The design for $v = 4$ is 01, 02, 03, 04, 1234, 1234.

REFERENCES

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